Progressions for the Common Core State Standards for Mathematics

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For more information about the Progressions, see `http://ime.math.arizona.edu/progressions`.

For discussion of the Progressions and related topics, see the Mathematical Musings blog: `http://mathematicalmusings.org`.

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Preface

The Common Core State Standards in mathematics began with progressions: narrative documents describing the progression of a topic across a number of grade levels, informed both by educational research and the structure of mathematics. These documents were then sliced into grade level standards. From that point on the work focused on refining and revising the grade level standards, thus, the early drafts of the progressions documents do not correspond to the 2010 Standards.

The Progressions for the Common Core State Standards are updated versions of those early progressions drafts, revised and edited to correspond with the Standards by members of the original Progressions work team, together with other mathematicians and education researchers not involved in the initial writing. They note key connections among standards, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of the mathematics.

Audience The Progressions are intended to inform teacher preparation and professional development, curriculum organization, and textbook content. Thus, their audience includes teachers and anyone involved with schools, teacher education, test development, or curriculum development. Members of this audience may require some guidance in working their way through parts of the mathematics in the Progressions. As with any written mathematics, understanding the Progressions may take time and discussion with others.

Revision of the draft Progressions was informed by comments and discussion at http://mathematicalmusings.org, Mathematical Musings (formerly The Tools for the Common Core blog). This blog is a venue for discussion of the Standards as well as the Progressions and is maintained by lead Standards writer Bill McCallum.

Scope Because they note key connections among standards and topics, the Progressions offer some guidance but not complete guidance about how topics might be sequenced and approached across and within grades. In this respect, the Progressions are an intermediate step between the Standards and a teachers manual for a grade-level textbook—a type of document that is uncommon in the United States.

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Other sources of information   Another important source of information about the Standards and their implications for curriculum is the *Publishers' Criteria for the Common Core State Standards for Mathematics*, available at http://bit.ly/2L7tMzx. In addition to giving criteria for evaluating K–12 curriculum materials, this document gives a brief and very useful orientation to the Standards in its short essay "The Structure is the Standards."

Illustrative Mathematics illustrates the range and types of mathematical work that students experience in a faithful implementation of the Common Core State Standards. This and other ongoing projects that involve the Standards writers and support the Common Core are listed at http://ime.math.arizona.edu/commoncore.

Understanding Language aims to heighten awareness of the critical role that language plays in the new Common Core State Standards and Next Generation Science Standards, to synthesize knowledge, and to develop resources that help ensure teachers can meet their students' evolving linguistic needs as the new Standards are implemented. See http://ell.stanford.edu.

In the context of the Common Core, teachers' needs for preparation and professional development are often substantial. The Conference Board of the Mathematical Sciences report *The Mathematical Education of Teachers II* gives recommendations for mathematical preparation and professional development, and for mathematicians' involvement in teachers' mathematical education. See www.cbmsweb.org. The American Statistical Association report *The Statistical Education of Teachers* gives recommendations for statistical preparation and professional development. See http://amstat.org/asa/files/pdfs/EDU-SET.pdf.

Modeling is discussed in *Guidelines for Assessment and Instruction in Mathematical Modeling Education* from the Consortium for Mathematics and Its Applications and the Society for Industrial and Applied Mathematics. See http://www.siam.org/reports/gaimme.php.

A statement of support for the Standards was signed by presidents of Conference Board of the Mathematical Sciences member societies (http://bit.ly/2IOPcDy). Statements of support from other organizations are listed at http://bit.ly/2Iq5dQZ.

Introduction

The college- and career-readiness goals of the Common Core State Standards of the Standards were informed by surveys of college faculty, studies of college readiness, studies of workplace needs, and reports and recommendations that summarize such studies.• Created to achieve these goals, the Standards are informed by the structure of mathematics as well as three areas of educational research: large-scale comparative studies, research on children's learning trajectories, and other research on cognition and learning.

References to work in these four areas are included in the "works consulted" section of the Standards document. This introduction outlines how the Standards have been shaped by each of these influences, describes the organization of the Standards, discusses how traditional topics have been reconceptualized to fit that organization, and mentions aspects of terms and usage in the Standards and the Progressions.

The structure of mathematics   One aspect of the structure of mathematics is reliance on a small collection of general properties rather than a large collection of specialized properties. For example, addition of fractions in the Standards extends the meanings and properties of addition of whole numbers, applying and extending key ideas used in addition of whole numbers to addition of unit fractions, then to addition of all fractions.• As number systems expand from whole numbers to fractions in Grades 3–5, to rational numbers in Grades 6–8, to real numbers in high school, the same key ideas are used to define operations within each system.

Another aspect of mathematics is the practice of defining concepts in terms of a small collection of fundamental concepts rather than treating concepts as unrelated. A small collection of fundamental concepts underlies the organization of the Standards. Definitions made in terms of these concepts become more explicit over the grades.• For example, subtraction can mean "take from," "find the unknown addend," or "find how much more (or less)," depending on context. However, as a mathematical operation subtraction

• These include the reports from Achieve, ACT, College Board, and American Diploma Project listed in the references for the Common Core State Standards as well as sections of reports such as the American Statistical Association's *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report: A PreK–12 Curriculum Framework* and the National Council on Education and the Disciplines' *Mathematics and Democracy, The Case for Quantitative Literacy*.

• In elementary grades, "whole number" is used with the meaning "non-negative integer" and "fraction" is used with the meaning "non-negative rational number."

• Note Standard for Mathematical Practice 6: "Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. . . . By the time they reach high school they have learned to examine claims and make explicit use of definitions."

can be defined in terms of the fundamental relation of addends and sum. Students acquire an informal understanding of this definition in Grade 1• and use it in solving problems throughout their mathematical work. The number line is another fundamental concept. In elementary grades, students represent whole numbers (2.MD.6), then fractions (3.NF.2) on number line diagrams. Later, they understand integers and rational numbers (6.NS.6), then real numbers (8.NS.2), as points on the number line.•

• Note 1.OA.4: "Understand subtraction as an unknown-addend problem." Similarly, 3.OA.6: "Understand division as an unknown-factor problem."

• For further discussion, see "Overview of School Algebra" in U.S. Department of Education, 2008, "Report of the Task Group on Conceptual Knowledge and Skills" in *Foundations for Success: The Final Report of the National Mathematics Advisory Panel.*

Large–scale comparative studies   One area of research compares aspects of educational systems in different countries. Compared to those of high-achieving countries, U.S. standards and curricula of recent decades were "a mile wide and an inch deep."•

• See Schmidt, Houang, & Cogan, 2002, "A Coherent Curriculum," *American Educator*, http://aft.org/pdfs/americaneducator/summer2002/curriculum.pdf.

In contrast, the organization of topics in high-achieving countries is more focused and more coherent. Focus refers to the number of topics taught at each grade and coherence is related to the way in which topics are organized. Curricula and standards that are focused have few topics in each grade. They are coherent if they are:

> articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential and hierarchical nature of the disciplinary content from which the subject matter derives.•

• Schmidt & Houang, 2012, "Curricular Coherence and the Common Core State Standards for Mathematics," *Educational Researcher*, http://edr.sagepub.com/content/41/8/294, p. 295.

Textbooks and curriculum documents from high–achieving countries give examples of such sequences of topics and performances.•

• For examples of "course of study" documents from other countries, see http://bit.ly/2IjRJ94. Some textbooks from other countries are readily available. The University of Chicago School Mathematics Project (http://bit.ly/18tEN7R) has translations of Japanese textbooks for grades 7–9 and Russian grades 1–3. Singapore Math (www.singaporemath.com) has textbooks from Singapore. Global Education Resources (GER, http://www.globaledresources.com) has translations of Japanese textbooks for grades 1–6 and translations of the teaching guides for grades 1–6 and 7–9. Portions of the teachers manuals for the Japanese textbooks have been translated and can be downloaded at Lesson Study Group at Mills College (http://bit.ly/12bZ1KQ). The first page of a two-page diagram showing connections of topics for Grades 1–6 in Japan can be seen at http://bit.ly/12EOjfN.

Research on children's learning trajectories   Within the United States, researchers who study children's learning have identified developmental sequences associated with constructs such as "teaching–learning paths," "learning progressions," or "learning trajectories." For example,

> A learning trajectory has three parts: a specific mathematical goal, a developmental path along which children develop to reach that goal, and a set of instructional activities that help children move along that path.•

• Clements & Sarama, 2009, *Learning and Teaching Early Math: The Learning Trajectories Approach*, Routledge, p. viii.

Findings from this line of research illuminate those of the large–scale comparative studies by giving details about how particular instructional activities help children develop specific mathematical abilities, identifying behavioral milestones along these paths.

The Progressions for the Common Core State Standards are not "learning progressions" in the sense described above. Well-documented learning progressions for all of K–12 mathematics do not exist. However, the Progressions for Counting and Cardinality, Operations and Algebraic Thinking, Number and Operations in Base Ten, Geometry, and Geometric Measurement are informed by such

learning progressions and are thus able to outline central instructional sequences and activities which have informed the Standards."•

Other research on cognition and learning   Other research on cognition, learning, and learning mathematics has informed the development of the Standards and Progressions in several ways. Fine-grained studies have identified cognitive features of learning and instruction for topics such as the equal sign in elementary and middle grades, proportional relationships, or connections among different representations of a linear function. Such studies have informed the development of standards in areas where learning progressions do not exist.• For example, it is possible for students in early grades to have a "relational" meaning for the equal sign, e.g., understanding $6 = 6$ and $7 = 8 - 1$ as correct equations (1.OA.7), rather than an "operational" meaning in which the right side of the equal sign is restricted to indicating the outcome of a computation. A relational understanding of the equal sign is associated with fewer obstacles in middle grades, and is consistent with its standard meaning in mathematics. Another example: Studies of students' interpretations of functions and graphs indicate specific features of desirable knowledge, e.g., that part of understanding is being able to identify and use the same properties of the same object in different representations. For instance, students identify the constant of proportionality (also known as the unit rate) in a graph, table, diagram, or equation of a proportional relationship (7.RP.2b) and can explain correspondences between its different representations (MP.1).

Studies in cognitive science have examined experts' knowledge, showing what the results of successful learning look like. Rather than being a collection of isolated facts, experts' knowledge is connected and organized according to underlying disciplinary principles.• So, for example, an expert's knowledge of multiplying whole numbers and mixed numbers, expanding binomials, and multiplying complex numbers is connected by common underlying principles rather than four separately memorized and unrelated special-purpose procedures. These findings from studies of experts are consistent with those of comparative research on curriculum. Both suggest that standards and curricula attend to "key ideas that determine how knowledge is organized and generated within that discipline."•

The ways in which content knowledge is deployed (or not) are intertwined with mathematical dispositions and attitudes.• For example, in calculating 30×9, a third grade might use the simpler form of the original problem (MP.1): calculating $3 \times 9 = 27$, then multiplying the result by 10 to get 270 (3.NBT.3). Formulation of the Standards for Mathematical Practice drew on the process standards of the National Council of Teachers of Mathematics *Principles and Standards for School Mathematics*, the strands of mathematical proficiency in the National Research Council's *Adding It Up*, and other distillations.•

• For more about research in this area, see the National Research Council's reports *Adding It Up: Helping Children Learn Mathematics*, 2001, and *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, 2009 (online at www.nap.edu); Sarama & Clements, 2009, *Early Childhood Mathematics Education Research*, Routledge; Clements et al., 2017, "The Research-based Balance in Early Childhood Mathematics: A Response to Common Core Criticisms," *Early Childhood Research Quarterly*; Clements et al., 2017, "What Is Developmentally Appropriate Teaching?," *Teaching Children Mathematics*; Clements et al., 2019, "Critiques of the Common Core in Early Math: A Research-Based Response," *Journal for Research in Mathematics Education*.

• For reports which summarize some research in these areas, see National Research Council, 2001, *Adding It Up: Helping Children Learn Mathematics*; National Council of Teachers of Mathematics, 2003, *A Research Companion to Principles and Standards for School Mathematics*; U.S. Department of Education, 2008, "Report of the Task Group on Learning Processes" in *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. For recommendations that reflect research in these areas, see the National Council of Teachers of Mathematics reports *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, 2006 and *Focus in High School Mathematics: Reasoning and Sense Making*, 2009.

• See the chapter on how experts differ from novices in the National Research Council's *How People Learn: Brain, Mind, Experience, and School* (online at http://www.nap.edu/catalog.php?record_id=9853).

• Schmidt & Houang, 2007, "Lack of Focus in the Intended Mathematics Curriculum: Symptom or Cause?" in *Lessons Learned: What International Assessments Tell Us About Math Achievement*, Brookings Institution Press.

• See the discussions of self-monitoring, metacognition, and heuristics in *How People Learn* and the Problem Solving Standard of *Principles and Standards for School Mathematics*.

• See Harel, 2008, "What is Mathematics? A Pedagogical Answer to a Philosophical Question" in Gold & Simons (eds.), *Proof and Other Dilemmas*, Mathematical Association of America; Cuoco, Goldenberg, & Mark, 1996, "Habits of Mind: An Organizing Principle for a Mathematics Curriculum," *Journal of Mathematical Behavior* or Cuoco, 1998, "Mathematics as a Way of Thinking about Things," in *High School Mathematics at Work*, National Academies Press, which can be read online at http://bit.ly/12Fa27m. *Adding It Up* can be read online at http://bit.ly/mbeQs1.

Organization of the Standards

An important feature of the Standards for Mathematical Content is their organization in groups of related standards. In K–8, these groups are called domains and in high school, they are called conceptual categories. The diagram in the margin shows K–8 domains which are important precursors of the conceptual category of algebra.• In contrast, many standards and frameworks in the United States are presented as parallel K–12 "strands." Unlike the diagram in the margin, a strands type of presentation has the disadvantage of deemphasizing relationships of topics in different strands.

Other aspects of the structure of the Standards are less obvious. The Progressions elaborate some features of this structure.• In particular:

- Grade–level coordination of standards across domains.

- Connections between standards for content and for mathematical practice.

- Key ideas that develop within one domain over the grades.

- Key ideas that change domains as they develop over the grades.

- Key ideas that recur in different domains and conceptual categories.

Grade–level coordination of standards across domains or conceptual categories   One example of how standards are coordinated is the following. In Grade 4 measurement and data, students solve problems involving conversion of measurements from a larger unit to a smaller unit.[4.MD.1] In Grade 5, this extends to conversion from smaller units to larger ones.[5.MD.1]

These standards are coordinated with the standards for operations on fractions. In Grade 4, expectations for multiplication are limited to multiplication of a fraction by a whole number (e.g., $3 \times \frac{2}{5}$) and its representation by number line diagrams, other visual models, and equations.[4.NF.4b, c] In Grade 5, fraction multiplication extends to multiplication of two non–whole number fractions.[5.NF.6]

Connections between content and practice standards   The Progressions provide examples of "points of intersection" between content and practice standards. For instance, standard algorithms for operations with multi–digit numbers can be viewed as expressions of regularity in repeated reasoning (MP.8). Such examples can be found by searching for "MP." in the Progressions.

Key ideas within domains   Within the domain of Number and Operations Base Ten, place value begins with the concept of ten ones in Kindergarten and extends through Grade 6, developing further

• For a more detailed diagram of relationships among the standards, see https://achievethecore.org/coherence-map.
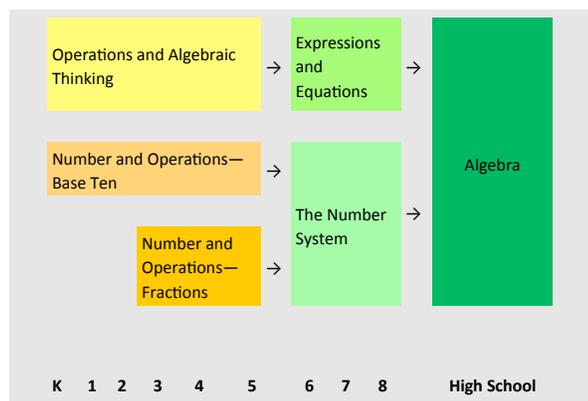
• Because the Progressions focus on key ideas and the Standards have different levels of grain-size, not every standard is included in some progression.

[4.MD.1] Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

[5.MD.1] Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

    b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

    c Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem.

[5.NF.6] Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

in the context of whole number and decimal representations and computations.

Key ideas that change domains   Some key concepts develop across domains and grades. For example, understanding number line diagrams begins in the domain of Measurement and Data in Grades 1 and 2 as students learn to measure lengths.• In Grades 3–5, it develops further in Number and Operations—Fractions. It continues in The Number System as students use number line diagrams to represent negative numbers in Grade 6 and irrational numbers in Grade 8.

Coordinated with the development of multiplication of fractions, measuring area begins in Grade 3 geometric measurement for rectangles with whole-number side lengths, extending to rectangles with fractional side lengths in Grade 5. Measuring volume begins in Grade 5 geometric measurement with right rectangular prisms with whole-number side lengths, extending to such prisms with fractional edge lengths in Grade 6 geometry.

Key recurrent ideas   Among key ideas that occur in more than one domain or conceptual category are:

- composing and decomposing.

- unit (including derived, superordinate, and subordinate unit).

These begin in elementary grades and continue through high school. Initially, students develop tacit knowledge of these ideas by using them. In later grades, their knowledge becomes more explicit.

A group of objects can be decomposed without changing its cardinality, and such decompositions can be represented in equations. For example, a group of 4 objects can be decomposed into a group of 1 and a group of 3, and represented with various equations, e.g., $4 = 1 + 3$ or $1 + 3 = 4$. Properties of operations allow numerical expressions to be decomposed and rearranged without changing their value. For example, the 3 in $1 + 3$ can be decomposed as $1 + 2$ and, using the associative property, the expression can be rearranged as $2 + 2$. Variants of this idea (often expressed as "transforming" or "rewriting" an expression) occur throughout K–8, extending to algebra and other conceptual categories in high school.

A one-, two-, or three-dimensional geometric figure can be decomposed and rearranged without changing, respectively, its length, area, or volume. For example, two copies of a square can be put edge to edge and be seen as composing a rectangle. A rectangle can be decomposed to form two triangles of the same shape. Variants of this idea (often expressed as "dissecting" and "rearranging") occur throughout K–8, extending to geometry and other conceptual categories in high school.

• Number line diagrams are difficult for young children because number line diagrams use length units, which are more difficult to see and count than are objects. Recent National Research Council reports recommend that number lines not be used in Kindergarten and Grade 1 (see pp. 167–168 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, http://www.nap.edu/read/12519/chapter/9). Note that early childhood materials and games often may use a number path in which numbers are put on shapes such as circles or squares. Because the numbers count separated things rather than consecutive length units, this is a count model, not the measurement model described here. Likewise talk of children using a "mental number line" may refer to use of the counting word sequence (a count model), not a measurement model.

In K–8, an important occurrence of units is in the base-ten system for numbers. A whole number can be viewed as a collection of ones, or organized in terms of its base-ten units. Ten ones compose a unit called a ten. That unit can be decomposed as ten ones. Understanding place value involves understanding that each place of a base-ten numeral represents an amount of a base-ten unit: ones, tens, hundreds, . . . , and tenths, hundredths, etc. The regularity in composing and decomposing of base-ten units is a major feature used and highlighted by algorithms for computing operations on whole numbers and decimals.

Units occur as units of measurement for length, area, and volume in geometric measurement and geometry. Students iterate these units in measurement, first physically and later mentally, e.g., placing copies of a length unit side by side to measure length, tiling a region with copies of an area unit to measure area, or packing a container with copies of a volume unit to measure volume. They understand that a length unit determines derived units for area and volume, e.g., a meter gives rise to a square meter and cubic meter.

Representing amounts in terms of units

	two units	notation	one unit	notation
Base-ten units	1 ten, 3 ones	13	13 ones	–
Measurement units	1 foot, 3 inches	1 ft, 3 in	15 inches	15 in
Fractional units	1 one, 3 fifths	$1\frac{3}{5}$	8 fifths	$\frac{8}{5}$
Base-ten units	1 one, 3 tenths	1.3	13 tenths	–

An amount may be represented in terms of one unit or in terms of two units, where one unit is a composition of the other.

Students learn to decompose a one ("a whole") into subordinate units: unit fractions of equal size. The whole is a length (possibly represented by an endpoint) on the number line or is a single shape or object. When possible, students are able to write a number in terms of those units in different ways, as a fraction, decimal, or mixed number. They expand their conception of unit by learning to view a group of objects as a unit and partition that unit into fractions of equal size.

Students learn early that groups of objects or numbers can be decomposed and reassembled without changing their cardinality. Later, students learn that specific length, area, or volume units can be decomposed into subordinate units of equal size, e.g., a meter can be decomposed into decimeters, centimeters, or millimeters.

Ideas of units and of decomposition and reassembly are used and extended in high school. For example, derived units may be created from two or more different units, e.g., miles per hour or vehicle-mile traveled. Shapes are decomposed and reassembled in order to determine certain attributes. For example, areas can be decomposed and reassembled as in the proof of the Pythagorean Theorem or angles can be decomposed and reassembled to yield trigonometric formulas.

Reconceptualized topics; changed notation and terminology

This section mentions some topics, terms, and notation that have been frequent in U.S. school mathematics, but do not occur in the Standards or Progressions.

"Number sentence" in elementary grades   "Equation" is used instead of "number sentence," allowing the same term to be used throughout K–12.

Notation for remainders in division of whole numbers   One aspect of attending to logical structure is attending to consistency. This has sometimes been neglected in U.S. school mathematics as illustrated by a common practice. The result of division within the system of whole numbers is frequently written like this:

$$84 \div 10 = 8 \text{ R } 4 \text{ and } 44 \div 5 = 8 \text{ R } 4.$$

Because the two expressions on the right are the same, students should conclude that $84 \div 10$ is equal to $44 \div 5$, but this is not the case. (Because the equal sign is not used appropriately, this usage is a non-example of Standard for Mathematical Practice 6.) Moreover, the notation 8 R 4 does not indicate a number.

Rather than writing the result of division in terms of a whole-number quotient and remainder, the relationship of whole-number quotient and remainder can be written like this:

$$84 = 8 \times 10 + 4 \text{ and } 44 = 8 \times 5 + 4.$$

Conversion and simplification   To achieve the expectations of the Standards, students need to be able to transform and use numerical and symbolic expressions. The skills traditionally labeled "conversion" and "simplification" are a part of these expectations. As noted in the statement of Standard for Mathematical Practice 1, students transform a numerical or symbolic expression in order to get the information they need, using conversion, simplification, or other types of transformations. To understand correspondences between different approaches to the same problem or different representations for the same situation, students draw on their understanding of different representations for a given numerical or symbolic expression as well as their understanding of correspondences between equations, tables, graphs, diagrams, and verbal descriptions.

Conversion and simplification of fractions   In Grade 3, students recognize and generate equivalences between fractions in simple cases (3.NF.3). Two important building blocks for understanding relationships between fraction and decimal notation occur in Grades 4 and 5. In Grade 4, students' understanding of decimal notation for fractions includes using decimal notation for fractions with denominators 10 and 100 (4.NF.5; 4.NF.6). In Grade 5, students' understanding of fraction notation for decimals includes using fraction notation for decimals to thousandths (5.NBT.3a).

Students identify correspondences between different approaches to the same problem (MP.1). In Grade 4, when solving word problems

that involve computations with simple fractions or decimals (e.g., 4.MD.2), one student might compute

$$\frac{1}{5} + \frac{12}{10}$$

as

$$.2 + 1.2 = 1.4,$$

another as

$$\frac{1}{5} + \frac{6}{5} = \frac{7}{5};$$

and yet another as

$$\frac{2}{10} + \frac{12}{10} = \frac{14}{10}.$$

Explanations of correspondences between

$$\frac{1}{5} + \frac{12}{10}, \quad .2 + 1.2, \quad \frac{1}{5} + \frac{6}{5}, \quad \text{and} \quad \frac{2}{10} + \frac{12}{10}$$

draw on understanding of equivalent fractions (3.NF.3 is one building block) and conversion from fractions to decimals (4.NF.5; 4.NF.6). This is revisited and augmented in Grade 7 when students use numerical and algebraic expressions to solve problems posed with rational numbers expressed in different forms, converting between forms as appropriate (7.EE.3).

In Grade 6, percents occur as rates per 100 in the context of finding parts of quantities (6.PR.3c). In Grade 7, students unify their understanding of numbers, viewing percents together with fractions and decimals as representations of rational numbers. Solving a wide variety of percentage problems (7.RP.3) provides one source of opportunities to build this understanding.

Simplification of algebraic expressions   In Grade 6, students apply properties of operations to generate equivalent expressions (6.EE.3). For example, they apply the distributive property to $3(2 + x)$ to generate $6 + 3x$. Traditionally, $6 + 3x$ is called the "simplification" of $3(2+x)$, however, students are not required to learn this terminology. Although the term "simplification" may suggest that the simplified form of an expression is always the most useful or always leads to a simpler form of a problem, this is not always the case. Thus, the use of this term may be misleading for students.

In Grade 7, students again apply properties of operations to generate equivalent expressions, this time to linear expressions with rational number coefficients (7.EE.1). Together with their understanding of fractions and decimals, students draw on their understanding of equivalent forms of an expression to identify and explain correspondences between different approaches to the same problem. For example, in Grade 7, this can occur in solving multi-step problems posed in terms of a mixture of fractions, decimals, and whole numbers (7.EE.4).

In high school, students apply properties of operations to solve problems, e.g., by choosing and producing an equivalent form of an expression for a quadratic or exponential function (A–SSE.3). As in earlier grades, the simplified form of an expression is one of its equivalent forms.

Terms and usage in the Standards and Progressions

In some cases, the Standards give choices or suggest a range of options. For example, standards like K.NBT.1, 4.NF.3c, and G–CO.12 give lists such as: "using objects or drawings"; "replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction"; "dynamic geometric software, compass and straightedge, reflective devices, and paper folding." Such lists are intended to suggest various possibilities rather than being comprehensive lists of requirements. The abbreviation "e.g." in a standard is frequently used as an indication that what follows is an example, not a specific requirement.

On the other hand, the Standards do impose some very important constraints. The structure of the Standards uses a particular definition of "fraction" for definitions and development of operations on fractions (see the Number and Operations—Fractions Progression). Likewise, the standards that concern ratio and rate rely on particular definitions of those terms. These are described in the Ratios and Proportional Relationships Progression.

In general, terms used in the Standards and Progressions are not intended as prescriptions for terms that teachers or students must use in the classroom. For example, students do not need to know the names of different types of addition situations, such as Put Together or Compare, although these can be useful for classroom discourse. Likewise, Grade 2 students might use the term "line plot," its synonym "dot plot," or describe this type of diagram in some other way.

The few standards that prescribe terms do so explicitly.

- 1.G.3: "describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*."

- 2.G.3: "describe the shares using the words *halves*, *thirds*, *half of*, *third of*, etc. and describe the whole as two halves, three thirds, four fourths."

- 6.RP.1: "use ratio language to describe a ratio relationship between two quantities."

- 6.RP.2: "use rate language in the context of a ratio relationship."

Counting and Cardinality, K

Overview

The domain of Counting and Cardinality is about understanding and using numbers. Counting and Cardinality underlies Operations and Algebraic Thinking as well as Number and Operations in Base Ten. It begins with early counting and telling how many in one group of objects. Addition, subtraction, multiplication, and division grow from these early roots. From its very beginnings, this progression involves important ideas that are neither trivial nor obvious; these ideas need to be taught, in ways that are interesting and engaging to young students.

Kindergarten

Several progressions originate in knowing number names and the count sequence.^{K.CC.1}

From saying the counting words to counting out objects   Students usually know or can learn to say the counting words up to a given number before they can use these numbers to count objects or to tell the number of objects. Students become fluent in saying the count sequence so that they have enough attention to focus on the pairings involved in counting objects. To count a group of objects, they pair each word said with one object.^{K.CC.4a} This is usually facilitated by an indicating act (such as pointing to objects or moving them) that keeps each word said in time paired to one and only one object located in space. Counting objects arranged in a line is easiest; with more practice, students learn to count objects in more difficult arrangements, such as rectangular arrays (they need to ensure they reach every row or column and do not repeat rows or columns); circles (they need to stop just before the object they started with); and scattered configurations (they need to make a single path through all of the objects).^{K.CC.5} Later, students can count out a given number of objects,^{K.CC.5} which is more difficult than just counting that many

K.CC.1 Count to 100 by ones and by tens.

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

    a When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object.

K.CC.5 Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

objects, because counting must be fluent enough for the student to have enough attention to remember the number of objects that is being counted out.

From subitizing to single-digit arithmetic fluency   Students come to quickly recognize the cardinalities of small groups without having to count the objects; this is called *perceptual subitizing.* Perceptual subitizing develops into *conceptual subitizing*—recognizing that a collection of objects is composed of two subcollections and quickly combining their cardinalities to find the cardinality of the collection (e.g., seeing a set as two subsets of cardinality 2 and saying "four"). Use of conceptual subitizing in adding and subtracting small numbers progresses to supporting steps of more advanced methods for adding, subtracting, multiplying, and dividing single-digit numbers (in several OA standards from Grade 1 to 3 that culminate in single-digit fluency).

From counting to counting on   Students understand that the last number name said in counting tells the number of objects counted.[K.CC.4b] Prior to reaching this understanding, a student who is asked "How many kittens?" may regard the counting performance itself as the answer, instead of answering with the cardinality of the set. Experience with counting allows students to discuss and come to understand the second part of K.CC.4b—that the number of objects is the same regardless of their arrangement or the order in which they were counted. This connection will continue in Grade 1 with the more advanced counting-on methods in which a counting word represents a group of objects that are added or subtracted and addends become embedded within the total[1.OA.6] (see page 23). Being able to count forward, beginning from a given number within the known sequence,[K.CC.2] is a prerequisite for such counting on. Finally, understanding that each successive number name refers to a quantity that is one larger[K.CC.4c] is the conceptual start for Grade 1 counting on. Prior to reaching this understanding, a student might have to recount entirely a collection of known cardinality to which a single object has been added.

From spoken number words to written base-ten numerals to base-ten system understanding   The Number and Operations in Base Ten Progression discusses the special role of 10 and the difficulties that English speakers face because the base-ten structure is not evident in all the English number words. See page 55.

From comparison by matching to comparison by numbers to comparison involving adding and subtracting   The standards about comparing numbers[K.CC.6,K.CC.7] focus on students identifying which of two groups has more than (or fewer than, or the same amount as) the other. Students first learn to match the objects in the two

[K.CC.4]Understand the relationship between numbers and quantities; connect counting to cardinality.

    b Understand that the last number name said tells the number of objects counted. The number of objects is the same regardless of their arrangement or the order in which they were counted.

[1.OA.6]Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

[K.CC.2]Count forward beginning from a given number within the known sequence (instead of having to begin at 1).

[K.CC.4]Understand the relationship between numbers and quantities; connect counting to cardinality.

    c Understand that each successive number name refers to a quantity that is one larger.

[K.CC.6]Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

[K.CC.7]Compare two numbers between 1 and 10 presented as written numerals.

groups to see if there are any extra and then to count the objects in each group and use their knowledge of the count sequence to decide which number is greater than the other (the number farther along in the count sequence). Students learn that even if one group looks as if it has more objects (e.g., has some extra sticking out), matching or counting may reveal a different result. Comparing num–bers progresses in Grade 1 to adding and subtracting in comparing situations (finding out "how many more" or "how many less"[1.OA.1] and not just "which is more" or "which is less").

[1.OA.1] Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Operations and Algebraic Thinking, K–5

Overview

The Operations and Algebraic Thinking Progression deals with the basic operations—the kinds of quantitative relationships they model and consequently the kinds of problems they can be used to solve as well as their mathematical properties and relationships. Although most of the standards organized under the OA heading involve whole numbers, the importance of this progression is much more general because it describes concepts, properties, and representations that extend to other number systems, to measurements, and to algebra. For example, if the mass of the sun is x kilograms, and the mass of the rest of the solar system is y kilograms, then the mass of the solar system as a whole is the sum $x + y$ kilograms. In this example of additive reasoning, it doesn't matter whether x and y are whole numbers, fractions, decimals, or even variables. Likewise, a property such as distributivity holds for all the number systems that students will study in K–12, including complex numbers.

The generality of the concepts involved in Operations and Algebraic Thinking means that students' work in this domain should be designed to help them extend arithmetic beyond whole numbers (see the Number and Operations—Fractions Progression and Number and Operations in Base Ten Progression) and understand and use expressions and equations in later grades (see the Expressions and Equations Progression).

Addition and subtraction are the first operations studied. Initially, the meaning of addition is separate from the meaning of subtraction, and students build relationships between addition and subtraction over time. Subtraction comes to be understood as reversing the actions involved in addition and as finding an unknown addend. Likewise, the meaning of multiplication is initially separate from the meaning of division, and students gradually perceive relationships between division and multiplication analogous to those

between addition and subtraction, understanding division as reversing the actions involved in multiplication and finding an unknown product.

Over time, students build their understanding of the properties of operations: commutativity and associativity of addition and multiplication, and distributivity of multiplication over addition. Initially, they build intuitive understandings of these properties, and they use these intuitive understandings in strategies to solve real–world and mathematical problems. Later, these understandings become more explicit and allow students to extend operations to the rational numbers (see the Number System Progression).

As the meanings and properties of operations develop, students develop computational methods in tandem. The Kindergarten and Grade 1 sections of this progression describe this development for single–digit addition and subtraction, culminating in methods that rely on properties of operations. The Number and Operations in Base Ten Progression describes how these methods combine with place value reasoning to extend computation to multi–digit numbers. The Number and Operations—Fractions Progression describes how the meanings of operations combine with fraction concepts to extend computation to fractions.

Students engage in the Standards for Mathematical Practice in grade–appropriate ways from Kindergarten to Grade 5. (See the Modeling Progression for discussion of modeling, models, and relationships of modeling with other mathematical practices.) Pervasive classroom use of these mathematical practices in each grade—some illustrated in this progression—affords students opportunities to develop understanding of operations and algebraic thinking.

Overview of Grades K–2

Students develop meanings for addition and subtraction as they encounter problem situations in Kindergarten, and they extend these meanings as they encounter increasingly difficult problem situations in Grade 1. They represent these problems in increasingly sophisticated ways. And they learn and use increasingly sophisticated computation methods to find answers. In each grade, the situations, representations, and methods are calibrated to be coherent and to foster growth from one grade to the next.

The main addition and subtraction situations students work with are listed in Table 1. The computation methods they learn to use are summarized in the margin and described in more detail in Appendix 1.

Methods for single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away. Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Level 2. Counting On. Embed an addend within the total (the addend is perceived simultaneously as an addend and as part of the total). Count this total but abbreviate the counting by omitting the count of this addend; instead, begin with the number word of this addend. Some method of keeping track (fingers, objects, mentally imaged objects, body motions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the remaining addend has been counted. The last number word is the total. For subtraction, the count is stopped when the total occurs in the count. The tracking method indicates the difference (seen as an unknown addend).

Level 3. Convert to an Easier Problem. Decompose an addend and compose a part with another addend.

See Appendix 1 for examples and further details.

Table 1. Addition and subtraction situations

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $$A + B = \square$$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $$A + \square = C$$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $$\square + B = C$$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $$C - B = \square$$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $$C - \square = A$$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $$\square - B = A$$

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $$A + B = \square$$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $$C = \square + \square$$	C apples are on the table. A are red and the rest are green. How many apples are green? $$A + \square = C$$ $$C - A = \square$$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version*. Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version*. Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $$A + \square = C$$ $$C - A = \square$$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $$A + B = \square$$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $$C - B = \square$$ $$\square + B = C$$

Adapted from the *Common Core State Standards for Mathematics*, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33. (The order of Both Addends Unknown and Addend Unknown reverses the order shown in the *Standards*.)

In each type, shown as a row, any one of the three quantities in the situation can be unknown, leading to the subtypes shown in each cell of the row. The table also shows some important language variants which, while mathematically the same, require separate attention. Other descriptions of the situations may use somewhat different names (see Appendix 2).

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

Kindergarten

Students act out adding and subtracting situations by representing quantities in the situation with objects, their fingers, and math drawings (MP.5).[K.OA.1] To do this, students must mathematize a real–world situation (MP.4), focusing on the quantities and their relationships rather than non–mathematical aspects of the situation. Situations can be acted out and/or presented with pictures or words. Math drawings facilitate reflection and discussion because they remain after the problem is solved. These concrete methods that show all of the objects are called Level 1 methods.

Students learn and use mathematical and non–mathematical language, especially when they make up problems and explain their representation and solution. The teacher can write expressions (e.g., $3-1$) to represent operations, as well as writing equations that represent the whole situation before the solution (e.g., $3 - 1 = \square$) or after (e.g., $3 - 1 = 2$). Expressions like $3 - 1$ or $2 + 1$ show the operation, and it is helpful for students to have experience just with the expression so they can conceptually chunk this part of an equation.

Working within 5   Students work with small numbers first, though many kindergarteners will enter school having learned parts of the Kindergarten standards at home or at a preschool program. Focusing attention on small groups in adding and subtracting situations can help students move from perceptual subitizing to conceptual subitizing in which they see and say the addends and the total,• e.g., "Two and one make three."

Students will generally use fingers for keeping track of addends and parts of addends for the Level 2 and 3 methods used in later grades, so it is important that students in Kindergarten develop rapid visual and kinesthetic recognition of numbers to 5 on their fingers. Students may bring from home different ways to show numbers with their fingers and to raise (or lower) them when counting. The three major ways used around the world are starting with the thumb, the little finger, or the pointing finger (ending with the thumb in the latter two cases). Each way has advantages physically or mathematically, so students can use whatever is familiar to them. The teacher can use the range of methods present in the classroom, and these methods can be compared by students to expand their understanding of numbers. Using fingers is not a concern unless it remains at Level 1 (direct modeling) in later grades.

Students in Kindergarten work with the following addition and subtraction situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown (see the dark cells in Table 2). Add To/Take From situations are action–oriented; they show changes from an initial state to a final state. These situations are readily modeled by equations because each aspect of the situation has a representation as number, operation ($+$ or $-$), or equal sign ($=$).•

K.OA.1 Represent addition and subtraction with objects, fingers, mental images, drawings,[2] sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

[2] Drawings need not show details, but should show the mathematics in the problem. (This applies wherever drawings are mentioned in the Standards.)

• Note on vocabulary: The term "total" is used here instead of the term "sum." "Sum" sounds the same as "some," but has the opposite meaning. "Some" is used to describe problem situations with one or both addends unknown, so it is better in the earlier grades to use "total" rather than "sum." Formal vocabulary for subtraction ("minuend" and "subtrahend") is not needed for Kindergarten, Grade 1, and Grade 2, and may inhibit students seeing and discussing relationships between addition and subtraction. At these grades, the terms "total" and "addend" are sufficient for classroom discussion.

• Here, the equal sign is used with the meaning of "becomes," rather than the more general "equals."

Table 2. Addition and subtraction situations by grade level

	Result Unknown	Change Unknown	Start Unknown
Add To	A bunnies sat on the grass. B more bunnies hopped there. How many bunnies are on the grass now? $$A + B = \square$$	A bunnies were sitting on the grass. Some more bunnies hopped there. Then there were C bunnies. How many bunnies hopped over to the first A bunnies? $$A + \square = C$$	Some bunnies were sitting on the grass. B more bunnies hopped there. Then there were C bunnies. How many bunnies were on the grass before? $$\square + B = C$$
Take From	C apples were on the table. I ate B apples. How many apples are on the table now? $$C - B = \square$$	C apples were on the table. I ate some apples. Then there were A apples. How many apples did I eat? $$C - \square = A$$	Some apples were on the table. I ate B apples. Then there were A apples. How many apples were on the table before? $$\square - B = A$$

	Total Unknown	Both Addends Unknown[1]	Addend Unknown[2]
Put Together /Take Apart	A red apples and B green apples are on the table. How many apples are on the table? $$A + B = \square$$	Grandma has C flowers. How many can she put in her red vase and how many in her blue vase? $$C = \square + \square$$	C apples are on the table. A are red and the rest are green. How many apples are green? $$A + \square = C$$ $$C - A = \square$$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare	*"How many more?" version*. Lucy has A apples. Julie has C apples. How many more apples does Julie have than Lucy? *"How many fewer?" version*. Lucy has A apples. Julie has C apples. How many fewer apples does Lucy have than Julie? $$A + \square = C$$ $$C - A = \square$$	*"More" version suggests operation.* Julie has B more apples than Lucy. Lucy has A apples. How many apples does Julie have? *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have? $$A + B = \square$$	*"Fewer" version suggests operation.* Lucy has B fewer apples than Julie. Julie has C apples. How many apples does Lucy have? *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have? $$C - B = \square$$ $$\square + B = C$$

Adapted from the *Common Core State Standards for Mathematics*, p. 88, which is based on *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, National Research Council, 2009, pp. 32–33. (To improve readability, the order of Both Addends Unknown and Addend Unknown reverses the order shown in the *Standards*.)

Darker shading indicates the four Kindergarten problem subtypes. Grade 1 and 2 students work with all subtypes and variants. Unshaded (white) problems illustrate the four difficult subtypes or variants that students should work with in Grade 1 but need not master until Grade 2. Other descriptions of the situations may use somewhat different names (see Appendix 2).

[1] This can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same number as." Such problems are not a problem subtype with one unknown, as is the Addend Unknown subtype to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.

[2] Either addend can be unknown; both variations should be included.

In Put Together/Take Apart situations, two quantities jointly compose a third quantity (the total), or a quantity can be decomposed into two quantities (the addends). This composition or decomposition may be physical or conceptual. These situations are acted out with objects initially and later children begin to move to conceptual mental actions of shifting between seeing the addends and seeing the total (e.g., seeing boys and girls or seeing children, or seeing red and green apples or all the apples).

The relationship between addition and subtraction in the Add To/Take From and the Put Together/Take Apart action situations is that of reversibility of actions: an Add To situation undoes a Take From situation and vice versa and a composition (Put Together) undoes a decomposition (Take Apart) and vice versa.

Put Together/Take Apart situations with Both Addends Unknown play an important role in Kindergarten because they allow students to explore various compositions that make each number.$^{K.OA.3}$ This will help students to build the Level 2 embedded number representations used to solve more advanced problem subtypes. As students decompose a given number to find all of the partners● that compose the number, the teacher can record each decomposition with an equation such as $5 = 4 + 1$, showing the total on the left and the two addends on the right.● Students can find patterns in all of the decompositions of a given number and eventually summarize these patterns for several numbers.

Equations with one number on the left and an operation on the right (e.g., $5 = 2 + 3$ to record a group of 5 things decomposed as a group of 2 things and a group of 3 things) allow students to understand equations as showing in various ways that the expressions on both sides have the same value.$^{MP.6}$

The Put Together/Take Apart Addend Unknown problem in Table 2 is shaded to show that it is a Grade 1 problem. Many kindergarten children can act out the simplest version of this problem type, a Take Apart problem where the known total is given first and the take apart action can be done easily. For example, for the problem

> 5 apples are on the table. 3 are red and the rest are green. How many are green?

a child can draw or make 5 objects, take apart or think of 3 of them as the red apples and then see the other 2 apples as the green apples. This easy Take Apart problem is similar to the Take Apart Both Addends Unknown problem shown in Table 2. It differs from the Take From Result Unknown problem problem in Table 2 because the addend is not actually taken away to leave only the other addend as the result. So it is easier for children to see the addends within the total in such Take Apart problems. But other variations of the Put Together/Take Apart Addend Unknown problem can be more difficult. If the known addend is given first, the problem is more readily modeled by Level 2 counting on. And some problems use class inclusion

$^{K.OA.3}$Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2 + 3$ and $5 = 4 + 1$).

● The two addends that make a total can also be called partners in Kindergarten and Grade 1 to help children understand that they are the two numbers that go together to make the total.

● For each total, two equations involving 0 can be written, e.g., $5 = 5 + 0$ and $5 = 0 + 5$. Once students are aware that such equations can be written, practice in decomposing is best done without such 0 cases.

$^{MP.6}$ Working toward "using the equal sign consistently and appropriately."

terms (for example, "cars," "trucks," "vehicles") that are less likely to be known by kindergarten children. So kindergarten children can explore the simplest Take Apart Addend Unknown problems, but full competence with this problem type is more appropriate for Grade 1.

Working within 10   Students expand their work in addition and subtraction from within 5 to within 10. They use the Level 1 methods developed for smaller totals as they represent and solve problems with objects, their fingers, and math drawings. Patterns such as "adding one is just the next counting word"[K.CC.4c] and "adding zero gives the same number" become more visible and useful for all of the numbers from 1 to 9. Patterns such as the $5 + n$ pattern used widely around the world play an important role in learning particular additions and subtractions, and later as patterns in steps in the Level 2 and 3 methods. Fingers can be used to show the same 5 patterns, but students should be asked to explain these relationships explicitly because they may not be obvious to all students.[MP.3] As the school year progresses, students internalize their external representations and solution actions, and mental images become important in problem representation and solution.

Student drawings show the relationships in addition and subtraction situations for larger numbers (6 to 9) in various ways, such as groupings, things crossed out, numbers labeling parts or totals, and letters or words labeling aspects of the situation. The symbols $+$, $-$, or $=$ may be in the drawing. Students should be encouraged to explain their drawings and discuss how different drawings are the same and different.[MP.1]

Later in the year, students solve addition and subtraction equations for numbers within 5, for example, $2 + 1 = \square$ or $3 - 1 = \square$, while still connecting these equations to situations verbally or with drawings. Experience with decompositions of numbers and with Add To and Take From situations enables students to begin to fluently add and subtract within 5.[K.OA.5]

Finally, composing and decomposing numbers from 11 to 19 into ten ones and some further ones builds from all this work.[K.NBT.1] This is a vital first step that kindergarteners must take toward understanding base-ten notation for numbers greater than 9. (See the NBT Progression.)

The Kindergarten standards can be stated succinctly, but they represent a great deal of focused and rich interactions in the classroom. This is necessary in order to enable all students to understand all of the numbers and concepts involved. Students who enter Kindergarten without knowledge of small numbers or of counting to ten will require extra teaching time in Kindergarten to meet the standards. Such time and support are vital for enabling all students to master the Grade 1 standards in Grade 1.

K.CC.4 Understand the relationship between numbers and quantities; connect counting to cardinality.

   c Understand that each successive number name refers to a quantity that is one larger.

$5 + n$ **pattern**
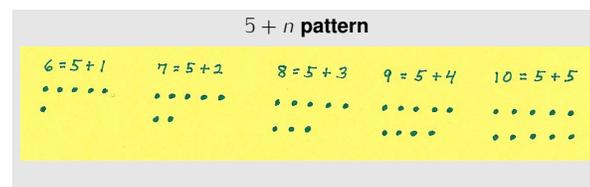
MP.3 Students explain their conclusions to others.

MP.1 Understand the approaches of others and identify correspondences.

K.OA.5 Fluently add and subtract within 5.

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Grade 1

Students extend their work in three major and interrelated ways, by:

- Representing and solving a new type of problem situation (Compare);

- Representing and solving the subtypes for all unknowns in all three types;

- Using Level 2 and Level 3 methods to extend addition and subtraction problem solving beyond 10, to problems within 20. In particular, the OA progression in Grade 1 deals with adding two single-digit addends, and related subtractions.•

Representing and solving a new type of problem (Compare)   In a Compare situation, two quantities are compared to find "How many more" or "How many less."•K.CC.6,K.CC.7 One reason Compare problems are more advanced than the other two major types is that in Compare problems, one of the quantities (the difference) is not present in the situation physically, and must be conceptualized and constructed in a representation, by showing the "extra" that when added to the smaller unknown makes the total equal to the bigger unknown or by finding this quantity embedded within the bigger unknown.

The language of comparisons is also difficult. For example, "Julie has three more apples than Lucy" tells both that Julie has more apples and that the difference is three. Many students "hear" the part of the sentence about who has more, but do not initially hear the part about how many more; they need experience hearing and saying a separate sentence for each of the two parts in order to comprehend and say the one-sentence form. Another language issue is that the comparing sentence might be stated in either of two related ways, using "more" or "less." Students need considerable experience with "less" to differentiate it from "more"; some children think that "less" means "more." Finally, as well as the basic "How many more/less" question form, the comparing sentence might take an active, equalizing and counterfactual form (e.g., "How many more apples does Lucy need to have as many as Julie?") or might be stated in a static and factual way as a question about how many things are unmatched (e.g., "If there are 8 trucks and 5 drivers, how many trucks do not have a driver?"). Extensive experience with a variety of contexts is needed to master these linguistic and situational complexities. Matching with objects and with drawings, and labeling each quantity (e.g., J or Julie and L or Lucy) is helpful. Later in Grade 1, tape diagrams can be used. These comparing diagrams can continue to be used for multi-digit numbers, fractions, decimals, and variables, thus connecting understandings of the numbers in these comparing

• Other Grade 1 problems within 20, such as $14 + 5$, are best viewed in the context of place value, i.e., associated with 1.NBT.4. See the NBT Progression.

• Compare problems build upon Kindergarten comparisons, in which students identified "Which is more?" or "Which is less?" without ascertaining the difference between the numbers.
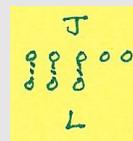
K.CC.6Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

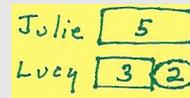K.CC.7Compare two numbers between 1 and 10 presented as written numerals.

Representing the difference in a Compare problem
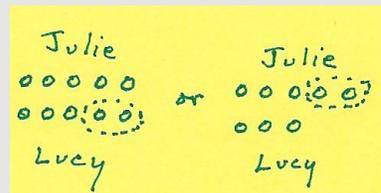
Compare problem solved by matching

Compare problem represented in tape diagram

situations with such situations for single-digit numbers. The labels can get more detailed in later grades.

Some textbooks represent all Compare problems with a subtraction equation, but that is not how many students think of the subtypes. Students represent Compare situations in different ways, often as an unknown addend problem (see Table 1). If textbooks and teachers model representations of or solution methods for Compare problems, these should reflect the variability students show. In all mathematical problem solving, what matters is the explanation a student gives to relate a representation to a context, and not the representation separated from its context.

Representing and solving the subtypes for all unknowns in all three types   In Grade 1, students solve problems of all twelve subtypes (see Table 1) including both language variants of Compare problems. Initially, the numbers in such problems are small enough that students can make math drawings showing all the objects in order to solve the problem. Students then represent problems with equations, called situation equations. For example, a situation equation for a Take From problem with Result Unknown might read $14 - 8 = \square$.

Put Together/Take Apart problems with Addend Unknown afford students the opportunity to see subtraction as "undoing" addition in a different way than as reversing the action, namely as finding an unknown addend.[1.OA.4] The meaning of subtraction as an unknown–addend addition problem is one of the essential understandings students will need in middle grades in order to extend arithmetic to negative rational numbers.

[1.OA.4] Understand subtraction as an unknown-addend problem.

Students next gain experience with the more difficult and more "algebraic" problem subtypes in which a situation equation does not immediately lead to the answer. For example, a student analyzing a Take From problem with Change Unknown might write the situation equation $14 - \square = 8$. This equation does not immediately lead to the answer. To make progress, the student can write a related equation called a solution equation—in this case, either $8 + \square = 14$ or $14 - 8 = \square$. These equations both lead to the answer by Level 2 or Level 3 strategies (see discussion in the next section).

Students thus begin developing an algebraic perspective many years before they will use formal algebraic symbols and methods. They read to understand the problem situation, represent the situation and its quantitative relationships with expressions and equations, and then manipulate that representation if necessary, using properties of operations and/or relationships between operations. Linking equations to concrete materials, drawings, and other representations of problem situations affords deep and flexible understandings of these building blocks of algebra. Learning where the total is in addition equations (alone on one side of the equal sign) and in subtraction equations (to the left of the minus sign) helps stu-

dents move from a situation equation to a related solution equation.

Because the language and conceptual demands are high, some students in Grade 1 may not master the most difficult subtypes of word problems, such as Compare problems that use language opposite to the operation required for solving (see the unshaded subtypes and variants in Table 2). Some students may also still have difficulty with the conceptual demands of Start Unknown problems. Grade 1 children should have an opportunity to solve and discuss such problems, but proficiency on grade level tests with these most difficult subtypes should wait until Grade 2 along with the other extensions of problem solving.

Using Level 2 and Level 3 strategies to extend addition and subtraction problem solving beyond 10, to problems within 20   As Grade 1 students extend the range of problem types and subtypes they can solve, they also extend the range of numbers they deal with[1.OA.6] and the sophistication of the methods they use to add and subtract within this larger range.[1.OA.1,1.OA.8]

The advance from Level 1 methods to Level 2 methods can be clearly seen in the context of situations with unknown addends. These are the situations that can be represented by an addition equation with one unknown addend, e.g., $9 + \square = 13$. Students can solve some unknown addend problems by trial and error or by knowing the relevant decomposition of the total. But a Level 2 counting on solution involves seeing the 9 as part of 13, and understanding that counting the 9 things can be "taken as done" if we begin the count from 9: thus the student may say,

> "Niiiiine,   ten,   eleven,   twelve,   thirteen."
>   1   2   3   4

Students keep track of how many they counted on (here, 4) with fingers, mental images, or physical actions such as head bobs. Elongating the first counting word ("Niiiiine . . .") is natural and indicates that the student differentiates between the first addend and the counts for the second addend. Counting on enables students to add and subtract easily within 20 because they do not have to use fingers to show totals of more than 10 which is difficult. Students might also use the commutative property to shorten tasks, by counting on from the larger addend even if it is second (e.g., for $4 + 9$, counting on from 9 instead of from 4).

Counting on should be seen as a thinking strategy, not a rote method. It involves seeing the first addend as embedded in the total, and it involves a conceptual interplay between counting and the cardinality in the first addend (shifting from the cardinal meaning of the first addend to the counting meaning). Finally, there is a level of abstraction involved in counting on, because students are counting the words rather than objects. Number words have become objects to students.

[1.OA.6] Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

[1.OA.1] Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

[1.OA.8] Determine the unknown whole number in an addition or subtraction equation relating three whole numbers.

Counting on can be used to add (find a total) or subtract (find an unknown addend). To an observer watching the student, adding and subtracting look the same. Whether the problem is $9 + 4$ or $13 - 9$, we will hear the student say the same thing: "Niiiiine, ten, eleven, twelve, thirteen" with four head bobs or four fingers unfolding. The differences are in what is being monitored to know when to stop, and what gives the answer.

Students in many countries learn counting forward methods of subtracting, including counting on. Counting on for subtraction is easier than counting down. Also, unlike counting down, counting on reinforces that subtraction is an unknown–addend problem. Learning to think of and solve subtractions as unknown addend problems makes subtraction as easy as addition (or even easier), and it emphasizes the relationship between addition and subtraction. The taking away meaning of subtraction can be emphasized within counting on by showing the total and then taking away the objects that are at the *beginning*. In a drawing, this taking away can be shown with a horizontal line segment suggesting a minus sign. So one can think of the $9 + \square = 13$ situation as "I took away 9. I now have 10, 11, 12, 13 [stop when I hear 13], so 4 are left because I counted on 4 from 9 to get to 13." Taking away objects at the end suggests counting down, which is more difficult than counting on. Showing 13 decomposed in groups of five as in the margin also supports students seeing how to use the Level 3 make–a–ten method; 9 needs 1 more to make 10 and there are 3 more in 13, so 4 from 9 to 13.

Level 3 methods involve decomposing an addend and composing it with the other addend to form an equivalent but easier problem. This relies on properties of operations.[1.OA.3] Students do not necessarily have to justify their representations or solutions using properties, but they can begin to learn to recognize these properties in action and discuss their use after solving.

There are a variety of methods to change to an easier problem. These draw on addition of three whole numbers.[1.OA.2] A known addition or subtraction can be used to solve a related addition or subtraction by decomposing one addend and composing it with the other addend. For example, a student can change $8 + 6$ to the easier $10 + 4$ by decomposing 6 as $2 + 4$ and composing the 2 with the 8 to make 10: $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$.

This method can also be used to subtract by finding an unknown addend: $14 - 8 = \square$, so $8 + \square = 14$, so $14 = 8 + 2 + 4 = 8 + 6$, that is $14 - 8 = 6$. Students can think as for adding above (stopping when they reach 14), or they can think of taking 8 from 10, leaving 2 with the 4, which makes 6. One can also decompose with respect to ten: $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$, but this can be more difficult than the forward methods.

Counting on to add and subtract

$9 + 4$
"Niiiiine, ten, eleven, twelve, thirteen."
　　　1　　2　　3　　　4

$13 - 9$
"Niiiiine, ten, eleven, twelve, thirteen."
　　　1　　2　　3　　　4

When counting on to add $9 + 4$, the student is counting the fingers or head bobs to know when to stop counting aloud, and the last counting word said gives the answer. For counting on to subtract $13 - 9$, the opposite is true: the student is listening to counting words to know when to stop, and the accumulated fingers or head bobs give the answer.

"Taking away" indicated with horizontal line segment and solving by counting on to 13
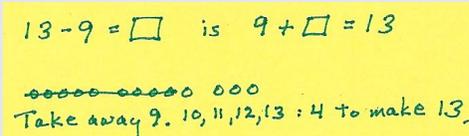
$13 - 9 = \square$　　is　　$9 + \square = 13$

Take away 9. 10, 11, 12, 13 : 4 to make 13.

[1.OA.3]Apply properties of operations as strategies to add and subtract.[3]

[3]Students need not use formal terms for these properties.

[1.OA.2]Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

These make–a–ten methods[•] have three prerequisites reaching back to Kindergarten:

a. knowing the partner that makes 10 for any number (K.OA.4 sets the stage for this),

b. knowing all decompositions for any number below 10 (K.OA.3 sets the stage for this), and

c. knowing all teen numbers as $10 + n$ (e.g., $12 = 10 + 2$, $15 = 10 + 5$, see K.NBT.1 and 1.NBT.2b).

The make–a–ten methods are more difficult in English than in East Asian languages in which teen numbers are spoken as *ten, ten one, ten two, ten three*, etc. In particular, prerequisite c is harder in English because of the irregularities and reversals in the teen number words. For example, "four" is spoken first in "fourteen," but this order is reversed in the numeral 14.

Another Level 3 method that works for certain numbers is a doubles ±1 or ±2 method: $6+7 = 6+(6+1) = (6+6)+1 = 12+1 = 13$. These methods do not connect with place value the way make–a–ten methods do.

The Add To and Take From Start Unknown situations are particularly challenging with the larger numbers students encounter in Grade 1. The situation equation $\Box + 6 = 15$ or $\Box - 6 = 9$ can be rewritten to provide a solution. Students might use the commutative property of addition to change $\Box + 6 = 15$ to $6 + \Box = 15$, then count on or use Level 3 methods to compose 4 (to make ten) plus 5 (ones in the 15) to find 9. Students might reverse the action in the situation represented by $\Box - 6 = 9$ so that it becomes $9+6 = \Box$. Or they might use their knowledge that the total is the first number in a subtraction equation and the last number in an addition equation to rewrite the situation equation as a solution equation: $\Box - 6 = 9$ becomes $9 + 6 = \Box$ or $6 + 9 = \Box$.

The difficulty levels in Compare problems differ from those in Put Together/Take Apart and Add To and Take From problems. Difficulties arise from the language issues mentioned before and especially from the opposite language variants where the comparing sentence suggests an operation opposite to that needed for the solution.[•]

As students progress to Level 2 and Level 3 methods, they no longer need representations that show each quantity as a group of objects. Students now move on to diagrams that use numbers and show relationships between these numbers. These can be extensions of drawings made earlier that did show each quantity as a group of objects. Add To/Take From situations at this point can continue to be represented by equations. Put Together/Take Apart situations can be represented as shown in the margin. Compare situations can be represented with tape diagrams showing the compared quantities (one smaller and one larger) and the difference. Other diagrams showing two numbers and the unknown can also be used. Such diagrams

● Computing 8 + 6 by making a ten:

   a. 8's partner to 10 is 2, so decompose 6 as 2 and its partner.

   b. 2's partner to 6 is 4.

   c. 10 + 4 is 14.

K.OA.4 For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.

K.OA.3 Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., $5 = 2+3$ and $5 = 4 + 1$).

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., $18 = 10 + 8$); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

1.NBT.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

  b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

● Bigger Unknown: *"Fewer" version suggests wrong operation.* Lucy has B fewer apples than Julie. Lucy has A apples. How many apples does Julie have?

Smaller Unknown. *"More" version suggests wrong operation.* Julie has B more apples than Lucy. Julie has C apples. How many apples does Lucy have?

Sum shown in tape, part-whole, and number-bond diagrams
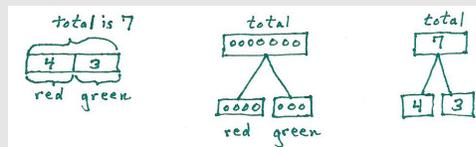
The tape diagram shows the addends as the tapes and the total (indicated by a bracket) as a composition of those tapes. The part-whole diagram and number-bond diagram capture the composing-decomposing action to allow the representation of the total at the top and the addends at the bottom either as drawn quantities or as numbers.

Sum shown in static diagrams
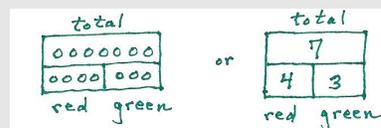
Students sometimes have trouble with static part-whole diagrams because these display a double representation of the total and the addends (the total 7 above and the addends 4 and 3 below), but at a given time in the addition or subtraction situation not all three quantities are present. The action of moving from the total to the addends (or from the addends to the total) in number-bond diagrams reduces this conceptual difficulty.

are a major step forward because the same diagrams can represent the adding and subtracting situations for all of the numbers students encounter in later grades (whether they are represented as multi-digit whole numbers, fractions, decimals, or variables; see, e.g., the Number and Operations—Fractions Progression, p. 144). Students can also continue to represent any situation with a situation equation and connect such equations to diagrams.[MP.1] Such connections can help students to solve the more difficult problem subtypes by understanding where the totals and addends are in the equation and rewriting the equation as needed.

Number line diagrams in the Standards   Number line diagrams are difficult for young children because number line diagrams use length units, which are more difficult to see and count than are objects. Recent National Research Council reports recommend that number lines not be used in Kindergarten and Grade 1.• The Standards follow these recommendations.

Number line diagrams are introduced in Grade 2 when students have had experience with length units on measuring tools (rulers, yardsticks, meter sticks, and measuring tapes, all of which are just special number line diagrams). Experience with these tools, and moving fingers along length units or otherwise focusing on the length units, can help children move from counting things to counting lengths. Even with such experiences, number line diagrams are difficult because the eye is drawn by the numbers that label the endpoints of lengths from 0 or by the marks above those numbers (see the example in the margin). The length units that need to be counted to tell the number of such lengths recede into the background. So children may often make off-by-1 errors if they use number line diagrams to add or subtract because they count the 0 and the other numbers or the marks instead of counting the length units as shown in the second example in the margin. Children first relate addition and subtraction to length units in Grade 2 by representing whole numbers as lengths from 0 on a number line diagram.[2.MD.6] See the Measurement and Data Progression.

• See pp. 167–168 of *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*, http://www.nap.edu/read/12519/chapter/9. Note that early childhood materials and games often may use a number path in which numbers are put on shapes such as circles or squares. Because the numbers count separated things rather than consecutive length units, this is a count model, not the measurement model described here. Likewise talk of children using a "mental number line" may refer to use of the counting word sequence (a count model), not a measurement model.

A number line diagram: Counting length units
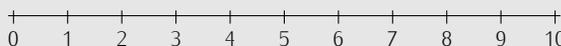
Measurement word reference

"These are five length units."

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Grade 2

Grade 2 students build upon their work in Grade 1 in two major ways.[2.OA.1] They represent and solve situational problems of all three types which involve addition and subtraction within 100 rather than within 20, and they represent and solve two-step situational problems of all three types.

Diagrams used in Grade 1 to show how quantities in the situation are related continue to be useful in Grade 2, and students continue to relate the diagrams to situation equations. Such relating helps students rewrite a situation equation like $\square - 38 = 49$ as $49 + 38 = \square$ because they see that the first number in the subtraction equation is the total. Each addition and subtraction equation has seven related equations. Students can write all of these equations, continuing to connect addition and subtraction, and their experience with equations of various forms.

Because there are so many problem situation subtypes, there are many possible ways to combine such subtypes to devise two-step problems. Because some Grade 2 students are still developing proficiency with the most difficult subtypes, two-step problems should not involve these subtypes. Most work with two-step problems should involve single-digit addends.

Most two-step problems made from two easy subtypes are easy to represent with an equation, as shown in the first two examples in the margin. But problems involving a comparison or two middle difficulty subtypes may be difficult to represent with a single equation and may be better represented by successive drawings or some combination of a diagram for one step and an equation for the other (see the last three examples). Students can make up any kinds of two-step problems and share them for solving.

The deep extended experiences students have with addition and subtraction in Kindergarten and Grade 1 culminate in Grade 2 with students becoming fluent in single-digit additions and the related subtractions using the mental Level 2 and 3 strategies as needed.[2.OA.2] So fluency in adding and subtracting single-digit numbers has progressed from numbers within 5 in Kindergarten to within 10 in Grade 1 to within 20 in Grade 2. The methods have also become more advanced.

The word *fluent* is used in the Standards to mean "fast and accurate." Fluency in each grade involves a mixture of just knowing some answers, knowing some answers from patterns (e.g., "adding 0 yields the same number"), and knowing some answers from the use of strategies. It is important to push sensitively and encouragingly toward fluency of the designated numbers at each grade level, recognizing that fluency will be a mixture of these kinds of thinking which may differ across students. The extensive work relating addition and subtraction means that a subtraction problem can frequently be solved by thinking of the related addition, especially for smaller numbers. It is also important that these patterns, strategies,

2.OA.1 Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Related addition and subtraction equations

$87 - 38 = 49$	$87 - 49 = 38$	$38 + 49 = 87$	$49 + 38 = 87$
$49 = 87 - 38$	$38 = 87 - 49$	$87 = 38 + 49$	$87 = 49 + 38$

Examples of two-step Grade 2 word problems

Two easy subtypes with the same operation, resulting in problems represented as, for example, $9 + 5 + 7 = \square$ or $16 - 8 - 5 = \square$ and perhaps by drawings showing these steps:

> Example for $9 + 5 + 7$: There were 9 blue balls and 5 red balls in the bag. Aki put in 7 more balls. How many balls are in the bag altogether?

Two easy subtypes with opposite operations, resulting in problems represented as, for example, $9 - 5 + 7 = \square$ or $16 + 8 - 5 = \square$ and perhaps by drawings showing these steps:

> Example for $9 - 5 + 7$: There were 9 carrots on the plate. The girls ate 5 carrots. Mother put 7 more carrots on the plate. How many carrots are there now?

One easy and one middle difficulty subtype:

> For example: Maria has 9 apples. Corey has 4 fewer apples than Maria. How many apples do they have in all?

> For example: The zoo had 7 cows and some horses in the big pen. There were 15 animals in the big pen. Then 4 more horses ran into the big pen. How many horses are there now?

Two middle difficulty subtypes:

> For example: There were 9 boys and some girls in the park. In all, 15 children were in the park. Then some more girls came. Now there are 14 girls in the park. How many more girls came to the park?

2.OA.2 Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

[2] See standard 1.OA.6 for a list of mental strategies.

and decompositions still be available in Grade 3 for use in multiplying and dividing and in distinguishing adding and subtracting from multiplying and dividing. So the important press toward fluency should also allow students to fall back on earlier strategies when needed. By the end of the K–2 grade span, students have sufficient experience with addition and subtraction to know single-digit sums from memory;[2.OA.2] as should be clear from the foregoing, this is not a matter of instilling facts divorced from their meanings, but rather as an outcome of a multi-year process that heavily involves the interplay of practice and reasoning.

Extensions to other domains and to later grades   In Grades 2 and 3, students continue and extend their work with adding and subtracting situations to length situations[2.MD.5,2.MD.6] (addition and subtraction of lengths is part of the transition from whole number addition and subtraction to fraction addition and subtraction) and to bar graphs.[2.MD.10,3.MD.3] Students solve two-step[3.OA.8] and multistep[4.OA.3] problems involving all four operations.

In Grades 3, 4, and 5, students extend their understandings of addition and subtraction problem types in Table 1 to situations that involve fractions. Importantly, the situational meanings for addition and subtraction remain the same for fractions as for whole numbers.

[2.OA.2]Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

[2]See standard 1.OA.6 for a list of mental strategies.

[2.MD.5]Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[2.MD.6]Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

[2.MD.10]Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

[3.MD.3]Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

[3.OA.8]Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]

[3]This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

[4.OA.3]Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

Summary of K–2 Operations and Algebraic Thinking

Kindergarten   Students in Kindergarten work with three kinds of problem situations: Add To with Result Unknown; Take From with Result Unknown; and Put Together/Take Apart with Total Unknown and Both Addends Unknown. The numbers in these problems involve addition and subtraction within 10. Students represent these problems with concrete objects and drawings, and they find the answers by counting (Level 1 method). More specifically,

- For Add To with Result Unknown, they make or draw the starting set of objects and the change set of objects, and then they count the total set of objects to give the answer.

- For Take From with Result Unknown, they make or draw the starting set and "take away" the change set; then they count the remaining objects to give the answer.

- For Put Together/Take Apart with Total Unknown, they make or draw the two addend sets, and then they count the total number of objects to give the answer.

Grade 1   Students in Grade 1 work with all of the problem situations, including all subtypes and language variants. The numbers in these problems involve additions involving single–digit addends, and the related subtractions. Students represent these problems with math drawings and with equations.

Students master the majority of the problem types. They might sometimes use trial and error to find the answer, or they might just know the answer based on previous experience with the given numbers. But as a general method they learn how to find answers to these problems by counting on (a Level 2 method), and they understand and use this method.[1.OA.5,1.OA.6] Students also work with Level 3 methods that change a problem to an easier equivalent problem.[1.OA.3,1.OA.6] The most important of these Level 3 methods involve making a ten, because these methods connect with the place value concepts students are learning in this grade (see the NBT Progression) and work for any numbers. Students also solve the easier problem subtypes with these Level 3 methods.

The four problem subtypes that Grade 1 students should work with, but need not master, are:

- Add To with Start Unknown

- Take From with Start Unknown

- Compare with Bigger Unknown using "fewer" language (misleading language suggesting the wrong operation)

- Compare with Smaller Unknown using "more" language (misleading language suggesting the wrong operation)

[1.OA.5] Relate counting to addition and subtraction (e.g., by counting on 2 to add 2).

[1.OA.6] Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

[1.OA.3] Apply properties of operations as strategies to add and subtract.[3]

[3] Students need not use formal terms for these properties.

[1.OA.6] Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).

Grade 2   Students in Grade 2 master all of the problem situations and all of their subtypes and language variants. The numbers in these problems involve addition and subtraction within 100. They represent these problems with diagrams and/or equations. For problems involving addition and subtraction within 20, more students master Level 3 methods; increasingly for addition problems, students might just know the answer (by end of Grade 2, students know all sums of two-digit numbers from memory[2.OA.2]). For other problems involving numbers to 100, Grade 2 students use their developing place value skills and understandings to find the answer (see the NBT Progression). Students work with two-step problems, especially with single-digit addends, but do not work with two-step problems in which both steps involve the most difficult problem subtypes and variants.

[2.OA.2] Fluently add and subtract within 20 using mental strategies.[2] By end of Grade 2, know from memory all sums of two one-digit numbers.

[2] See standard 1.OA.6 for a list of mental strategies.

Grade 3

Students focus on understanding the meaning and properties of multiplication and division and on finding products and related quotients of single-digit numbers.[3.OA.1–7] These skills and understandings are crucial; students will rely on them for years to come as they learn to multiply and divide with multi-digit whole number and to add, subtract, multiply and divide with fractions and with decimals. Note that mastering this material, and reaching fluency in single-digit multiplications and related divisions with understanding[3.OA.7] may be quite time-consuming because there are no general strategies for multiplying or dividing all single-digit numbers as there are for addition and subtraction. Instead, there are many patterns and strategies dependent upon specific numbers. So it is imperative that extra time and support be provided if needed.

Common types of multiplication and division situations. Common multiplication and division situations are shown in Table 3. There are three major types, shown as rows of Table 3. The Grade 3 standards focus on Equal Groups and on Arrays.[•] As with addition and subtraction, each multiplication or division situation involves three quantities, each of which can be the unknown. Because there are two factors and one product in each situation (product = factor × factor), each type has one subtype solved by multiplication (Unknown Product) and two unknown factor subtypes solved by division.

In Equal Groups, the roles of the factors differ. One factor is the number of objects in a group (like any quantity in addition and subtraction situations), and the other is a multiplier that indicates the number of groups. So, for example, 4 groups of 3 objects is arranged differently than 3 groups of 4 objects. Thus there are two kinds of division situations depending on which factor is the unknown (the number of objects in each group or the number of groups). In the Array situations, the roles of the factors do not differ. One factor tells the number of rows in the array, and the other factor tells the number of columns in the situation. But rows and columns depend on the orientation of the array. If an array is rotated 90°, the rows become columns and the columns become rows. This is useful for seeing the commutative property for multiplication[3.OA.5] in rectangular arrays and areas. This property can be seen to extend to Equal Groups situations when Equal Groups situations are related to arrays by arranging each group in a row and putting the groups under each other to form an array. Array situations can be seen as Equal Groups situations if each row or column is considered as a group. Relating Equal Groups situations to arrays, and indicating rows or columns within arrays, can help students see that a corner object in an array (or a corner square in an area model) is not double counted: at a given time, it is counted as part of a row or as a part of a column but not both.

Table 3. Multiplication and division situations

	$A \times B = \square$	$A \times \square = C$ and $C \div A = \square$	$\square \times B = C$ and $C \div B = \square$
Equal Groups of Objects	Unknown Product There are A bags with B plums in each bag. How many plums are there in all?	Group Size Unknown If C plums are shared equally into A bags, then how many plums will be in each bag?	Number of Groups Unknown If C plums are to be packed B to a bag, then how many bags are needed?
Arrays of Objects		*Equal groups language*	
	Unknown Product There are A rows of apples with B apples in each row. How many apples are there?	Unknown Factor If C apples are arranged into A equal rows, how many apples will be in each row?	Unknown Factor If C apples are arranged into equal rows of B apples, how many rows will there be?
		Row and column language	
	Unknown Product The apples in the grocery window are in A rows and B columns. How many apples are there?	Unknown Factor If C apples are arranged into an array with A rows, how many columns of apples are there?	Unknown Factor If C apples are arranged into an array with B columns, how many rows are there?
Compare		$A > 1$	
	Larger Unknown A blue hat costs \$$B$. A red hat costs A times as much as the blue hat. How much does the red hat cost?	Smaller Unknown A red hat costs \$$C$ and that is A times as much as a blue hat costs. How much does a blue hat cost?	Multiplier Unknown A red hat costs \$$C$ and a blue hat costs \$$B$. How many times as much does the red hat cost as the blue hat?
		$A < 1$	
	Smaller Unknown A blue hat costs \$$B$. A red hat costs A as much as the blue hat. How much does the red hat cost?	Larger Unknown A red hat costs \$$C$ and that is A of the cost of a blue hat. How much does a blue hat cost?	Multiplier Unknown A red hat costs \$$C$ and a blue hat costs \$$B$. What fraction of the cost of the blue hat is the cost of the red hat?

Adapted from the *Common Core State Standards for Mathematics*, p. 89.

Equal Groups problems can also be stated in terms of columns, exchanging the order of A and B, so that the same array is described. For example: There are B columns of apples with A apples in each column. How many apples are there?

In the row and column situations (as with their area analogues), number of groups and group size are not distinguished.

Multiplicative Compare problems appear first in Grade 4, with whole-number values for A, B, and C, and with the "times as much" language in the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit fraction language change the subject of the comparing sentence, e.g., "A red hat costs A times as much as the blue hat" results in the same comparison as "A blue hat costs $\frac{1}{A}$ times as much as the red hat," but has a different subject.

Division problems of the form $A \times \square = C$ are about finding an unknown multiplicand. For Equal Groups and Compare situations, these involve what is called the sharing, partitive, how-many-in-each-group, or what-is-the-unit interpretation of division. As discussed on p. 31, Array situations can be seen as Equal Groups situations, thus, also as examples of the sharing interpretation of division for problems about finding an unknown multiplicand.

Division problems of the form $\square \times B = C$ are about finding an unknown multiplier. For Equal Groups and Compare situations, these involve what is called the measurement, quotitive, how-many-groups, or how-many-units interpretation of division. As discussed on p. 31, Array situations can be seen as Equal Groups situations, thus, also as examples of the measurement interpretation of division for problems about finding an unknown multiplier.

As noted in Table 3, row and column language can be difficult. The Array problems given in the table are of the simplest form in which a row is a group and Equal Groups language is used ("with 6 apples in each row"). Such problems are a good transition between the Equal Groups and Array situations and can support the generalization of the commutative property discussed above. Problems in terms of "rows" and "columns," e.g., "The apples in the grocery window are in 3 rows and 6 columns," are difficult because of the distinction between the number of things *in a* row and the number *of* rows. There are 3 rows but the number of columns (6) tells how many are in each row. There are 6 columns but the number of rows (3) tells how many are in each column. Students do need to be able to use and understand these words, but this understanding can grow over time while students also learn and use the language in the other multiplication and division situations.

Variations of each type that use measurements instead of discrete objects are given in the Geometric Measurement Progression. Grade 2 standards focus on length measurement[2.MD.1–4] and Grade 3 standards focus on area measurement.[3.MD.5–7] The measurement examples are more difficult than are the examples about discrete objects, so these should follow problems about discrete objects. Area problems where regions are partitioned by unit squares are foundational for Grade 3 standards because area is used to represent single–digit multiplication and division strategies,[3.MD.7] multi–digit multiplication and division in Grade 4, and multiplication and division of fractions in Grades 5 and 6.[5.NBT.6] The distributive property is central to all of these uses and will be discussed later.

The top row of Table 3 shows the usual order of writing multiplications of Equal Groups in the United States. The equation $3 \times 6 = \square$ means how many are in 3 groups of 6 things each: three sixes. But in many other countries the equation $3 \times 6 = \square$ means how many are 3 things taken 6 times (6 groups of 3 things each): six threes. Some students bring this interpretation of multiplication equations into the classroom. So it is useful to discuss the different interpretations and allow students to use whichever is used in their home. This is a kind of linguistic commutativity that precedes the reasoning discussed above arising from rotating an array. These two sources of commutativity can be related when the rotation discussion occurs.

Levels in problem representation and solution   Multiplication and division problem representations and solution methods can be considered as falling within three levels related to the levels for addition and subtraction (see Appendix 1). Level 1 is making and counting all of the quantities involved in a multiplication or division. As before, the quantities can be represented by objects or with a diagram, but a diagram affords reflection and sharing when it is drawn on the board and explained by a student. The Grade 2 standards 2.OA.3
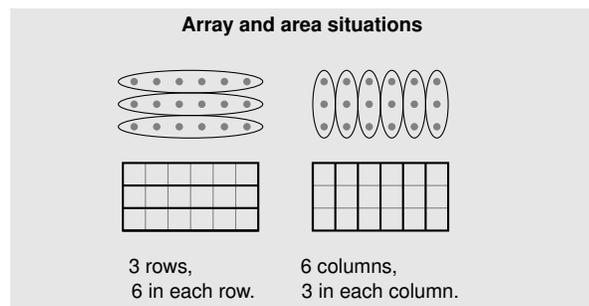
Array and area situations

3 rows,
6 in each row.

6 columns,
3 in each column.

2.MD.1 Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

2.MD.2 Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

2.MD.4 Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

3.MD.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7 Relate area to the operations of multiplication and addition.

5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

and 2.OA.4 are at this level but set the stage for Level 2. Standard 2.OA.3 relates doubles additions up to 20 to the concept of odd and even numbers and to counting by 2s (the easiest count-by in Level 2) by pairing and counting by 2s the things in each addend. 2.OA.4 focuses on using addition to find the total number of objects arranged in rectangular arrays (up to 5 by 5).

Level 2 is repeated counting on by a given number, such as for 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. The count-bys give the running total. The number of 3s said is tracked with fingers or a visual or physical (e.g., head bobs) pattern. For 8×3, you know the number of 3s and count by 3 until you reach 8 of them. For $24 \div 3$, you count by 3 until you hear 24, then look at your tracking method to see how many 3s you have. Because listening for 24 is easier than monitoring the tracking method for 8 3s to stop at 8, dividing can be easier than multiplying.

The difficulty of saying and remembering the count-by for a given number depends on how closely related it is to 10, the base for our written and spoken numbers. For example, the count-by sequence for 5 is easy, but the count-by sequence for 7 is difficult. Decomposing with respect to a ten can be useful in going over a decade within a count-by. For example in the count-by for 7, students might use the following mental decompositions of 7 to compose up to and then go over the next decade, e.g., $14 + 7 = 14 + 6 + 1 = 20 + 1 = 21$. The count-by sequence can also be said with the factors, such as "one times three is *three*, two times three is *six*, three times three is *nine*, etc." Seeing as well as hearing the count-bys and the equations for the multiplications or divisions can be helpful.

Level 3 methods use the associative property or the distributive property to compose and decompose. These compositions and decompositions may be additive (as for addition and subtraction) or multiplicative. For example, students multiplicatively compose or decompose:

4×6 is easier to count by 3 eight times:

$$4 \times 6 = 4 \times (2 \times 3) = (4 \times 2) \times 3 = 8 \times 3.$$

Students may know a product 1 or 2 ahead of or behind a given product and say:

I know 6×5 is 30, so 7×5 is $30 + 5$ more which is 35.

This implicitly uses the distributive property:

$$7 \times 5 = (6 + 1) \times 5 = 6 \times 5 + 1 \times 3 = 30 + 5 = 35.$$

Students may decompose a product that they do not know in terms of two products they know (for example, 4×7 shown in the margin).

Supporting Level 2 methods with arrays

Small arrays (up to 5×5) support seeing and beginning to learn the Level 2 count-bys for the first five equal groups of the small numbers 2 through 5 if the running total is written to the right of each row (e.g., 3, 6, 9, 12, 15). Students may write repeated additions and then count by ones without the objects, often emphasizing each last number said for each group. Grade 3 students can be encouraged to move as early as possible from equal groups or array models that show all of the quantities to similar representations using diagrams that show relationships of numbers because diagrams are faster and less error-prone and support methods at Level 2 and Level 3. Some demonstrations of methods or of properties may need to fall back to initially showing all quantities along with a diagram.

Composing up to, then over the next decade

7	14	21	28	35	42	49	56	63	70
	$6+1$		$2+5$	$5+2$		$1+6$	$4+3$		

There is an initial $3 + 4$ for $7 + 7$ that completes the reversing pattern of the partners of 7 involved in these mental decompositions with respect to the decades.

Decomposing 4×7

$$
\begin{aligned}
4 \times 7 &= 4 \times (5 + 2) \\
&= (4 \times 5) + (4 \times 2) \\
&= 20 + 8 \\
&= 28
\end{aligned}
$$

Students may not use the properties explicitly (for example, they might omit the second two steps), but classroom discussion can identify and record properties in student reasoning. An area diagram can support such reasoning.

The $5 + n$ pattern students used earlier for additions can now be extended to show how 6, 7, 8, and 9 times a number are $5 + 1$, $5 + 2$, $5 + 3$, and $5 + 4$ times that number. These patterns are particularly easy to do mentally for the numbers 4, 6, and 8. The 9s have particularly rich patterns based on $9 = 10 - 1$. The pattern of the tens digit in the product being 1 less than the multiplier, the ones digit in the product being 10 minus the multiplier, and that the digits in nines products sum to 9 all come from this pattern.

There are many opportunities to describe and reason about the many patterns involved in the Level 2 count-bys and in the Level 3 composing and decomposing methods. There are also patterns in multiplying by 0 and by 1. These need to be differentiated from the patterns for adding 0 and adding 1 because students often confuse these three patterns: $n + 0 = n$ but $n \times 0 = 0$, and $n \times 1$ is the pattern that does not change n (because $n \times 1 = n$). Patterns make multiplication by some numbers easier to learn than multiplication by others, so approaches may teach multiplications and divisions in various orders depending on what numbers are seen as or are supported to be easiest.

Multiplications and divisions can be learned at the same time and can reinforce each other. Level 2 methods can be particularly easy for division, as discussed above. Level 3 methods may be more difficult for division than for multiplication.

Throughout multiplication and division learning, students gain fluency and begin to know certain products and unknown factors. All of the understandings of multiplication and division situations, of the levels of representation and solving, and of patterns need to culminate by the end of Grade 3 in fluent multiplying and dividing of all single–digit numbers and 10.[3.OA.7] Such fluency may be reached by becoming fluent for each number (e.g., the 2s, the 5s, etc.) and then extending the fluency to several, then all numbers mixed together. Organizing practice so that it focuses most heavily on understood but not yet fluent products and unknown factors can speed learning. To achieve this by the end of Grade 3, students must begin working toward fluency for the easy numbers as early as possible. Because an unknown factor (a division) can be found from the related multiplication, the emphasis at the end of the year is on knowing from memory all products of two one–digit numbers. As should be clear from the foregoing, this isn't a matter of instilling facts divorced from their meanings, but rather the outcome of a carefully designed learning process that heavily involves the interplay of practice and reasoning. All of the work on how different numbers fit with the base–ten numbers culminates in these "just know" products and is necessary for learning products. Fluent dividing for all single–digit numbers, which will combine "just knows," knowing

Supporting reasoning with area diagram

$$7 = 5 + 2$$

$$20 + 8 = 28$$
$$4 \times 5 + 4 \times 2 = 4 \times 7$$
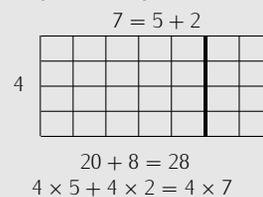
The $5 + n$ pattern for multiplying the numbers 4, 6, **and** 8

	n	$4 \times n$		$6 \times n$		$8 \times n$	
1	$5 + 1$	4	24	6	36	8	48
2	$5 + 2$	8	28	12	42	16	56
3	$5 + 3$	12	32	18	48	24	64
4	$5 + 4$	16	36	24	54	32	72
5	$5 + 5$	20	40	30	60	40	80

Patterns in multiples of 9

$1 \times 9 = 9$

$2 \times 9 = 2 \times (10 - 1) = (2 \times 10) - (2 \times 1) = 20 - 2 = 18$

$3 \times 9 = 3 \times (10 - 1) = (3 \times 10) - (3 \times 1) = 30 - 3 = 27,$ etc

3.OA.7 Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

from a multiplication, patterns, and best strategy, is also part of this vital standard.

Using a letter for the unknown quantity, the order of operations, and two-step word problems with all four operations   Students in Grade 3 begin the step to algebraic language by using a letter for the unknown quantity in expressions or equations for one- and two-step problems.[3.OA.8] But the symbols of arithmetic, $\times$ or $*$ for multiplication and $\div$ for division, continue to be used in Grades 3, 4, and 5.•

Understanding and using the associative and distributive properties (as discussed above) requires students to know two conventions for reading an expression that has more than one operation:

> *Parentheses.* Operations inside parentheses are done before operations outside parentheses (the parentheses can be thought of as hands curved around the symbols and grouping them).

> *Precedence.* If a multiplication or division is written next to an addition or subtraction, imagine parentheses around the multiplication or division (it is done before these operations). At Grades 3 through 5, parentheses can usually be used for such cases so that fluency with this rule can wait until Grade 6.

These conventions can seem to be a central aspect of algebra. But actually they are just simple "rules of the road" that, along with performing operations from left to right• in the absence of parentheses and precedence, allow expressions involving more than one operation to be interpreted unambiguously and thus are connected with the mathematical practice of communicating precisely (MP.6). Use of parentheses is important in displaying structure and thus is connected with the mathematical practice of making use of structure (MP.7).•

Together with the meaning of the equal sign, these conventions are important in expressing the associative and distributive properties. For example, this instance of the distributive property

$$3 \times (10 + 5) = 3 \times 10 + 3 \times 5$$

says that $10 + 5$ multiplied by 3 yields the same number as multiplying 10 by 3 adding it to 5 multiplied by 3. Thus, the calculation described by $3 \times (10 + 5)$ can be replaced by the calculation described by $3 \times 10 + 3 \times 5$, and vice versa.

The associative and distributive properties are at the heart of Grades 3 to 5 because they are used in the Level 3 multiplication and division strategies, in multi-digit and decimal multiplication and division, and in all operations with fractions.

[3.OA.8]Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.[3]

[3]This standard is limited to problems posed with whole numbers and having whole-number answers; students should know how to perform operations in the conventional order when there are no parentheses to specify a particular order (Order of Operations).

• Use of / to indicate division is not suggested until the connection between fractions and division has been discussed, see the Grade 5 section of the Number and Operations—Fractions Progression. Note that, if used to represent an unknown quantity, the letter x may be difficult to distinguish from the multiplication symbol $\times$.

• Performed from right to left: $10 - 2 + 5$ is $10 - 7$, which is 3. But, from left to right: $10 - 2 + 5$ is $8 + 5$, which is 13.

Making use of structure (MP.7) to make computation easier

$$13 + 29 + 77 + 11 = (13 + 77) + (29 + 11)$$

Here, an expression with parentheses is used to describe an approach that students might take but not necessarily its steps or symbolism.

• This discussion and the footnote for 3.OA.8 above are about reading, rather than writing, expressions. Use of parentheses, but not necessarily fluency with parentheses, is expected in Grade 5 (see 5.OA.1). However reading expressions with parentheses may begin earlier. As illustrated on the next page, equations that represent a word problem may not require use of parentheses or multiple operation symbols.

As with two-step problems at Grade 2 which involve only addition and subtraction (2.OA.1 and 2.MD.5), the Grade 3 two-step word problems vary greatly in difficulty and ease of representation. More difficult problems may require two steps of representation and solution rather than one. Students may vary widely in how they represent such problems, with some students just writing steps or making one drawing and not using equations. Use of two-step problems involving easy or middle difficulty adding and subtracting within 1,000 or one such adding or subtracting with one step of multiplication or division can help to maintain fluency with addition and subtraction while giving the needed time to the major Grade 3 multiplication and division standards.

A two-step problem with diagram showing problem situation and equations showing the two parts

Carla has 4 packages of silly bands. Each package has 8 silly bands in it. Agustin is supposed to get 15 fewer silly bands than Carla. How many silly bands should Agustin get?

C is the number of silly bands that Carla has.
A is the number of silly bands that Agustin has.

$C = 4 \times 8 = 32$

$$A + 15 = C$$
$$A + 15 = 32$$
$$A = 17$$

Students may be able to solve this problem without writing such equations. For example:

Carla has 4×8 silly bands.
$32 - 15 = 17$
So Agustin has 17 silly bands.

Or

Agustin's plus 15 is as many as Carla has.
$15 + ? = 32$
So Agustin has 17 silly bands.

Grade 4

Multiplicative Compare   Consider two diving boards, one 40 feet high, the other 8 feet high. Students in earlier grades learned to compare these heights in an additive sense—"This one is 32 feet higher than that one"—by solving additive Compare problems[2.OA.1] and using addition and subtraction to solve word problems involving length.[2.MD.5] Students in Grade 4 learn to compare these quantities multiplicatively as well: "This one is 5 times as high as that one."[4.OA.1,4.OA.2,4.MD.1,4.MD.2] In an additive comparison, the underlying question is *what amount would be added to one quantity* in order to result in the other. In a multiplicative comparison, the underlying question is *what factor would multiply one quantity* in order to result in the other. Multiplicative Compare situations are shown in Table 3.

Language can be difficult in multiplicative Compare problems. The language used in the three examples in Table 3 is fairly simple, e.g., "A red hat costs 3 times as much as the blue hat." Saying the comparing sentence in the opposite way is more difficult. It could be said using division, e.g., "The cost of a red hat divided by 3 is the cost of a blue hat." It could also be said using a unit fraction, e.g., "A blue hat costs one–third as much as a red hat"; note however that multiplying by a fraction in not an expectation of the Standards in Grade 4. In any case, many languages do not use either of these options for saying the opposite comparison. They use the terms *three times more than* and *three times less than* to describe opposite multiplicative comparisons. These did not used to be acceptable usages in English because they mix the multiplicative and additive comparisons and are ambiguous. If the cost of a red hat is three times more than a blue hat that costs $5, does a red hat cost $15 (three times as much) or $20 (three times more than: a difference that is three times as much)? However, the terms *three times more than* and *three times less than* are now appearing frequently in newspapers and other written materials. It is recommended to discuss these complexities with Grade 4 students while confining problems that appear on tests or in multi–step problems to the well-defined multiplication language in Table 3. The tape diagram for the additive Compare situation that shows a smaller and a larger tape can be extended to the multiplicative Compare situation.

Fourth graders extend problem solving to multi–step word problems using the four operations posed with whole numbers. The same limitations discussed for two–step problems concerning representing such problems using equations apply here. Some problems might easily be represented with a single equation, and others will be more sensibly represented by more than one equation or a diagram and one or more equations. Numbers can be those in the Grade 4 standards, but the number of steps should be no more than three and involve only easy and medium difficulty addition and subtraction problems.

Tape diagram used to solve the Compare problem in Table 3

B is the cost of a blue hat in dollars
R is the cost of a red hat in dollars

$3 \times B = R$

$3 \times \$6 = \$18$

Tape diagram used to solve a Compare problem

A big penguin will eat 3 times as much fish as a small penguin. The big penguin will eat 420 grams of fish. All together, how much will the two penguins eat?
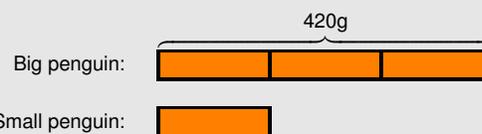
B is the number of grams the big penguin eats
S is the number of grams the small penguin eats

$$3 \times S = B$$
$$3 \times S = 420$$
$$S = 140$$

$$S + B = 140 + 420$$
$$= 560$$

Remainders   In problem situations, students must interpret and use remainders with respect to context.[4.OA.3] For example, what is the smallest number of busses that can carry 250 students, if each bus holds 36 students? The whole number quotient in this case is 6 and the remainder is 34; the equation $250 = 6 \times 36 + 34$ expresses this result and corresponds to a picture in which 6 busses are completely filled while a seventh bus carries 34 students. Notice that the answer to the stated question (7) differs from the whole number quotient.

On the other hand, suppose 250 pencils were distributed among 36 students, with each student receiving the same number of pencils. What is the largest number of pencils each student could have received? In this case, the answer to the stated question (6) is the same as the whole number quotient. If the problem had said that the teacher got the remaining pencils and asked how many pencils the teacher got, then the remainder would have been the answer to the problem.

Factors, multiples, and prime and composite numbers   Students extend the idea of decomposition to multiplication and learn to use the term *multiple*.[4.OA.4] Any whole number is a multiple of each of its factors, so for example, 21 is a multiple of 3 and a multiple of 7 because $21 = 3 \times 7$. A number can be multiplicatively decomposed into equal groups and expressed as a product of these two factors (called factor pairs). A *prime* number has only one and itself as factors. A *composite* number has two or more factor pairs. Students examine various patterns in factor pairs by finding factor pairs for all numbers 1 to 100 (e.g., no even number other than 2 will be prime because it always will have a factor pair including 2). To find all factor pairs for a given number, students can search systematically, by checking if 2 is a factor, then 3, then 4, and so on, until they start to see a "reversal" in the pairs (for example, after finding the pair 6 and 9 for 54, students will next find the reverse pair, 9 and 6; all subsequent pairs will be reverses of previously found pairs). Students understand and use of the concepts and language in this area, but need not be fluent in finding all factor pairs. Determining whether a given whole number in the range 1 to 100 is a multiple of a given one–digit number is a matter of interpreting prior knowledge of division in terms of the language of multiples and factors.

Generating and analyzing patterns   This standard[4.OA.5] begins a small focus on reasoning about number or shape patterns, connecting a rule for a given pattern with its sequence of numbers or shapes. Patterns that consist of repeated sequences of shapes or growing sequences of designs can be appropriate for the grade. For example, students could examine a sequence of dot designs in which each design has 4 more dots than the previous one and they could reason

[4.OA.3] Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.

A note on notation

The result of division within the system of whole numbers is frequently written as:

$$84 \div 10 = 8\,\mathrm{R}\,4 \quad \text{and} \quad 44 \div 5 = 8\,\mathrm{R}\,4.$$

Because the two expressions on the right are the same, students should conclude that $84 \div 10$ is equal to $44 \div 5$, but this is not the case. (Because the equal sign is not used appropriately, this usage is a non-example of Standard for Mathematical Practice 6.) Moreover, the notation $8\,\mathrm{R}\,4$ does not indicate a number.

Rather than writing the result of division solely in terms of a whole-number quotient and remainder, the relationship of whole-number quotient and remainder can be written as:

$$84 = 8 \times 10 + 4 \quad \text{and} \quad 44 = 8 \times 5 + 4.$$

In Grade 5, students can begin to use fraction or decimal notation to express the result of division, e.g., $84 \div 10 = 8\frac{4}{10}$. See the Number and Operations—Fractions Progression.

[4.OA.4] Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

[4.OA.5] Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.

about how the dots are organized in the design to determine the total number of dots in the 100th design.

In examining numerical sequences, fourth graders can explore rules of repeatedly adding the same whole number or repeatedly multiplying by the same whole number. Properties of repeating patterns of shapes can be explored with division. For example, to determine the 100th shape in a pattern that consists of repetitions of the sequence "square, circle, triangle," the fact that when we divide 100 by 3 the whole number quotient is 33 with remainder 1 tells us that after 33 full repeats, the 99th shape will be a triangle (the last shape in the repeating pattern), so the 100th shape is the first shape in the pattern, which is a square.

Notice that the Standards do not require students to infer or guess the underlying rule for a pattern (see margin), but rather ask them to generate a pattern from a given rule and identify features of the given pattern that were not explicit in the rule itself.

Generating and analyzing patterns

Generating the pattern: Starting with 0, using the rule "plus 2."

0, 2, 4, 6, 8, 10, 12, . . .

Analyzing the pattern.

The numbers are always even.

The rule says to add but we can get the same result by multiplying by 2 starting with 0: 2×0, $2 \times 1, 2 \times 2, 2 \times 3, \ldots$

Generating the pattern: Starting with 0, using the rule "plus 3."

0, 3, 6, 9, 12, 15, 18, . . .

Analyzing the pattern.

The numbers alternate between odd and even.

The numbers are always multiples of 3, even though the rule says plus 3.

The pattern can be written like this:

$3 \times 0, 3 \times 1, 3 \times 2, 3 \times 3, \ldots$

Generating the pattern: Starting with 1, using the rule "times 3."

1, 3, 9, 27, . . .

Analyzing the pattern.

The numbers are always odd.

The pattern can be written like this:

$1, 3, 3 \times 3, 3 \times 3 \times 3, 3 \times 3 \times 3 \times 3, \ldots$

The problem with patterns

Students are asked to continue the pattern 2, 4, 6, 8, . . . Here are some legitimate responses:

- Cody: I am thinking of a "plus 2 pattern," so it continues 10, 12, 14, 16, . . .
- Ali: I am thinking of a repeating pattern, so it continues 2, 4, 6, 8, 2, 4, 6, 8, . . .
- Suri: I am thinking of the units digit in the multiples of 2, so it continues 0, 2, 4, 6, 8, 0, 2, . . .
- Erica: If $g(n)$ is any polynomial, then $f(n) = 2n + (n-1)(n-2)(n-3)(n-4)g(n)$ describes a continuation of this sequence.
- Zach: I am thinking of that high school cheer, "2, 4, 6, 8. Who do we appreciate?"

Because the task provides no structure, all of these answers must be considered correct. Without any structure, continuing the pattern is simply speculation—a guessing game. Because there are many ways to continue a sequence, patterning problems should provide enough structure so that the sequence is well defined.

Grade 5

As preparation for the Expressions and Equations Progression in the middle grades, students in Grade 5 begin working more formally with expressions.[5.OA.1, 5.OA.2] They write expressions to express a calculation, e.g., writing $2 \times (8 + 7)$ to express the calculation "add 8 and 7, then multiply by 2." They also evaluate and interpret expressions, e.g., using their conceptual understanding of multiplication to interpret $3 \times (18932 + 921)$ as being three times as large as $18932 + 921$, without having to calculate the indicated sum or product. Thus, students in Grade 5 begin to think about numerical expressions in ways that prefigure their later work with variable expressions (e.g., three times an unknown length is $3 \times L$). In Grade 5, this work should be viewed as exploratory rather than for attaining mastery; for example, expressions should not contain nested grouping symbols,• and they should be no more complex than the expressions one finds in an application of the associative or distributive property, e.g., $(8 + 27) + 2$ or $(6 \times 30) + (6 \times 7)$. Note however that the numbers in expressions need not always be whole numbers.

Students extend their Grade 4 pattern work by working briefly with two numerical patterns that can be related and examining these relationships within sequences of ordered pairs and in the graphs in the first quadrant of the coordinate plane.[5.OA.3] This work prepares students for studying proportional relationships and functions in middle grades.

• Any type of grouping symbols (e.g., parentheses, brackets, or braces) may be used. The Standards do not dictate a fixed order in which particular types of grouping symbols must be used.

Generating and plotting sequences of ordered pairs

Starting with 0 *and using the rule "plus 3."*

    0, 3, 6, 9, 12, 15, 18, . . .

Starting with 0 *and using the rule "plus 2."*

    0, 2, 4, 6, 8, 10, 12, . . .

Forming ordered pairs consisting of corresponding terms.

0	3	6	9	12	15	18
0	2	4	6	8	10	12

Graphing the ordered pairs on a coordinate plane.
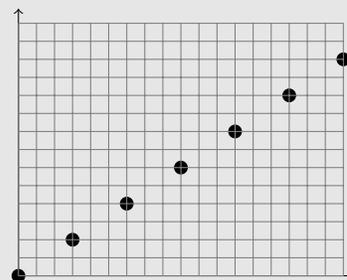
Identifying apparent relationships between corresponding terms.

    Analysis 1
    The first pattern is: 2×0, 2×1, 2×2, 2×3, 2×4, . . .

    The second pattern is: 3×0, 3×1, 3×2, 3×3, 3×4, . . .

    To make a number in the first pattern into the corresponding number in the second pattern, you divide by 2 and multiply by 3.

    Analysis 2

$3 \times n$	0	3	6	9	12	15	18
$2 \times n$	0	2	4	6	8	10	12

    When the number in the second pattern is $3 \times n$, the number in the first pattern is $2 \times n$.
    $\frac{3}{2} \times (2 \times n) = 3 \times n$

Connections to NF and NBT in Grades 3 through 5

Students extend their whole number work with adding and sub-
tracting and multiplying and dividing situations to fractions. Each
of these extensions can begin with problems that include all of the
subtypes of the situations in Tables 1 and 2. The operations of addi-
tion, subtraction, multiplication, and division continue to be used in
the same way in these problem situations when they are extended
to fractions (although making these extensions is not automatic or
easy for all students). The connections described for Kindergarten
through Grade 3 among word problem situations, representations
for these problems, and use of properties in solution methods are
equally relevant for these new kinds of numbers. Students use the
new kinds of numbers in geometric measurement and data problems
and extend to some two-step and multi-step problems involving all
four operations. In order to keep the difficulty level from becom-
ing extreme, there should be a tradeoff between the algebraic or
situational complexity of any given problem and its computational
difficulty taking into account the kinds of numbers involved.

As students' notions of quantity evolve and generalize from dis-
crete to continuous during Grades 3–5, their notions of multiplication
evolve and generalize. This evolution deserves special attention be-
cause it begins in the domain of Operations and Algebraic Thinking
but ends in the domain of Fractions. Thus, the concept of multipli-
cation begins in Grade 3 with an entirely discrete notion of "equal
groups."[3.OA.1] By Grade 4, students can also interpret a multipli-
cation equation as a statement of comparison involving the notion
"times as much."[4.OA.1] This notion easily includes continuous quan-
tities, e.g., $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4
times as much as $\frac{3}{4}$ cup of flour.[4.NF.4,4.MD.2] By Grade 5, when stu-
dents multiply fractions in general,[5.NF.4] products can be larger or
smaller than either factor, and multiplication can be seen as an op-
eration that "stretches or shrinks" by a scale factor.[5.NF.5] This view
of multiplication as scaling is the appropriate notion for reasoning
multiplicatively with continuous quantities.

Where this progression is heading

The properties of and relationships between operations that stu-
dents worked with in Grades K–5 will become even more prominent
in extending arithmetic to systems that include negative numbers;
meanwhile the meanings of the operations will continue to evolve,
e.g., subtraction also can be seen as "adding the opposite." See the
Number System Progression.

In Grade 6, students will begin to view expressions not just as
calculation recipes but as entities in their own right, which can be
described in terms of their parts. For example, students see $8 \times (5+2)$
as the product of 8 with the sum $5 + 2$. In particular, students

[3.OA.1] Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

[4.OA.1] Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

[4.NF.4] Apply and extend previous understandings of multiplica-
tion to multiply a fraction by a whole number.

[4.MD.2] Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or deci-
mals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measure-
ment quantities using diagrams such as number line diagrams that feature a measurement scale.

[5.NF.4] Apply and extend previous understandings of multiplica-
tion to multiply a fraction or whole number by a fraction.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

   a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without per-
forming the indicated multiplication.

   b Explaining why multiplying a given number by a frac-
tion greater than 1 results in a product greater than the given number (recognizing multiplication by whole num-
bers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

must use the conventions for parentheses and order of operations to *interpret* expressions, not just to evaluate them. Viewing expressions as entities created from component parts is essential for seeing the structure of expressions in later grades and using structure to reason about expressions and functions. See the Expressions and Equations Progression.

As noted above, the foundation for these later competencies is laid in Grade 5 when students write expressions to record a "calculation recipe" without actually evaluating the expression,[5.OA.2] use parentheses to formulate expressions,[5.OA.1] and examine patterns and relationships numerically and visually on a coordinate plane graph. Before Grade 5, student thinking that also builds toward the Grade 6 Expressions and Equations work is focusing on the expressions on each side of an equation, relating each expression to the situation, and discussing the situational and mathematical vocabulary involved to deepen the understandings of expressions and equations.

In Grades 6 and 7, students begin to explore the systematic algebraic methods used for solving algebraic equations. Central to these methods are the relationships between addition and subtraction and between multiplication and division, emphasized in several parts of this progression and prominent also in the Number System Progression. Students' varied work throughout elementary school with equations with unknowns in all locations and in writing equations to decompose a given number into many pairs of addends or many pairs of factors are also important foundations for understanding equations and for solving equations with algebraic methods. Of course, any method of solving, whether systematic or not, relies on an understanding of what solving itself is—namely, a process of answering a question: which values from a specified set, if any, make the equation true?[6.EE.5]

Students represent and solve word problems with equations involving one unknown in Grades K through 5. The unknown was expressed by a □ or other symbol in K–2 and by a letter in Grades 3 to 5. Grade 6 students continue the K–5 focus on representing a problem situation using an equation (a situation equation) and then (for the more difficult situations) writing an equivalent equation that is easier to solve (a solution equation). Grade 6 students discuss their reasoning more explicitly by focusing on the structures of expressions and using the properties of operations explicitly. Some of the math drawings that students have used in K through 5 to represent problem situations continue to be used in the middle grades. These can help students throughout the grades deepen the connections they make among the situation and problem representations by a diagram and/or by an equation, and support the informal K–5 and increasingly formal 6–8 solution methods arising from understanding the structure of expressions and equations.

[5.OA.2] Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.

[5.OA.1] Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

[6.EE.5] Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Appendix 1. Methods for single-digit addition and subtraction problems

Level 1. Direct Modeling by Counting All or Taking Away.

Represent situation or numerical problem with groups of objects, a drawing, or fingers. Model the situation by composing two addend groups or decomposing a total group. Count the resulting total or addend.

Adding ($8 + 6 = \square$): Represent each addend by a group of objects. Put the two groups together. Count the total. Use this strategy for Add To/Result Unknown and Put Together/Total Unknown.

Subtracting ($14 - 8 = \square$): Represent the total by a group of objects. Take the known addend number of objects away. Count the resulting group of objects to find the unknown added. Use this strategy for Take From/Result Unknown.

Levels	$8 + 6 = 14$	$14 - 8 = 6$
Level 1: Count all		
Level 2: Count on		

Level 3:
Recompose
  Make a ten (general): one addend breaks apart to make 10 with the other addend
 	
  Make a ten (from 5's within each addend)		

Doubles ± n	$6 + 8$ $= 6 + 6 + 2$ $= 12 + 2 = 14$	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone.
Children are much more successful with counting on; it makes subtraction as easy as addition.

Level 2. Counting On.

Embed an addend within the total (the addend is perceived simul-
taneously as an addend and as part of the total). Count this total
but abbreviate the counting by omitting the count of this addend;
instead, begin with the number word of this addend. Some method
of keeping track (fingers, objects, mentally imaged objects, body mo-
tions, other count words) is used to monitor the count.

For addition, the count is stopped when the amount of the re-
maining addend has been counted. The last number word is the
total. For subtraction, the count is stopped when the total occurs in
the count. The tracking method indicates the difference (seen as an
unknown addend).

Counting on can be used to find the total or to find an addend.
These look the same to an observer. The difference is what is mon-
itored: the total or the known addend. Some students count down
to solve subtraction problems, but this method is less accurate and
more difficult than counting on. Counting on is not a rote method. It
requires several connections between cardinal and counting mean-
ings of the number words and extended experience with Level 1
methods in Kindergarten.

Adding (e. g., $8 + 6 = \square$) uses counting on to find a
total: One counts on from the first addend (or the larger
number is taken as the first addend). Counting on is
monitored so that it stops when the second addend has
been counted on. The last number word is the total.

Finding an unknown addend (e.g., $8 + \square = 14$): One
counts on from the known addend. The keeping track
method is monitored so that counting on stops when
the known total has been reached. The keeping track
method tells the unknown addend.

Subtracting ($14 - 8 = \square$): One thinks of subtracting
as finding the unknown addend, as $8 + \square = 14$ and uses
counting on to find an unknown addend (as above).

The problems in Table 2 which are solved by Level 1 methods in
Kindergarten can also be solved using Level 2 methods: counting on
to find the total (adding) or counting on to find the unknown addend
(subtracting).

The middle difficulty (lightly shaded) problem types in Table 2
for Grade 1 are directly accessible with the embedded thinking of
Level 2 methods and can be solved by counting on.

Finding an unknown addend (e.g., $8 + \square = 14$) is
used for Add To/Change Unknown, Put Together/Take
Apart/Addend Unknown, and Compare/Difference Unknown.
It is also used for Take From/Change Unknown ($14 - \square =$

8) after a student has decomposed the total into two ad-
dends, which means they can represent the situation as
$14 - 8 = \square$.

Adding or subtracting by counting on is used by some
students for each of the kinds of Compare problems (see
the equations in Table 2). Grade 1 students do not neces-
sarily master the Compare Bigger Unknown or Smaller
Unknown problems with the misleading language in the
bottom row of Table 2.

Solving an equation such as $6 + 8 = \square$ by counting on from
8 relies on the understanding that $8 + 6$ gives the same total, an
implicit use of the commutative property without the accompanying
written representation $6 + 8 = 8 + 6$.

Level 3. Convert to an Easier Equivalent Problem.

Decompose an addend and compose a part with another addend.
These methods can be used to add or to find an unknown addend
(and thus to subtract). These methods implicitly use the associative
property.

Adding

Make a ten. E.g, for $8 + 6 = \square$,

$$8 + \underline{6} = 8 + \underline{2 + 4} = 10 + 4 = 14,$$

so $8 + 6$ becomes $10 + 4$.
Doubles plus or minus 1. E.g., for $6 + 7 = \square$,

$$6 + \underline{7} = 6 + \underline{6 + 1} = 12 + 1 = 13,$$

so $6 + 7$ becomes $12 + 1$.

Finding an unknown addend

Make a ten. E. g., for $8 + \square = 14$,

$$8 + \underline{2} = 10 \text{ and } \underline{4} \text{ more makes } 14. \ \underline{2 + 4} = 6.$$

So $8 + \square = 14$ is done as two steps: how many up to ten and how
many over ten (which can be seen in the ones place of 14).
Doubles plus or minus 1. E.g., for $6 + \square = 13$,

$$6 + \underline{6 + 1} = 12 + 1. \ \underline{6 + 1} = 7.$$

So $6 + \square = 13$ is done as two steps: how many up to 12 $(6 + 6)$ and
how many from 12 to 13.

Subtracting

Thinking of subtracting as finding an unknown addend. E.g., solve $14 - 8 = \square$ or $13 - 6 = \square$ as $8 + \square = 14$ or $6 + \square = 13$ by the above methods (make a ten or doubles plus or minus 1).

Make a ten by going down over ten. E.g., $14 - 8 = \square$ can be done in two steps by going down over ten: $14 - 4$ (to get to 10) $- 4 = 6$.

The Level 1 and Level 2 problem types can be solved using these Level 3 methods.

Level 3 problem types can be solved by representing the situation with an equation or drawing, then re-representing to create a situation solved by adding, subtracting, or finding an unknown addend as shown above by methods at any level, but usually at Level 2 or 3. Many students only show in their writing part of this multi-step process of re-representing the situation.

Students re-represent Add To/Start Unknown $\square + 6 = 14$ situations as $6 + \square = 14$ by using the commutative property (formally or informally).

Students re-represent Take From/Start Unknown $\square - 8 = 6$ situations by reversing as $6 + 8 = \square$, which may then be solved by counting on from 8 or using a Level 3 method.

At Level 3, the Compare misleading language situations can be solved by representing the known quantities in a diagram that shows the bigger quantity in relation to the smaller quantity. The diagram allows the student to find a correct solution by representing the difference between quantities and seeing the relationship among the three quantities. Such diagrams are the same diagrams used for the other versions of compare situations; focusing on which quantity is bigger and which is smaller helps to overcome the misleading language.

Some students may solve Level 3 problem types by doing the above re-representing but use Level 2 counting on.

As students move through levels of solution methods, they increasingly use equations to represent problem situations as situation equations and then to re-represent the situation with a solution equation or a solution computation. They relate equations to diagrams, facilitating such re-representing. Labels on diagrams can help connect the parts of the diagram to the corresponding parts of the situation. But students may know and understand things that they may not use for a given solution of a problem as they increasingly do various representing and re-representing steps mentally.

Appendix 2. Categorization of addition and subtraction situations

The history of research on addition and subtraction word problems is complex, and includes decades of work from researchers around the world. (See, for example, the *Handbook*s *of Research on Mathematics Teaching and Learning*.) In the U.S., such work has been done by researchers in mathematics education, cognitive development, and cognitive psychology, including researchers whose work has informed Cognitively Guided Instruction (CGI). Different terminologies and classifications have been used. Distinctions have primarily focused on the mathematical and action structure of the situations, linguistic variations, and specific contexts or order of the sentences that might affect performance. Although a group of such researchers met to stabilize terminology and distinctions, minor variations continue to occur.

Major categories   Most category systems use the three major categories in the *Standards* Table 1:

- Add To/Take From (called Change Plus and Change Minus in *Mathematics Learning in Early Childhood*);

- Put Together/Take Apart (sometimes called Collection or Combine);

- Compare.

All problems in these three major categories involve three quantities. Each of these quantities can be the unknown quantity. This is the most fundamental distinction in the research literature.

Subcategories   The second most important distinction is between addition and subtraction situations. The three major categories differ considerably in how this distinction is made and how fundamental it is.

Add To/Take From.   Addition and subtraction actions in Add To/Take From (Change) situations are quite different. These are the earliest meanings of addition and subtractions for children. For this reason, in many categorizations, including Table 1 of the *Standards*, Add To/Take From has two subcategories: Add To and Take From. These have also been called Change Plus and Change Minus or Join and Separate. The equal sign in the equations for these situations has the action meaning "become."

Put Together/Take Apart. The action for Put Together/Take Apart is more subtle and may be only conceptual (e.g., as in the apple problem in Table 1 of the *Standards*: considering the apples by color and then disregarding color to make the total). Also, for this major category there is not a fundamental distinction between the situational role of the addends, although one addend must occur first in the word

problem. In contrast, in Add To/Take From problems, one addend is first in the situation and the other addend is added to or taken from that first addend. For this reason, some researchers, including (sometimes) CGI researchers, have distinguished two subcategories of this major category—Unknown Total and Unknown Addend—but this classification can obscure the understanding that all major categories involve three unknown quantities and either addend can be unknown. The case in which both addends are unknown was used in Table 1 of the *Standards* because it is one of the prerequisites for the important make-a-ten strategy; these prerequisites are K.OA.4, K.OA.3, and K.NBT.1.

Compare. Compare situations have no situational addition or subtraction action. In fact, such situations only mention two quantities: a bigger quantity and a smaller quantity that are compared to find how much bigger or smaller one quantity is than the other. This third quantity must be conceptually constructed by comparing the two given quantities; this quantity is the difference. There are always two opposite but equivalent ways to state the comparison: Using "more" or "less/fewer" when the difference is unknown (and other linguistic variations of this distinction). For example:

Lucy has two apples. Julie has five apples.

How many more apples does Julie have than Lucy?
   or
How many fewer apples does Lucy have than Julie?

When the difference is known, the language used to state the comparison can suggest the solution operation: saying "more" for a bigger unknown situation, where you need to add the difference to the smaller quantity (Lucy has two apples. Julie has three more apples than Lucy), or saying "less" (or "fewer") for a smaller unknown situation, where you need to subtract the difference from the known bigger quantity to find the smaller unknown quantity (Julie has five apples. Lucy has three fewer apples than Julie). Such problems are easier for students than problems in which the comparing sentence suggests the wrong operation, for example, Lucy has two apples. Lucy has three fewer apples than Julie. Table 1 of the *Standards* distinguishes two subtypes, according to language variation, for each of the three Compare subcategories. Table 2 in the Operations and Algebraic Thinking Progression is more specific about how the language variations affect problem solving, and it indicates that this is the reason that the two subtypes suggesting the wrong operation are not for mastery in Grade 1.

The CGI Compare subtypes distinguish between whether or not the quantity in the comparing sentence is unknown (Compare Quantity Unknown vs. Referent Quantity Unknown). Compare Quantity Unknown is easier than Referent Quantity Unknown because using the unknown quantity as the subject of the comparing sentence

means that the comparing action can be done by starting with the known quantity. But this linguistic analysis does not reveal the underlying mathematical situation: two quantities are being compared, and one is smaller and the other one is bigger. The linguistic analysis also is not the fundamental way in which children and adults (for two-step problems) solve comparing situations. The key to success, especially for the more difficult misleading language versions at the bottom of Table 2 in this progression, is deciding which quantity is the bigger and which is the smaller by thinking about the comparing sentence; making a drawing to show this can be very helpful even for adults. It is important for classroom discourse to focus on the most important issues and to decide which quantity is the bigger and which is the smaller. This is the basis for equations that children write to show comparing situations.

Children write many different kinds of equations for compare situations, and teachers and programs should not focus on a subtraction equation as the most important or only equation, as some textbooks have done in the past (children are more likely to write an unknown addend equation). Also, according to the CGI categorization, a Compare problem switches subtypes when an equivalent comparing sentence is used (changing "more" to "fewer" (or "less") or vice versa). So the same situation in the world changes its CGI subtype when the comparing sentence changes its subject. This is also problematic because one major problem solving strategy for the more difficult version is to say the opposite comparing sentence to avoid the misdirecting language. This strategy seems more straightforward to teachers if using a comparing sentence that is equivalent mathematically does not change the problem classification. Generating comparing sentences that do not change the situation is also helpful to children in becoming fluent with the meanings of "less" and "fewer," which some children initially think mean "more." Only using the word "more" in the CGI problem type tables does not inform teachers of the importance for children of the need to practice with the words "less" and "fewer," and how relating equivalent comparing sentences can be helpful to solving the most difficult Compare variations.

Standards **categories**   A decision that had to be made for the *Standards* categorization was whether to choose names for the problem types that were easier for children or for adults (teachers and researchers). The decision was made to choose terms that were easier for children (with the thought that they might also be easier for teachers). Add To/Take From and Put Together/Take Apart use action words that are easier for children than the older terms Join/Separate (or Change Plus/Minus) and Part-Part-Whole (or Combine). Terms for the problem types do not have to be used in the classroom, but such use can facilitate discussion about the types. But as this summary has indicated, these terms are not part of the official math-

ematical vocabulary. Rather, they are descriptions of the situations whose structures children need to understand (MP.7). Any useful names can be discussed, including those elicited from children. Choosing terms that were easier for children also meant that the CGI terms Compare Quantity Unknown and Referent Quantity Unknown were not optimal because they concern problem difficulty (a major focus of the research being done at the time) rather than problem representation and solution. The *Standards* categorization was intended to help teachers understand children's problem representation and solution, so the focus on Bigger Unknown and Smaller Unknown was more appropriate.

The distinctions between the types are often easier for children than for teachers to make, especially between the Add To/Take From and Put Together/Take Apart types because children in the younger grades tend to pay more attention to the situation while teachers tend to be more abstract, just thinking of addition or subtraction. For example, children understand commutativity earlier for Put Together/Take Apart problems than for Add To/Take From problems because the roles of the addends are not so different.

Number and Operations in Base Ten, K–5

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding

in computations, decomposing 1 ten into 10 ones and composing a
ten from 10 ones.

The power of the base–ten system is in repeated bundling by
ten: 10 tens make a unit called a *hundred*. Repeating this process of
creating new units by bundling in groups of ten creates units called
thousand, *ten thousand*, *hundred thousand* . . . In learning about
decimals, children partition a one into 10 equal–sized smaller units,
each of which is a tenth. Each base–ten unit can be understood in
terms of any other base–ten unit. For example, one hundred can be
viewed as a tenth of a thousand, 10 tens, 100 ones, or 1,000 tenths.
Algorithms• for operations in base ten draw on such relationships
among the base–ten units.

Computations   Standard algorithms• for base-ten computations with
the four operations rely on decomposing numbers written in base–
ten notation into base–ten units. The properties of operations then
allow any multi–digit computation to be reduced to a collection of
single–digit computations. These single–digit computations some-
times require the composition or decomposition of a base–ten unit.

Beginning in Kindergarten, the requisite abilities develop grad-
ually over the grades. Experience with addition and subtraction
within 20 is a Grade 1 standard[1.OA.6] and fluency is a Grade 2
standard.[2.OA.2] Computations within 20 that "cross 10," such as $9+8$
or $13-6$, are especially relevant to NBT because they afford the
development of the Level 3 make-a-ten strategies for addition and
subtraction described in the OA Progression. From the NBT per-
spective, make–a–ten strategies are (implicitly) the first instances of
composing or decomposing a base–ten unit. Such strategies are a
foundation for understanding in Grade 1 that addition may require
composing a ten[1.NBT.4] and in Grade 2 that subtraction may involve
decomposing a ten.[2.NBT.7]

Strategies and algorithms   The Standards distinguish strategies•
from algorithms. Work with computation begins with use of strate-
gies and "efficient, accurate, and generalizable methods." (See Grade
1 critical areas 1 and 2, Grade 2 critical area 2; Grade 4 critical area
1.) For each operation, the culmination of this work is signaled in
the Standards by use of the term "standard algorithm."

Initially, students compute using objects or drawings and strate-
gies based on place value, properties of operations, and/or the re-
lationship between addition and subtraction (or multiplication and
division). They relate their strategies to written methods and ex-
plain the reasoning used (for addition within 100 in Grade 1; for
addition and subtraction within 1000 in Grade 2) or illustrate and
explain their calculations with equations, rectangular arrays, and/or
area models (for multiplication and division in Grade 4).

Students' initial experiences with computation also include de-
velopment, discussion, and use of "efficient, accurate, and general-

• From the Standards glossary:

> **Computation algorithm.**   A set of predefined
> steps applicable to a class of problems that gives
> the correct result in every case when the steps are
> carried out correctly. See also: computation strat-
> egy.

In mathematics, an algorithm is defined by its steps and not by the
way those steps are recorded in writing. This progression gives
examples of different recording methods and discusses their ad-
vantages and disadvantages.

• The Standards do not specify a particular standard algorithm for
each operation. This progression gives examples of algorithms
that could serve as the standard algorithm and discusses their
advantages and disadvantages.

[1.OA.6] Add and subtract within 20, demonstrating fluency for ad-
dition and subtraction within 10. Use strategies such as counting
on; making ten (e.g., $8+6=8+2+4=10+4=14$); decom-
posing a number leading to a ten (e.g., $13-4=13-3-1=$
$10-1=9$); using the relationship between addition and subtrac-
tion (e.g., knowing that $8+4=12$, one knows $12-8=4$); and
creating equivalent but easier or known sums (e.g., adding $6+7$
by creating the known equivalent $6+6+1=12+1=13$).

[2.OA.2] Fluently add and subtract within 20 using mental
strategies.[2] By end of Grade 2, know from memory all sums of
two one-digit numbers.

[2] See standard 1.OA.6 for a list of mental strategies.

[1.NBT.4] Add within 100, including adding a two-digit number and
a one-digit number, and adding a two-digit number and a multiple
of 10, using concrete models or drawings and strategies based
on place value, properties of operations, and/or the relationship
between addition and subtraction; relate the strategy to a writ-
ten method and explain the reasoning used. Understand that
in adding two-digit numbers, one adds tens and tens, ones and
ones; and sometimes it is necessary to compose a ten.

[2.NBT.7] Add and subtract within 1000, using concrete models or
drawings and strategies based on place value, properties of op-
erations, and/or the relationship between addition and subtrac-
tion; relate the strategy to a written method. Understand that
in adding or subtracting three-digit numbers, one adds or sub-
tracts hundreds and hundreds, tens and tens, ones and ones;
and sometimes it is necessary to compose or decompose tens or
hundreds.

• From the Standards glossary:

> **Computation strategy.**   Purposeful manipula-
> tions that may be chosen for specific problems,
> may not have a fixed order, and may be aimed
> at converting one problem into another. See also:
> computation algorithm.

Examples of computation strategies are given in this progression
and in the Operations and Algebraic Thinking Progression.

izable methods." So from the beginning, students see, discuss, and explain methods that can be generalized to all numbers represented in the base-ten system. Initially, they may use written methods that include extra helping steps to record the underlying reasoning. These helping step variations can be important initially for understanding. Over time, these methods can and should be abbreviated into shorter written methods compatible with fluent use of standard algorithms.
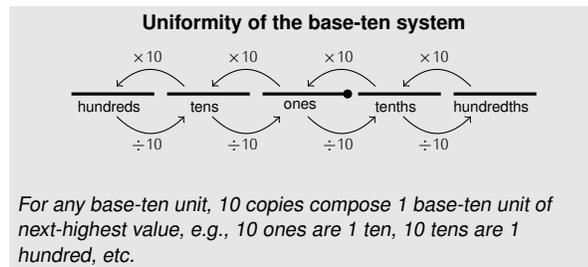
Students may also develop and discuss mental or written calculation methods that cannot be generalized to all numbers or are less efficient than other methods.

Mathematical practices   The Standards for Mathematical Practice are central in supporting students' progression from understanding and use of strategies to fluency with standard algorithms. The initial focus in the Standards on understanding and explaining such calculations, with the support of visual models, affords opportunities for students to see mathematical structure as accessible, important, interesting, and useful.

Students learn to see a number as composed of its base-ten units (MP.7). They learn to use this structure and the properties of operations to reduce computing a multi-digit sum, difference, product, or quotient to a collection of single-digit computations in different base-ten units. (In some cases, the Standards refer to "multi-digit" operations rather than specifying numbers of digits. The intent is that sufficiently many digits should be used to reveal the standard algorithm for each operation in all its generality.) Repeated reasoning (MP.8) that draws on the uniformity of the base-ten system is a part of this process. For example, in addition computations students generalize the strategy of making a ten to composing 1 base-ten unit of next-highest value from 10 like base-ten units.

Students abstract quantities in a situation (MP.2) and use objects, drawings, and diagrams (MP.4) to help conceptualize (MP.1), solve (MP.1, MP.3), and explain (MP.3) computational problems. They explain correspondences between different methods (MP.1) and construct and critique arguments about why those methods work (MP.3). Drawings, diagrams, and numerical recordings may raise questions related to precision (MP.6), e.g., does that 1 represent 1 one or 1 ten?, and to probe into the referents for symbols used (MP.2), e.g., does that 1 represent the number of apples in the problem?

Some methods may be advantageous in situations that require quick computation, but less so when uniformity is useful. Thus, comparing methods offers opportunities to raise the topic of using appropriate tools strategically (MP.5). Comparing methods can help to illustrate the advantages of standard algorithms: standard algorithms are general methods that minimize the number of steps needed and, once, fluency is achieved, do not require new reasoning.

Uniformity of the base-ten system

For any base-ten unit, 10 copies compose 1 base-ten unit of next-highest value, e.g., 10 ones are 1 ten, 10 tens are 1 hundred, etc.

Kindergarten

In Kindergarten, teachers help children lay the foundation for understanding the base-ten system by drawing special attention to 10. Children learn to view the whole numbers 11 through 19 as ten ones and some more ones. They decompose 10 into pairs such as $1 + 9$, $2+8$, $3+7$ and find the number that makes 10 when added to a given number such as 3 (see the OA Progression for further discussion).

Work with numbers from 11 to 19 to gain foundations for place value[K.NBT.1]   Children use objects, math drawings,• and equations to describe, explore, and explain how the "teen numbers," the counting numbers from 11 through 19, are ten ones and some more ones. Children can count out a given teen number of objects, e.g., 12, and group the objects to see the ten ones and the two ones. It is also helpful to structure the ten ones into patterns that can be seen as ten objects, such as two fives (see the OA Progression).

A difficulty in the English-speaking world is that the words for teen numbers do not make their base-ten meanings evident. For example, "eleven" and "twelve" do not sound like "ten and one" and "ten and two." The numbers "thirteen, fourteen, fifteen, …, nineteen" reverse the order of the ones and tens digits by saying the ones digit first. Also, "teen" must be interpreted as meaning "ten" and the prefixes "thir" and "fif" do not clearly say "three" and "five." In contrast, the corresponding East Asian number words are "ten one, ten two, ten three," and so on, fitting directly with the base-ten structure and drawing attention to the role of ten. Children could learn to say numbers in this East Asian way in addition to learning the standard English number names. Difficulties with number words beyond nineteen are discussed in the Grade 1 section. Children do count by tens in Kindergarten[K.CC.1] to develop their understanding of and fluency with the pattern of decade words so that they can build all two-digit counting words.

The numerals $11, 12, 13, \ldots, 19$ need special attention for children to understand them. The first nine numerals $1, 2, 3, \ldots, 9$, and 0 are essentially arbitrary marks. These same marks are used again to represent larger numbers. Children need to learn the differences in the ways these marks are used. For example, initially, a numeral such as 16 looks like "one, six," not "1 ten and 6 ones." Layered place value cards can help children see the 0 "hiding" under the ones place and that the 1 in the tens place really is 10 (ten ones). By working with teen numbers in this way, students gain a foundation for viewing 10 ones as a new unit called a ten in Grade 1.

The 0 in 10 uses the understanding of "0 as a count of no objects,"[K.CC.3] but also is the first use of 0 as a placeholder when a digit in a place means 0 units, here 0 ones. In Kindergarten, children mostly use the meaning of 10 as a counting number after 9 and before 11, but also gain foundational knowledge about 0 as placeholder, an understanding that will be extended in Grade 1.

[K.NBT.1]Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.
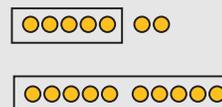
• Math drawings are simple drawings that make essential mathematical features and relationships salient while suppressing details that are not relevant to the mathematical ideas.

Number-bond diagram and equation

$$17 = 10 + 7$$
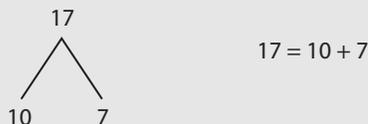
Decompositions of teen numbers can be recorded with diagrams or equations.

5- and 10-frames

Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column.

[K.CC.1]Count to 100 by ones and by tens.

Place value cards
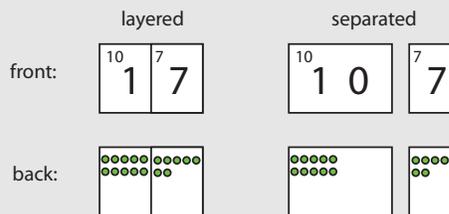
Children can use layered place value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings. When any of the number arrangements is turned over, the one card is hidden under the tens card. Children can see this and that they need to move the ones dots above and on the right side of the tens card.

Because their sense of left and right may not yet be well developed, children may need illustrations in the classroom that show the tens quantities on the left and ones quantities on the right to support correct visual building of the place value system. Working with the cards givens them experience in building it with their motor system as well. The cards can be made on one page: numbers on one side and quantities on the back. On each card, the tens quantity should be on the left side of the back.

[K.CC.3]Write numbers from 0 to 20. Represent a number of objects with a written numeral 0–20 (with 0 representing a count of no objects).

Grade 1

In first grade, students learn to view ten ones as a unit called a "ten." The ability to compose and decompose this unit flexibly and to view the numbers 11 to 19 as composed of one ten and some ones allows development of efficient, general base-ten methods for addition and subtraction. Students see a two-digit numeral as representing some tens and some ones, and they add and subtract using this understanding.

Extend the counting sequence and understand place value   Via structured learning time, discussion, and practice students learn patterns in spoken number words and in written numerals, and how the two are related.

Grade 1 students take the important step of viewing ten ones as a unit called a "ten."[1.NBT.2a] They learn to view the numbers 11 through 19 as composed of 1 ten and some ones.[1.NBT.2b] They learn to view the decade numbers $10, \ldots, 90$, in written and in spoken form, as 1 ten, …, 9 tens.[1.NBT.2c] More generally, first graders learn that the two digits of a two-digit number represent amounts of tens and ones, e.g., 67 represents 6 tens and 7 ones. Saying 67 as "6 tens, 7 ones" as well as "sixty-seven" can help students focus on the tens and ones structure of written numerals.

The number words continue to require attention at first grade because of their irregularities. The decade words, "twenty," "thirty," "forty," etc., must be understood as indicating 2 tens, 3 tens, 4 tens, etc. Many decade number words sound much like teen number words. For example, "fourteen" and "forty" sound very similar, as do "fifteen" and "fifty," and so on to "nineteen" and "ninety." As discussed in the Kindergarten section, the number words from 13 to 19 give the number of ones before the number of tens. From 20 to 100, the number words switch to agreement with written numerals by giving the number of tens first. Because the decade words do not clearly indicate they mean a number of tens ("-ty" does mean tens but not clearly so) and because the number words "eleven" and "twelve" do not cue students that they mean "1 ten and 1" and "1 ten and 2," children frequently make count errors such as "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve."

Grade 1 students use their base-ten work to help them recognize that the digit in the tens place is more important for determining the size of a two-digit number.[1.NBT.3] They use this understanding to compare two two-digit numbers, indicating the result with the symbols $>$, $=$, and $<$. Correctly placing the $<$ and $>$ symbols is a challenge for early learners. Accuracy can improve if students think of putting the wide part of the symbol next to the larger number.•

1.NBT.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

   a 10 can be thought of as a bundle of ten ones—called a "ten."

   b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

   c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

Part of a numeral list

91	101	111
92	102	112
93	103	113
94	104	114
95	105	115
96	106	116
97	107	117
98	108	118
99	109	119
100	110	120

In the classroom, a list of the numerals from 1 to 120 can be shown in columns of 10 to help highlight the base-ten structure, e.g., in the leftmost column, the 9s (indicating 9 tens) are lined up and the ones increase by 1 from 91 to 99. The numbers $101, \ldots, 120$ may be especially difficult for children to write because they want to write the counting number they hear (e.g., one hundred six is 1006). But each place of a written numeral must have exactly one digit in it. Omitting a digit or writing more than one digit in a place moves other digits to the left or right of their correct places. A digit can be 0, which can be thought of as using 0 as a placeholder.
  Layered place value cards can help students learn the place value meanings.

Place value cards
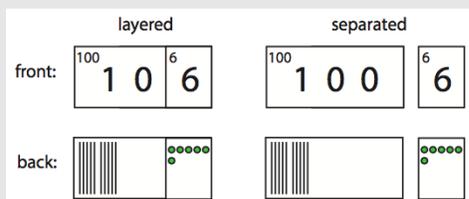
1.NBT.3 Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols $>$, $=$, and $<$.

• The widespread eating analogy (the alligator or big fish eats the little fish) is problematic because it is external to the symbols themselves and can be scary for some children, especially little ones. Explanations such as "the bigger part of the symbol is next to the bigger number" stay within the realm of mathematics.

Use place value understanding and properties of operations to add and subtract   First graders use their base-ten work to compute sums within 100 with understanding.[1.NBT.4] Concrete objects, cards, or drawings afford connections with written numerical work and discussions and explanations in terms of tens and ones.   In particular, showing composition of a ten with objects or drawings affords connection of the visual ten with the written numeral 1 that indicates 1 ten.

Combining tens and ones separately as illustrated in the margin can be extended to the general method of combining like base-ten units. The margin illustrates combining ones, then tens. Like base-ten units can be combined in any order, but going from smaller to larger eliminates the need to go back to a given place to add in a new unit. For example, in computing $46 + 37$ by combining tens, then ones (going left to right), one needs to go back to add in the new 1 ten: "4 tens and 3 tens is 7 tens, 6 ones and 7 ones is 13 ones which is 1 ten and 3 ones, 7 tens and 1 ten is 8 tens. The total is 8 tens and 3 ones: 83."

Students may also develop sequence methods that extend their Level 2 single-digit counting on strategies (see the OA Progression) to counting on by tens and ones, or mixtures of such strategies in which they add instead of count the tens or ones. Using objects or drawings that show the ones as rows of five plus extra ones (see the margin) can support students' extension of the Level 3 make-a-ten methods discussed in the OA Progression for single-digit numbers.

First graders also engage in mental calculation, such as mentally finding 10 more or 10 less than a given two-digit number without having to count by ones.[1.NBT.5] They may explain their reasoning by saying that they have one more or one less ten than before. Drawings and layered cards can afford connections with place value and be used in explanations.

In Grade 1, children learn to compute differences of two-digit numbers for limited cases.[1.NBT.6] Differences of multiples of 10, such as $70 - 40$ can be viewed as 7 tens minus 4 tens and represented with objects, e.g., objects bundled in tens, or drawings. Children use the relationship between subtraction and addition when they view $80 - 70$ as an unknown addend addition problem, $70 + \square = 80$, and reason that 1 ten must be added to 70 to make 80, so $80 - 70 = 10$.

First graders are not expected to compute differences of two-digit numbers other than multiples of ten. Deferring such work until Grade 2 allows two-digit subtraction with and without decomposing to occur in close succession, highlighting the similarity between these two cases. This helps students to avoid making the generalization "in each column, subtract the larger digit from the smaller digit, independent of whether the larger digit is in the subtrahend or minuend," e.g., making the error $82 - 45 = 43$.

[1.NBT.4]Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

Adding tens and ones separately

46
+37
————
8 3

combine ones
view $6 + 7$ as 1 ten and 3 ones

combine 4 tens and 3 tens with the newly composed ten (shown on the addition line)

This method is an application of the commutative and associative properties. The diagrams can help children with understanding and explaining the steps (MP.1). Advantages of writing the 1 below the addends are discussed in the Grade 2 margin (p. 59).

Counting on by tens

46
+37
————
8 3

56 66 76   77 78 79 80 81
82 83

starting from 46 count on 3 tens then count on 7 ones

Counting on by tens from 46, beginning 56, 66, 76, then counting on by ones. This method can be generalized, but the complexity of the counting on required and the lack of efficiency becomes apparent as the number of digits in the addends increases.

[1.NBT.5]Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

[1.NBT.6]Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Grade 2

At Grade 2, students extend their base-ten understanding to hundreds. They now add and subtract within 1000, with composing and decomposing, and they understand and explain the reasoning of the processes they use. They become fluent with addition and subtraction within 100.

Understand place value   In Grade 2, students extend their understanding of the base-ten system by viewing 10 tens as forming a new unit called a "hundred."[2.NBT.1a] This lays the groundwork for understanding the structure of the base-ten system as based in repeated bundling in groups of 10 and understanding that the unit associated with each place is 10 of the unit associated with the place to its right.

Representations such as manipulative materials, math drawings, and layered three-digit place value cards afford connections between written three-digit numbers and hundreds, tens, and ones. Number words and numbers written in base-ten numerals and as sums of their base-ten units can be connected with representations in drawings and place value cards, and by saying numbers aloud and in terms of their base-ten units, e.g., 456 is "Four hundred fifty six" and "four hundreds five tens six ones."[2.NBT.3] Unlayering place value cards (see pp. 55–56) reveals the expanded form of the number.

Unlike the decade words, the hundreds words explicitly indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may still only think of it as the number said after 499 or reached after 500 counts of 1.
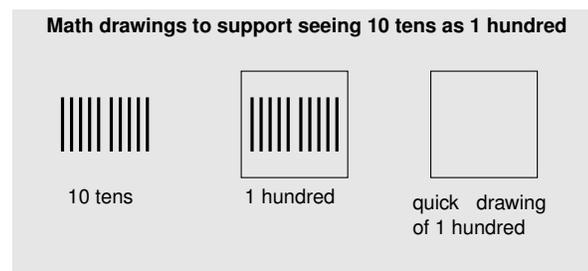
A major task for Grade 2 is learning the counting sequence from 100 to 1,000. As part of learning and using the base-ten structure, students count by ones within various parts of this sequence, especially the more difficult parts that "cross" tens or hundreds.

Building on their place value work, students continue to develop proficiency with mental computation.[2.NBT.8] They extend this to skip-counting by 5s, 10s, and 100s to emphasize and experience the tens and hundreds within the sequence and to prepare for multiplication.[2.NBT.2]•

Comparing magnitudes of two-digit numbers uses the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers uses the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number. For this reason, three-digit numbers are compared by first inspecting the hundreds place (e.g. $845 > 799$; $849 < 855$).[2.NBT.4] Drawings help support these understandings.

[2.NBT.1] Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

> a 100 can be thought of as a bundle of ten tens—called a "hundred."

Math drawings to support seeing 10 tens as 1 hundred

10 tens   1 hundred   quick drawing of 1 hundred

[2.NBT.3] Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

[2.NBT.8] Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

[2.NBT.2] Count within 1000; skip-count by 5s, 10s, and 100s.

• Because 2.NBT.2 is designed to prepare students for multiplication, there is no need to start skip-counting at numbers that are not multiples of 5.

[2.NBT.4] Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, $=$, and $<$ symbols to record the results of comparisons.

[2.NBT.5] Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

[2.NBT.6] Add up to four two-digit numbers using strategies based on place value and properties of operations.

Use place value understanding and properties of operations to add and subtract   Students fluently add and subtract within 100.[2.NBT.5] They also add and subtract within 1000.[2.NBT.7] They explain why addition and subtraction strategies work, using place value and the properties of operations, and may support their explanations with drawings or objects.[2.NBT.9] Because adding and subtracting within 100 is a special case of adding and subtracting within 1000, methods within 1000 will be discussed before fluency within 100.

Drawings can support students in understanding and explaining written methods. The drawing in the margin shows addends de–composed into their base–ten units (here, hundreds, tens, and ones). The quick drawings of the units show each hundred as a single unit rather than as ten tens (see illustration on p. 58), generalizing the approach that students used in Grade 1 of showing a ten as a single unit rather than as 10 separate ones. The putting together of like quick drawings illustrates adding like units as specified in 2.NBT.7: add ones to ones, tens to tens, and hundreds to hundreds. The draw–ing shows newly composed units within drawn boundaries. Steps of adding like units and composing new units shown in the draw–ing can be connected with corresponding steps in written methods. Connecting drawings with numerical calculations also facilitates dis–cussing how different written methods may show steps in different locations or different orders (MP.1 and MP.3). The associative and the commutative properties enable adding like units to occur.

Two written methods for addition within 1000 are shown in the margin. The first explicitly shows the hundreds, tens, and ones that are being added; this can be helpful conceptually to students. The second method is shorter and explicitly shows the adding of the single digits in each place and how this approach can continue on to places on the left. These methods can be related to drawings to show the place value meanings in each step (MP.1).

The first written method is a helping step variation that gener–alizes to all numbers in base ten but becomes impractical because of writing so many zeros. Students can move from this method to the second method (or another shorter method) by seeing how the steps of the two methods are related. Some students might make this transition in Grade 2, some in Grade 3, but all need to make it by Grade 4 where fluency requires a shorter method.

This first method can be seen as related to oral counting-on or written adding–on methods in which an addend is decomposed into hundreds, tens, and ones. These are successively added to the other addend, with the student saying or writing successive totals. These methods require keeping track of what parts of the decomposed addend have been added, and skills of mentally counting or adding hundreds, tens, and ones correctly. For example, beginning with hundreds: 278 plus 100 is 378 ("I've used all of the hundreds"), 378 plus 30 is 408 and plus 10 (to add on all of the 40) is 418, and 418 plus 7 is 425. One way to keep track: draw the 147 and cross out parts as they are added on. Counting-on and adding–on methods

2.NBT.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the re-lationship between addition and subtraction.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of op-erations, and/or the relationship between addition and subtrac-tion; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or sub-tracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

2.NBT.9 Explain why addition and subtraction strategies work, using place value and the properties of operations.[3]

[3] Explanations may be supported by drawings or objects.

Drawing: Combining like units and composing new units
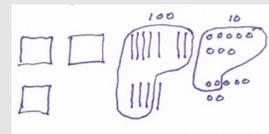
The student drawing shows the base-ten units of 278 and 147 in three wide columns. The units of 278 are shown above like units of 147. Boundaries around ten tens and ten ones indicate the newly composed hundred and the newly composed ten, which can then be drawn in the next-left columns.

Addition: Newly composed units recorded in separate rows

278	278	278	278
+ 1 4 7	+ 1 4 7	+ 1 4 7	+ 1 4 7
	3 0 0	3 0 0	3 0 0
		1 1 0	1 1 0
			1 5
			4 2 5

The computation shown proceeds from left to right, but could go right to left. Working from left to right has two advantages: Many students prefer it because they read from left to right; working first with the largest units yields a closer approximation earlier.

Addition: Newly composed units recorded in the same row

278	278	278	278
+ 1 4 7	+ 1 4 7	+ 1 4 7	+ 1 4 7
	1 5	1 2 5	4 2 5

	Add ones, 8 + 7. Record these 15 ones: 1 on the line in tens column; 5 below in ones place.	Add tens, 7 + 4 + 1. Record these 12 tens: 1 on the line in hundreds column; 2 below in tens place.	Add hundreds, 2 + 1 + 1. Record these 4 hundreds below in hundreds place.

Digits representing newly composed units are below the addends, on the line. This placement has several advantages. It is easier to write teen numbers in their usual order (e.g., 1, then 5) rather than "write 5 and carry 1" (5, then 1). Each two-digit partial sum (e.g., "15") is written with first digit near second, suggesting their origin (see the drawing). Students add pairs of original digits first, then the easy-to-add "1," avoiding the need to hold an altered digit in memory as when the 1 is written above the addends. The original digits are unchanged. The three multi-digit numbers (addends and total) can be seen clearly.

become even more difficult with numbers over 1000. If they arise from students, they should be discussed. But the major focus for addition within 1000 needs to be on methods such as those on page 59 that are simpler for students and lead toward fluency (e.g., recording new units in separate rows shown) or are sufficient for fluency (e.g., recording new units in one row).

Drawings and steps for a generalizable method of subtracting within 1000 are shown in the margin. The total 425 does not have enough tens or ones to subtract the 7 tens or 8 ones in 278. Therefore one hundred is decomposed to make ten tens and one ten is decomposed to make ten ones. These decompositions can be done and written in either order; starting from the left is shown because many students prefer to operate in that order. In the middle step, one hundred and one ten have been decomposed in two steps (making 3 hundreds, 11 tens, 15 ones) so that the 2 hundreds, 7 tens, and 8 ones in 278 can be subtracted. These subtractions of like units can also be done in any order. When students alternate decomposing and subtracting like units, they may forget to decompose entirely or in a given column after they have just subtracted (e.g., after subtracting 8 from 15 to get 7, they move left to the tens column and see a 1 on the top and a 7 on the bottom and write 6 because they are in subtraction mode, having just subtracted the ones).

Students can also subtract within 1000 by viewing a subtraction as an unknown addend problem, e.g., $278 + ? = 425$. Counting-on and adding-on methods such as those described above for addition can be used. But as with addition, the major focus needs to be on methods that lead toward fluency or are sufficient for fluency (e.g., recording in the same row as shown in the second example on p. 59).

In Grade 1, students have added within 100 using objects or drawings and used at least one method that is generalizable to larger numbers (such as between 101 and 1000). In Grade 2, they can make that generalization, using drawings for understanding and explanation as discussed above. This extension could be done first for two-digit numbers (e.g., $78 + 47$) so that students can see and discuss composing both ones and tens without the complexity of hundreds in the drawings or numbers (imagine the margin examples for $78 + 47$). After computing totals that compose both ones and tens for two-digit numbers, the type of problems required for fluency in Grade 2 (totals within 100) seem easy, e.g., $28 + 47$ requires only composing a new ten from ones. This is now easier to do without drawings: one just records the new ten before it is added to the other tens or adds it to them mentally.

A similar approach can be taken for subtraction: first using objects or drawings to solve subtractions within 100 that involve decomposing one ten, then rather quickly solving subtractions that require two decompositions. Spending a long time on subtraction within 100 can stimulate students to count on or count down, which, as discussed above, are methods that are considerably more difficult with numbers above 100. Problems with different types of decompo-

Subtraction: Decomposing where needed first

All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

sitions could be included so that students solve problems requiring two, one, and no decompositions. Then students can spend time on subtractions that include multiple hundreds (totals from 201 to 1000). Relative to these experiences, the objectives for fluency at this grade are easy: focusing within 100 just on the two cases of one decomposition (e.g., $72 - 28$) or no decomposition (e.g., $78 - 32$) without drawings. Having many experiences decomposing smaller top digits to get enough to subtract can help children maintain accuracy with cases such as $72 - 38$ where they might thoughtlessly subtract 2 from 8 to get 46 instead of decomposing 72 as 60 and 12, then subtracting to get 34.

Students also add up to four two-digit numbers using strategies based on place value and properties of operations.[2.NBT.6] This work affords opportunities for students to see that they may have to compose more than one ten, and as many as three new tens. It is also an opportunity for students to reinforce what they have learned by informally using the commutative and associative properties. They could mentally add all of the ones, then write the new tens in the tens column, and finish the computation in writing. They could successively add each addend or add the first two and last two addends and then add these totals. Carefully chosen problems could suggest strategies that depend on specific numbers. For example, $38 + 47 + 93 + 62$ can be easily added by adding the first and last numbers to make 100, adding the middle two numbers to make 140, and increasing 140 by 100 to make 240. Students also can develop special strategies for particularly easy computations such as $398 + 529$, where the 529 gives 2 to the 398 to make 400, leaving 400 plus 527 is 927. But the major focus in Grade 2 needs to remain on the methods that work for all numbers and generalize readily to numbers beyond 1000.

2.NBT.6 Add up to four two-digit numbers using strategies based on place value and properties of operations.

Grade 3

At Grade 3, the major focus is multiplication (see the Operations and Algebraic Thinking Progression), so students' work with addition and subtraction is limited to maintenance of fluency within 1000 for some students and building fluency to within 1000 for others.

Use place value understanding and properties of operations to perform multi–digit arithmetic   Students fluently add and subtract within 1000 using methods based on place value, properties of operations, and/or the relationship of addition and subtraction.[3.NBT.2] They focus on methods that generalize readily to larger numbers so that these methods can be extended to 1,000,000 in Grade 4 and fluency can be reached with such larger numbers. Fluency within 1000 implies that students use written methods without objects or drawings, though objects or drawings can be used with explanations to overcome errors and to continue to build understanding as needed.

   Students use their place value understanding to round numbers to the nearest 10 or 100.[3.NBT.1] They need to understand that when moving to the right across the places in a number (e.g., 456), the digits represent smaller units. When rounding to the nearest 10 or 100, the goal is to approximate the number by the closest number with no ones or no tens and ones (e.g., so 456 to the nearest ten is 460; and to the nearest hundred is 500). Rounding to the unit represented by the leftmost place is typically the sort of estimate that is easiest for students and often is sufficient for practical purposes.[•] Rounding to the unit represented by a place in the middle of a number may be more difficult for students (the surrounding digits are sometimes distracting). Rounding two numbers before computing can take as long as just computing their sum or difference.

   The special role of 10 in the base–ten system is important in understanding multiplication of one–digit numbers with multiples of 10.[3.NBT.3] For example, the product 3×50 can be represented as 3 groups of 5 tens, which is 15 tens, which is 150. This reasoning relies on the associative property of multiplication:[•]

$$3 \times 50 = 3 \times (5 \times 10) = (3 \times 5) \times 10 = 15 \times 10 = 150.$$

It is an example of how to explain an instance of a calculation pattern for these products: calculate the product of the non–zero digits, then shift the product one place to the left to make the result ten times as large.

3.NBT.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

3.NBT.1 Use place value understanding to round whole numbers to the nearest 10 or 100.

• The intent of this standard is that students round numbers in ways that are useful for problem solving. Therefore rounding three-digit numbers to hundreds and two-digit numbers to tens should be emphasized. Note that there are different methods of tie-breaking, e.g., 15 might be rounded to 20 or to 10, and that the Standards do not specify a method.

3.NBT.3 Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

• Use of parentheses, but not necessarily fluency with parentheses, is expected in Grade 5 (see 5.OA.1); however reading expressions with parentheses may begin earlier. See the Grade 3 section of the Operations and Algebraic Thinking Progression for further discussion.

Grade 4

At Grade 4, students extend their work in the base–ten system. They use standard algorithms to fluently add and subtract. They use methods based on place value and properties of operations supported by suitable representations to multiply and divide with multi-digit numbers.

Generalize place value understanding for multi–digit whole numbers   In the base–ten system, the value of each place is 10 times the value of the place to the immediate right.[4.NBT.1] Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left.

To read numerals between 1,000 and 1,000,000, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base–thousand unit (thousand, million, billion, trillion, etc.). Thus, 457,000 is read "four hundred fifty seven thousand."[4.NBT.2] The same methods students used for comparing and rounding numbers in Grade 3 apply to these numbers,[4.NBT.3] because of the uniformity of the base–ten system.•

Decimal notation and fractions   Students in Grade 4 work with fractions having denominators 10 and 100.[4.NF.5] Because it involves partitioning into 10 equal parts and treating the parts as numbers called one tenth and one hundredth, work with these fractions can be used as preparation to extend the base–ten system to non-whole numbers.

Using the unit fractions $\frac{1}{10}$ and $\frac{1}{100}$, non-whole numbers like $23\frac{7}{10}$ can be written in an expanded form that extends the form used with whole numbers: $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$. As with whole–number expansions in the base–ten system, each unit in this decomposition is ten times the unit to its right, reflecting the uniformity of the base–ten system. This can be connected with the use of base–ten notation to represent $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$ as 23.7. Using decimals allows students to apply familiar place value reasoning to fractional quantities.[4.NF.6] The Number and Operations—Fractions Progression discusses decimals to hundredths and comparison of decimals[4.NF.7] in more detail, as well as decimal–fraction conversion in the Standards.

The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "oneths" place to its right in order to create symmetry with respect to the decimal point.

each piece is

1
$1
dollar   dime   penny
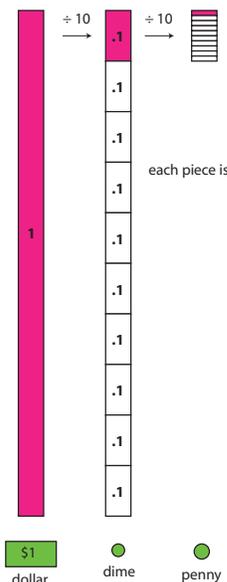
[4.NBT.1]Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.

10×30 **represented as** 3 **tens each taken** 10 **times**

Each of the 3 tens becomes a hundred and moves to the left. In the product, the 3 in the tens place of 30 is shifted one place to the left to represent 3 hundreds. In 300 divided by 10 the 3 is shifted one place to the right in the quotient to represent 3 tens.

[4.NBT.2]Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

[4.NBT.3]Use place value understanding to round multi-digit whole numbers to any place.

• The intent is that students round numbers in ways that are useful for problem solving. Rounding numbers to the leftmost one or two places can be useful. Rounding to other places need not be practiced.

[4.NF.5]Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[4]

[4]Students who can generate equivalent fractions can develop strategies for adding fractions with unlike denominators in general. But addition and subtraction with unlike denominators in general is not a requirement at this grade.

[4.NF.6]Use decimal notation for fractions with denominators 10 or 100.

[4.NF.7]Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

However, because one is the basic unit from which the other base-ten units are derived, the symmetry occurs instead with respect to the ones place, as illustrated in the margin.

Ways of reading decimals aloud vary. Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five." (Decimals smaller than one may be written with or without a zero before the decimal point.) Decimals with many non-zero digits are more easily read aloud in this manner. (For example, the number π, which has infinitely many non-zero digits, begins 3.1415 . . .)

Other ways to read 0.15 aloud are "1 tenth and 5 hundredths" and "15 hundredths," just as 1,500 is sometimes read "15 hundred" or "1 thousand, 5 hundred." Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

Just as 15 is understood as 15 ones and as 1 ten and 5 ones in computations with whole numbers, 0.15 is viewed as 15 hundredths and as 1 tenth and 5 hundredths in computations with decimals.

It takes time to develop understanding and fluency with the different forms. Layered cards for decimals can help students understand how 2 tenths and 7 hundredths make 27 hundredths, as well as emphasizing symmetry with respect to the ones place. Place value cards can be layered with the whole-number or decimal places farthest from the ones place on the bottom (see illustration of the whole-number cards on p. 55). These places are then covered by each place toward the ones place: Tenths go on top of hundredths, and tens go on top of hundreds (for example, .2 goes on top of .07 to make .27, and 20 goes on top of 700 to make 720).

Use place value understanding and properties of operations to perform multi-digit arithmetic   Students fluently add and subtract multi-digit numbers through 1,000,000 using the standard algorithm.[4.NBT.4] Work with the larger numbers allows students to consolidate their understanding of the uniformity of the base-ten system (see p. 54): as with the first four places (ones, tens, hundreds, thousands) a digit in any place represents a value that is ten times the value that it represents in the place to its right. Because students in Grade 2 and Grade 3 have been using at least one method that readily generalizes to 1,000,000, this extension does not have to take a long time. Thus, students will have time for the major NBT focus for this grade: multiplication and division.

In fourth grade, students compute products of one-digit numbers and multi-digit numbers (up to four digits) and products of two two-digit numbers.[4.NBT.5] They divide multi-digit numbers (up to four digits) by one-digit numbers. As with addition and subtraction, students should use methods they understand and can explain. Visual representations such as area and array diagrams that students draw and connect to equations and other written numerical work are useful for this purpose, which is why 4.NBT.5 explicitly states that they

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Symmetry with respect to the ones place
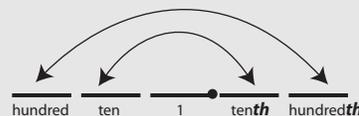
hundred   ten   1   ten*th*   hundred*th*

4.NBT.4 Fluently add and subtract multi-digit whole numbers using the standard algorithm.

4.NBT.5 Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Multiplication: Area model illustrating partial products

Each part of the region above corresponds to one of the terms in the computation below.

$$8 \times 549 \quad = \quad 8 \times (500 + 40 + 9)$$
$$= \quad 8 \times 500 + 8 \times 40 + 8 \times 9.$$

An area model can be used for any multiplication situation after students have discussed how to show an Equal Groups or a Compare situation with an area model by making the length of the rectangle represent the size of the equal groups or the larger compared quantity imagining things inside the square units to make an array (but not drawing them), and understanding that the dimensions of the rectangle are the same as the dimensions of the imagined array, e.g., an array illustrating 8×549 would have 8 rows and 549 columns. (See the Grade 3 section of the Operations and Algebraic Thinking Progression for discussion of Equal Groups and Compare situations.)

Multiplication: Recording methods

The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from $8 \times 9 = 72$ is written diagonally to the left of the 2 rather than above the 4 in 549. The colors indicate correspondences with the area model above.

are to be used to illustrate and explain the calculation. By reasoning repeatedly (MP.8) about the connection between diagrams and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

One component of understanding general methods for multiplication is understanding how to compute products of one-digit numbers and multiples of 10, 100, and 1000. This extends work in Grade 3 on products of one-digit numbers and multiples of 10. We can calculate 6×700 by calculating 6×7 and then shifting the result to the left two places (by placing two zeros at the end to show that these are hundreds) because 6 groups of 7 hundred is 6×7 hundreds, which is 42 hundreds, or 4,200. Students can use this place value reasoning, which can also be supported with diagrams of arrays or areas, as they develop and practice using the patterns in relationships among products such as 6×7, 6×70, 6×700, and 6×7000. Products of 5 and even numbers, such as 5×4, 5×40, 5×400, 5×4000 and 4×5, 4×50, 4×500, 4×5000 might be discussed and practiced separately afterwards because they may seem at first to violate the patterns by having an "extra" 0 that comes from the one-digit product.

Another part of understanding general base-ten methods for multi-digit multiplication is understanding the role played by the distributive property. This allows numbers to be decomposed into base-ten units, products of the units to be computed, then combined. By decomposing the factors into base-ten units and applying the distributive property, multiplication computations are reduced to single-digit multiplications and products of numbers with multiples of 10, of 100, and of 1000. Students can connect diagrams of areas or arrays to numerical work to develop understanding of general base-ten multiplication methods.

Computing products of two two-digit numbers requires using the distributive property several times when the factors are decomposed into base-ten units. For example,

$$
\begin{aligned}
36 \times 94 &= (30 + 6) \times (94) \\
&= 30 \times 94 + 6 \times 94 \\
&= 30 \times (90 + 4) + 6 \times (90 + 4) \\
&= 30 \times 90 + 30 \times 4 + 6 \times 90 + 6 \times 4.
\end{aligned}
$$

The four products in the last line correspond to the four rectangles in the area model in the margin. Their factors correspond to the factors in written methods. When written methods are abbreviated, some students have trouble seeing how the single-digit factors are related to the two-digit numbers whose product is being computed (MP.2). They may find it helpful initially to write each two-digit number as the sum of its base-ten units (e.g., writing next to the calculation $94 = 90 + 4$ and $36 = 30 + 6$) so that they see what the single digits are. Some students also initially find it helpful to write what they are multiplying in front of the partial products

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Illustrating partial products with an area model

The products of base-ten units are shown as parts of a rectangular region. Such area models can support understanding and explaining of different ways to record multiplication. For students who struggle with the spatial demands of other methods, a useful helping step method is to make a quick sketch like this with the lengths labeled and just the partial products, then to add the partial products outside the rectangle.

Methods that compute partial products first
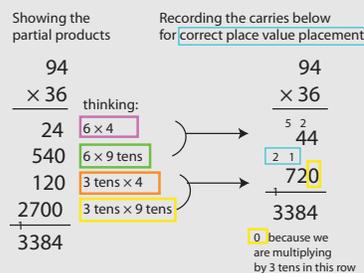
These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94. The digits 2 and 1 are surrounded by a blue box. The 1 from $30 \times 4 = 120$ is placed correctly in the hundreds place and the digit 2 from $30 \times 90 = 2700$ is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 (surrounded by a yellow box) in the ones place of the second row of the method on the right is there because the whole row of digits is produced by multiplying by 30 (not 3). Colors on the left correspond with the area model above.

Methods that alternate multiplying and adding
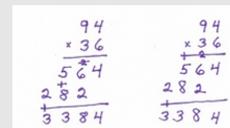
These methods put the newly composed units from a partial product in the correct column, then they are added to the next partial product. These alternating methods are more difficult than the methods above that show the four partial products. The first method can be used in Grade 5 division when multiplying a partial quotient times a two-digit divisor.

Not shown is the recording method in which the newly composed units are written above the top factor (e.g., 94). This puts the hundreds digit of the tens times ones product in the tens column (e.g., the 1 hundred in 120 from 30×4 above the 9 tens in 94). This placement violates the convention that students have learned: a digit in the tens place represents tens, not hundreds.

(e.g., $6 \times 4 = 24$). These helping steps can be dropped when they are no longer needed. At any point before or after their acquisition of fluency, some students may prefer to multiply from the left because they find it easier to align the subsequent products under this biggest product.

In Grade 4, multi-digit division requires working with remainders. In preparation for working with remainders, students can compute sums of a product and a number, such as $4 \times 8 + 3$. In multi-digit division, students will need to find the greatest multiple less than a given number. For example, when dividing by 6, the greatest multiple of 6 less than 50 is $6 \times 8 = 48$. Students can think of these "greatest multiples" in terms of putting objects into groups. For example, when 50 objects are shared among 6 groups, the largest whole number of objects that can be put in each group is 8, and 2 objects are left over. (Or when 50 objects are allocated into groups of 6, the largest whole number of groups that can be made is 8, and 2 objects are left over.) The equation $6 \times 8 + 2 = 50$ (or $8 \times 6 + 2 = 50$) corresponds with this situation. As illustrated in the margin, this can be viewed as allocating objects (the divisor is the number of groups or the number of objects in each group) or as finding the side length of a rectangle (the divisor is the length of the other side).

General methods for computing quotients of multi-digit numbers and one-digit numbers rely on the same understandings as for multiplication, but cast in terms of division.[4.NBT.6] One component is quotients of multiples of 10, 100, or 1000 and one-digit numbers. For example, $42 \div 6$ is related to $420 \div 6$ and $4200 \div 6$. Students can draw on their work with multiplication and they can also reason that $4200 \div 6$ means partitioning 42 hundreds into 6 equal groups, so there are 7 hundreds in each group.

Another component of understanding general methods for multi-digit division computation is the idea of decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units (see p. 67). As with multiplication, this relies on the distributive property.

Cases involving 0 in division may require special attention.

Cases involving 0 in division

Array and area situations

3 rows,
6 in each row.

6 columns,
3 in each column.

In Grade 3, students begin to connect array and area situations (see the Operations and Algebraic Thinking Progression, p. 33). In Grade 4, they extend these connections to think flexibly about an area diagram for multi-digit division like that on page 67. Students can think of the quotient as the unknown number of objects in a row (number of objects in each group), the unknown number of columns (number of groups), or the unknown side length.

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

A note on notation

The result of division within the system of whole numbers is frequently written as:

$$84 \div 10 = 8\,R\,4 \quad \text{and} \quad 44 \div 5 = 8\,R\,4.$$

Because the two expressions on the right are the same, students should conclude that $84 \div 10$ is equal to $44 \div 5$, but this is not the case. (Because the equal sign is not used appropriately, this usage is a non-example of Standard for Mathematical Practice 6.) Moreover, the notation $8\,R\,4$ does not indicate a number.

Rather than writing the result of division solely in terms of a whole-number quotient and remainder, the relationship of whole-number quotient and remainder can be written as:

$$84 = 8 \times 10 + 4 \quad \text{and} \quad 44 = 8 \times 5 + 4.$$

In Grade 5, students can begin to use fraction or decimal notation to express the result of division, e.g., $84 \div 10 = 8\frac{4}{10}$. See the Number and Operations—Fractions Progression.

Division as finding an unknown side length with two written methods

? hundreds + ? tens + ? ones

7	966
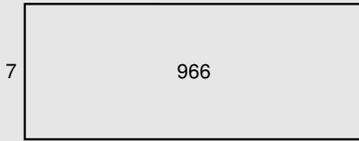
Find the unknown length of the rectangle; first find the hundreds, then the tens, then the ones.

Note that color correspondence differs from that on pp. 64–65.
As on pp. 64–65, each partial product has a different color.
Unlike pp. 64–65, one factor (the partial quotient) of each partial product is also in color.

Method A	**Method B**
Method A shows each partial quotient.	*Method B abbreviates partial quotients.*

100   +   ?

7	700	

The length has 1 hundred, making a rectangle with area 700.

$$
\begin{array}{r}
100 \\
7{\overline{\smash{)}966}} \\
-700 \\
\hline
266
\end{array}
$$

$$
\begin{array}{r}
1 \\
7{\overline{\smash{)}966}} \\
-7 \\
\hline
26
\end{array}
$$

Method A records the difference of the areas as $966 - 700 = 266$, showing the remaining area (266). Only hundreds are subtracted; the tens and ones digits do not change.

Method B records only the hundreds digit (2) of the difference and "brings down" the unchanged tens digit (6). These digits represent: 2 hundreds + 6 tens = 26 tens.

100   +   30 + ?

7	700	210	

The length has 3 tens, making a rectangle with area 210.

$$
\begin{array}{r}
30 \\
100 \\
7{\overline{\smash{)}966}} \\
-700 \\
\hline
266 \\
-210 \\
\hline
56
\end{array}
$$

$$
\begin{array}{r}
13 \\
7{\overline{\smash{)}966}} \\
-7 \\
\hline
26 \\
-21 \\
\hline
56
\end{array}
$$

Method A records the difference of the areas as $266 - 210 = 56$. Only hundreds and tens are subtracted; the ones digit does not change.

Method B records only the tens digit (5) of the difference and "brings down" the ones digit (6). These digits represent: 5 tens + 6 ones = 56 ones.

100   +   30 + 8

7	700	210	56

The length has 8 ones, making an area of 56. The original rectangle can now be seen as composed of three smaller rectangles with areas of the amounts that were subtracted from 966.

$966 \div 7$ *can be viewed as finding the unknown side length of a rectangular region with area 966 square units and a side of length 7 units. The divisor, partial quotients (100, 30, 8), and final quotient (138) represent quantities in length units and the dividend represents a quantity in area units.*

$$
\begin{array}{r}
\left.\begin{array}{r} 8 \\ 30 \\ 100 \end{array}\right\} 138 \\
7{\overline{\smash{)}966}} \\
-700 \\
\hline
266 \\
-210 \\
\hline
56 \\
-56 \\
\hline
0
\end{array}
$$

Method A's final step is adding partial quotients.

$$
\begin{array}{r}
138 \\
7{\overline{\smash{)}966}} \\
-7 \\
\hline
26 \\
-21 \\
\hline
56 \\
-56 \\
\hline
0
\end{array}
$$

Method B already shows the unknown side length. Place values of digits that represent partial quotients can be said explicitly when explaining Method B, e.g., "7 hundreds subtracted from the 9 hundreds is 2 hundreds."

The relationships shown in the diagram and the two written methods can be summarized by equations.
$$966 = 700 + 210 + 56 = 7 \times 100 + 7 \times 30 + 7 \times 8$$
$$= 7 \times (100 + 30 + 8) = 7 \times 138$$

Grade 5

In Grade 5, students extend their understanding of the base-ten system to decimals to the thousandths place, building on their Grade 4 work with tenths and hundredths. They become fluent with the standard multiplication algorithm with multi-digit whole numbers. They reason about dividing whole numbers with two-digit divisors, and reason about adding, subtracting, multiplying, and dividing decimals to hundredths.

Understand the place value system   Students extend their understanding of the base-ten system to the relationship between adjacent places, how numbers compare,• and how numbers round for decimals to thousandths.
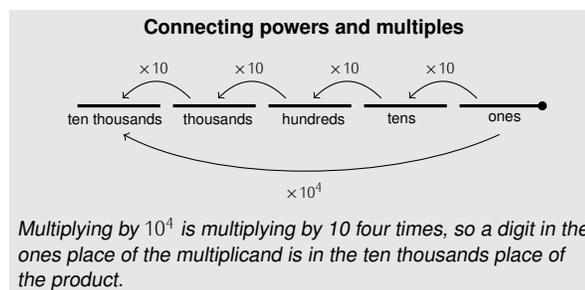
New at Grade 5 is the use of whole-number exponents to denote powers of 10. Students understand why multiplying by a power of 10 shifts the digits of a whole number or decimal that many places to the left. For example, multiplying by 10^4 is multiplying by 10 four times. Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times larger) because in the base-ten system the value of each place is 10 times the value of the place to its right.[5.NBT.1] So multiplying by 10 four times shifts every digit 4 places to the left. Patterns in the number of 0s in products of a whole number and a power of 10 and the location of the decimal point in products of decimals with powers of 10 can be explained in terms of place value.[5.NBT.2] Because students have developed their understandings of and computations with decimals in terms of multiples (consistent with 4.OA.4) rather than powers, connecting the terminology of multiples with that of powers affords connections between understanding of multiplication and exponentiation.

Perform operations with multi-digit whole numbers and with decimals to hundredths   At Grade 5, students fluently compute products of whole numbers using the standard algorithm.[5.NBT.5] Underlying this algorithm are the properties of operations and the base-ten system (see the Grade 4 section).

Division in Grade 5 extends Grade 4 methods to two-digit divisors.[5.NBT.6]• Students continue to decompose the dividend into base-ten units and find the quotient place by place, starting from the highest place. They illustrate and explain their calculations using equations, rectangular arrays, and/or area models. Estimating the quotients is a new aspect of dividing by a two-digit number. Even if students round the dividend appropriately, the resulting estimate may need to be adjusted up or down. Sometimes multiplying the ones of a two-digit divisor composes a new thousand, hundred, or ten. These newly composed units can be written as part of the division computation, added mentally, or as part of a separate mul-

• This extends previous work with fractions. See the Grade 4 section of Number and Operations—Fractions Progression.

5.NBT.1 Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.

Connecting powers and multiples

Multiplying by 10^4 is multiplying by 10 four times, so a digit in the ones place of the multiplicand is in the ten thousands place of the product.

5.NBT.2 Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

5.NBT.5 Fluently multiply multi-digit whole numbers using the standard algorithm.

5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

• After they interpret a fraction as division of the numerator by the denominator (5.NF.3), students begin using fraction or decimal notation to express the results of division.

tiplication computation. Students who need to write decomposed units when subtracting need to remember to leave space to do so.

Because of the uniformity of the structure of the base–ten system, students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers.[5.NBT.7] Like base–ten units must be added and subtracted, so students need to attend to aligning the corresponding places correctly (this also aligns the decimal points). It can help to put 0s in places so that all numbers show the same number of places to the right of the decimal point. A whole number is not usually written with a decimal point, but a decimal point followed by one or more 0s can be inserted on the right (e.g., 16 can also be written as 16.0 or 16.00). The process of composing and decomposing a base–ten unit is the same for decimals as for whole numbers and the same methods of recording numerical work can be used with decimals as with whole numbers. For example, students can write digits representing newly composed units on the addition line, and they can decompose units wherever needed before subtracting.

General methods used for computing products of whole numbers extend to products of decimals. Because the expectations for decimals are limited to thousandths and expectations for factors are limited to hundredths at this grade level, students will multiply tenths with tenths and tenths with hundredths, but they need not multiply hundredths with hundredths. Before students consider decimal multiplication more generally, they can study the effect of multiplying by 0.1 and by 0.01 to explain why the product is ten or a hundred times as small as the multiplicand (moves one or two places to the right). They can then extend their reasoning to multipliers that are single–digit multiples of 0.1 and 0.01 (e.g., 0.2 and 0.02, etc.).

There are several lines of reasoning that students can use to explain the placement of the decimal point in other products of decimals. Students can think about the product of the smallest base–ten units of each factor. For example, a tenth times a tenth is a hundredth, so 3.2×7.1 will have an entry in the hundredth place. Note, however, that students might place the decimal point incorrectly for 3.2×8.5 unless they take into account the 0 in the ones place of 32×85. (Or they can think of 0.2×0.5 as 10 hundredths.) They can also think of the decimals as fractions or as whole numbers divided by 10 or 100.[5.NF.3] When they place the decimal point in the product, they have to divide by a 10 from each factor or 100 from one factor. For example, to see that $0.6 \times 0.8 = 0.48$, students can use fractions: $\frac{6}{10} \times \frac{8}{10} = \frac{48}{100}$.[5.NF.4] Students can also reason that when they carry out the multiplication without the decimal point, they have multiplied each decimal factor by 10 or 100, so they will need to divide by those numbers in the end to get the correct answer. Also, students can use reasoning about the sizes of numbers to determine the placement of the decimal point. For example, 3.2×8.5 should be close to 3×9, so 27.2 is a more reasonable product for 3.2×8.5 than 2.72 or 272. This estimation–based method is not reliable in

$$
\begin{array}{r}
1 \left.\right\} \\
10 \left.\right\} 61 \\
_{(30)}\ \ 50 \left.\right\} \\
27\overline{)1655} \\
-1350 \\
\overline{305} \\
-270 \\
\overline{35} \\
-27 \\
\overline{8}
\end{array}
$$

Computing $1655 \div 27$*: Rounding 27 to 30 produces the underestimate 50 at the first step, but this method of writing partial quotients allows the division process to be continued.*

5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

all cases, however, especially in cases students will encounter in later grades. For example, it is not easy to decide where to place the decimal point in 0.023×0.0045 based on estimation. Students can summarize the results of their reasoning such as those above as specific numerical patterns and then as one general overall pattern such as "the number of decimal places in the product is the sum of the number of decimal places in each factor."

General methods used for computing quotients of whole numbers extend to decimals with the additional issue of placing the decimal point in the quotient. As with decimal multiplication, students can first examine the cases of dividing by 0.1 and 0.01 to see that the quotient becomes 10 times or 100 times as large as the dividend (see also the Number and Operations—Fractions Progression). For example, students can view $7 \div 0.1 = \square$ as asking how many tenths are in 7.[5.NF.7b] Because it takes 10 tenths to make 1, it takes 7 times as many tenths to make 7, so $7 \div 0.1 = 7 \times 10 = 70$. Or students could note that 7 is 70 tenths, so asking how many tenths are in 7 is the same as asking how many tenths are in 70 tenths, which is 70. In other words, $7 \div 0.1$ is the same as $70 \div 1$. So dividing by 0.1 moves the number 7 one place to the left, the quotient is ten times as big as the dividend. As with decimal multiplication, students can then proceed to more general cases. For example, to calculate $7 \div 0.2$, students can reason that 0.2 is 2 tenths and 7 is 70 tenths, so asking how many 2 tenths are in 7 is the same as asking how many 2 tenths are in 70 tenths. In other words, $7 \div 0.2$ is the same as $70 \div 2$; multiplying both the 7 and the 0.2 by 10 results in the same quotient. Or students could calculate $7 \div 0.2$ by viewing 0.2 as 2×0.1, so they can first divide 7 by 2, which is 3.5, and then divide that result by 0.1, which makes 3.5 ten times as large, namely 35. Dividing by a decimal less than 1 results in a quotient larger than the dividend[5.NF.5] and moves the digits of the dividend one place to the left. Students can summarize the results of their reasoning as specific numerical patterns, then as one general overall pattern such as "when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places."

5.NF.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

   b Interpret division of a whole number by a unit fraction, and compute such quotients.

5.NF.5 Interpret multiplication as scaling (resizing), by:

   a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

   b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Where this progression is heading

At Grade 6, students extend their fluency with the standard algorithms, using these for all four operations with decimals and to compute quotients of multi-digit numbers. At Grade 6 and beyond, students may occasionally compute with numbers larger than those specified in earlier grades as required for solving problems, but the Standards do not specify mastery with such numbers.

In Grade 6, students extend the base-ten system to negative numbers. In Grade 7, they begin to do arithmetic with such numbers. See the Number System Progression.

By reasoning about the standard division algorithm, students learn in Grade 7 that every fraction can be represented with a decimal that either terminates or repeats. In Grade 8, students learn informally that every number has a decimal expansion, and that those with a terminating or repeating decimal representation are rational numbers (i.e., can be represented as a quotient of integers). There are numbers that are not rational (irrational numbers), such as the square root of 2. (It is not obvious that the square root of 2 is not rational, but this can be proved.) In fact, surprisingly, it turns out that most numbers are not rational. Irrational numbers can always be approximated by rational numbers.

In Grade 8, students build on their work with rounding and exponents when they begin working with scientific notation. This allows them to express approximations of very large and very small numbers compactly by using exponents and generally only approximately by showing only the most significant digits. For example, the Earth's circumference is approximately 40,000,000 m. In scientific notation, this is 4×10^7 m.

The Common Core Standards are designed so that ideas used in base-ten computation, as well as in other domains, can support later learning. For example, use of the distributive property occurs together with the idea of combining like units in the NBT and NF standards. In their work with expressions, students use these ideas again when they collect like terms, e.g., $5b + 3b = (5 + 3)b = 8b$ in Grade 6 (see the Expressions and Equations Progression). High school calculations with polynomials and complex numbers also draw on these ideas (see the Algebra Progression and the Number Progression).

The distributive property and like units: Multiplication of whole numbers and polynomials

52×73		$(5x + 2)(7x + 3)$	
$=$	$(5 \times 10 + 2) \times (7 \times 10 + 3)$	$= \quad (5x + 2)(7x + 3)$	decomposing as like units (powers of 10 or of x)
$=$	$(5 \times 10) \times (7 \times 10 + 3) + 2 \times (7 \times 10 + 3)$	$= \quad 5x(7x + 3) + 2(7x + 3)$	using the distributive property
$=$	$35 \times 10^2 + 15 \times 10 + 14 \times 10 + 2 \times 3$	$= \quad 35x^2 + 15x + 14x + 2 \cdot 3$	using the distributive property again
$=$	$35 \times 10^2 + 29 \times 10 + 6$	$= \quad 35x^2 + 29x + 6$	combining like units (powers of 10 or powers of x)

Measurement and Data, K–5

Overview

As students work with data in Grades K–5, they build foundations for their study of statistics and probability in Grades 6 and beyond, and they strengthen and apply what they are learning in arithmetic. Kindergarten work with data uses counting and order relations. First and second graders solve addition and subtraction problems in a data context. In Grades 3–5, work with data is closely related to the number line, fraction concepts,• fraction arithmetic, and solving problems that involve the four operations. The end of this overview lists these and other notable connections between data work and arithmetic in Grades K–5.

Categorical data The K–5 data standards run along two paths. One path deals with *categorical data* and focuses on *bar graphs* as a way to represent and analyze such data. Categorical data come from sorting objects into categories—for example, sorting a jumble of alphabet blocks to form two stacks, a stack for vowels and a stack for consonants. In this case there are two categories (Vowels and Consonants). The Standards follow the *Guidelines for Assessment and Instruction in Statistics Education Report* in reserving the term "categorical data" for non-numerical categories.•

Students' work with categorical data in early grades will support their later work with bivariate categorical data and two-way tables in eighth grade (this is discussed further at the end of the Categorical Data Progression).

Measurement data The other path deals with *measurement data*. As the name suggests, measurement data comes from taking measurements. For example, if every child in a class measures the length of his or her hand to the nearest centimeter, then a set of measurement data is obtained. Other ways to generate measurement data might include measuring liquid volumes with graduated cylinders or measuring room temperatures with a thermometer. In each case,

• In the Standards, the word "fraction" is used to refer to a type of number, which may be written in the form numerator over denominator ("in fraction notation" or "as a fraction" in conventional terminology), or in decimal notation ("as a decimal"), or—if it is greater than 1—in the form whole number followed by a number less than 1 written as a fraction ("as a mixed number").

• The *Guidelines for Assessment and Instruction in Statistics Education Report* was published in 2007 by the American Statistical Association, http://www.amstat.org/education/gaise.

the Standards call for students to represent measurement data with a *line plot*. This is a type of display that positions the data along the appropriate scale, drawn as a number line diagram. These plots have two names in common use, "dot plot" (because each observation is represented as a dot) and "line plot" (because each observation is represented above a number line diagram).

The number line diagram in a line plot corresponds to the scale on the measurement tool used to generate the data. In a context involving measurement of liquid volumes, the scale on a line plot could correspond to the scale etched on a graduated cylinder. In a context involving measurement of temperature, one might imagine a picture in which the scale on the line plot corresponds to the scale printed on a thermometer. In the last two cases, the correspondence may be more obvious when the scale on the line plot is drawn vertically.

Students should understand that the numbers on the scale of a line plot indicate the total number of measurement units from the zero of the scale. (For discussion of the conceptual and procedural issues involved, see the Grade 2 section of the Geometric Measurement Progression.)

Students need to choose appropriate representations (MP.5), labeling axes to clarify the correspondence with the quantities in the situation and specifying units of measurement (MP.6). Measuring and recording data require attention to precision (MP.6). Students should be supported as they learn to construct picture graphs, bar graphs, and line plots. Grid paper should be used for assignments as well as assessments. This may help to minimize errors arising from the need to track across a graph visually to identify values. Also, a template can be superimposed on the grid paper, with blanks provided for the student to write in the graph title, scale labels, category labels, legend, and so on. It might also help if students write relevant numbers on graphs during problem solving.

In students' work with data, context is important. As noted in the *Guidelines for Assessment and Instruction in Statistics Education Report*, "data are not just numbers, they are numbers with a context. In mathematics, context obscures structure. In data analysis, context provides meaning." In keeping with this perspective, students should work with data in the context of science, social science, health, and other subjects, always interpreting data plots in terms of the data they represent (MP.2).

Example of a line plot

Height of Saplings (inches)
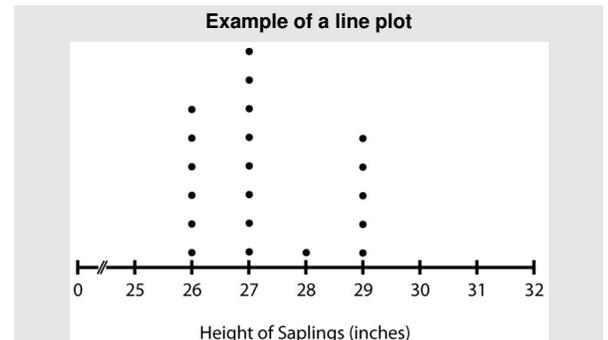
Note that the break in the scale between 0 and 25 indicates that marks between 0 and 25 are not shown.

Example of a scale on a measurement tool
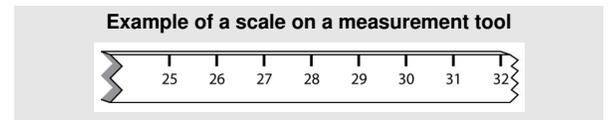
Standard	Notable connections
Categorical data, K–3	
K.MD.3. Classify objects into given categories, count the number of objects in each category and sort[1] the categories by count. *Limit category counts to be less than or equal to 10.*	• K.CC. Counting to tell the number of objects • K.CC. Comparing numbers
1.MD.4. Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.	• 1.OA. Problems involving addition and subtraction within 20 ○ Put Together, Take Apart, Compare[2] ○ problems that call for addition of three whole numbers
2.MD.10. Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.	• 2.OA. Problems involving addition and subtraction within 100 ○ Put Together, Take Apart, Compare[2]
3.MD.3. Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems[2] using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent 5 pets.*	• 3.OA.3. Problems involving multiplication and division within 100 • 3.OA.8. Two-step problems using the four operations • 3.G.1. Categories of shapes
Measurement data, 2–5	
2.MD.9. Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.	• 1.MD.2. Length measurement • 2.MD.6. Number line
3.MD.4. Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.	• 3.NF.2. Fractions on a number line (Expectations limited to denominators 2, 3, 4, 6, and 8.)
4.MD.4. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*	• 4.NF.3,4. Problems involving fraction arithmetic (Expectations limited to denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.)
5.MD.2. Make a line plot to display a data set of measurements in fractions of a unit $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)$. Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*	• 5.NF.1,2,4,6,7. Problems involving fraction arithmetic (Division of a fraction by a fraction is not a requirement at this grade.)

[1] Here, "sort the categories" means "order the categories," i.e., show the categories in order according to their respective counts.

[2] For discussion and examples of these problem types, see the Overview of K–2 in the Operations and Algebraic Thinking Progression.

Categorical Data, K–3

Kindergarten

Students in Kindergarten classify objects into categories, initially specified by the teacher and perhaps eventually elicited from students. For example, in a science context, the teacher might ask students in the class to sort pictures of various organisms into two piles: organisms with wings and those without wings. Students can then count the number of specimens in each pile.[K.CC.5] Students can use these category counts and their understanding of cardinality to say whether there are more specimens with wings or without wings.[K.CC.6,K.CC.7]

A single group of specimens might be classified in different ways, depending on which attribute has been identified as the attribute of interest. For example, some specimens might be insects, while others are not insects. Some specimens might live on land, while others live in water.

[K.CC.5]Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.

[K.CC.6]Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.

[K.CC.7]Compare two numbers between 1 and 10 presented as written numerals.

Grade 1

Students in Grade 1 begin to organize and represent categorical data. For example, if a collection of specimens is sorted into two piles based on which specimens have wings and which do not, students might represent the two piles of specimens on a piece of paper, by making a group of marks for each pile, as shown below (the marks could also be circles, for example). The groups of marks should be clearly labeled to reflect the attribute in question.

The work shown in the figure is the result of an intricate process. At first, we have before us a jumble of specimens with many attributes. Then there is a narrowing of attention to a single attribute (wings or not). Then the objects might be arranged into piles. The arranging of objects into piles is then mirrored in the arranging of marks into groups. In the end, each mark represents an object; its position in one column or the other indicates whether or not that object has a given attribute.

There is no single correct way to represent categorical data—and the Standards do not require Grade 1 students to use any specific format. However, students should be familiar with mark schemes like the one shown in the figure. Another format that might be useful in Grade 1 is a picture graph in which one picture represents one object. (Note that picture graphs are not an expectation in the Standards until Grade 2.) If different students devise different ways to represent the same data set, then the class might discuss relative strengths and weaknesses of each scheme (MP.5).

Students' data work in Grade 1 has important connections to addition and subtraction, as noted in the overview. Students in Grade 1 can ask and answer questions about categorical data based on a representation of the data. For example, with reference to the figure above, a student might ask how many specimens there were altogether, representing this problem by writing an equation such as $7 + 8 = \square$. Students can also ask and answer questions leading to other kinds of addition and subtraction problems (1.OA), such as Compare problems[1.OA.1] or problems involving the addition of three numbers (for situations with three categories).[1.OA.2]
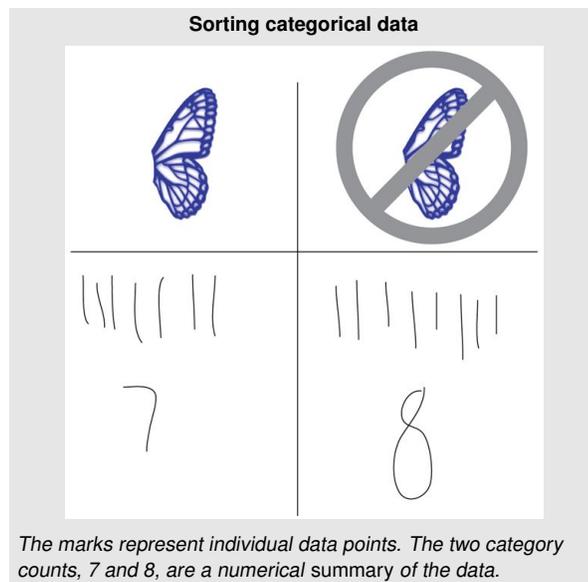
Sorting categorical data

The marks represent individual data points. The two category counts, 7 and 8, are a numerical summary of the data.

[1.OA.1] Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

[1.OA.2] Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Grade 2

Students in Grade 2 draw a picture graph and a bar graph (with single–unit scale) to represent a data set with up to four categories. They solve simple Put Together, Take Apart, and Compare problems using information presented in a bar graph.[2.MD.10, 2.OA.1]

The margin shows an activity in which students make a bar graph to represent categorical data, then solve addition and subtraction problems based on the data. Students might use scissors to cut out the pictures of each organism and then sort the organisms into piles by category. Category counts might be recorded efficiently in the form of a table.

A bar graph representing categorical data displays no additional information beyond the category counts. In such a graph, the bars are a way to make the category counts easy to interpret visually. Thus, the word problem in part 4 could be solved without drawing a bar graph, just by using the category counts. The problem could even be cast entirely in words, without the accompanying picture: "There are 9 insects, 4 spiders, 13 vertebrates, and 2 organisms of other kinds. How many more spiders would there have to be in order for the number of spiders to equal the number of vertebrates?" Of course, in solving this problem, students would not need to participate in categorizing data or representing it.

Scales in bar graphs   Consider the two bar graphs shown in the margin, in which the bars are oriented vertically. (Bars in a bar graph can also be oriented horizontally, in which case the following discussion would be modified in the obvious way.) Both of these bar graphs represent the same data set.

These examples illustrate that the horizontal axis in a bar graph of categorical data is not a scale of any kind; position along the hor–izontal axis has no numerical meaning. Thus, the horizontal position and ordering of the bars are not determined by the data. (To mini–mize potential confusion, it might help to avoid presenting students with categorical data in which category names use numerals, e.g., "Candidate 1," "Candidate 2," "Candidate 3." This will ensure that the only numbers present in the display are on the count scale.)

However, the vertical axes in these graphs do have numerical meaning. In fact, the vertical axes in these graphs are segments of number line diagrams. We might think of the vertical axis as a "count scale" (a scale showing counts in whole numbers)—as opposed to a measurement scale, which can be subdivided to represent fractions of a unit of measurement.

Because the count scale in a bar graph is a segment of a number line diagram, answering a question such as "How many more birds are there than spiders?" involves understanding differences on a number line diagram.[2.MD.10]

When drawing bar graphs on grid paper, the tick marks on the count scale should be drawn at intersections of the gridlines. The

[2.MD.10]Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

[2.OA.1]Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, tak-ing from, putting together, taking apart, and comparing, with un-knowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.

Activity for representing categorical data
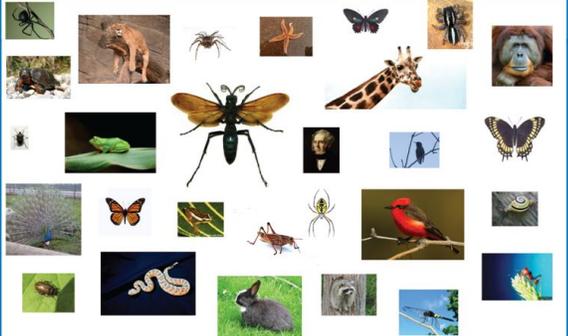
1. How many organisms in the picture belong to each of the following categories: (a) insects (six legs); (b) spiders (eight legs); (c) vertebrates (backbone); (d) other.
2. To check your answer, do your counts add up to the correct total?
3. When you are sure your counts are correct, show them as a bar graph.
4. Alexa added more spiders to the picture until the number of spiders was the same as the number of vertebrates. How many spiders did she add?

Students might reflect on the way in which the category counts in part 1 of the activity enable them to efficiently solve the word problem in part 4. (The word problem in part 4 would be difficult to solve directly using just the array of images.)
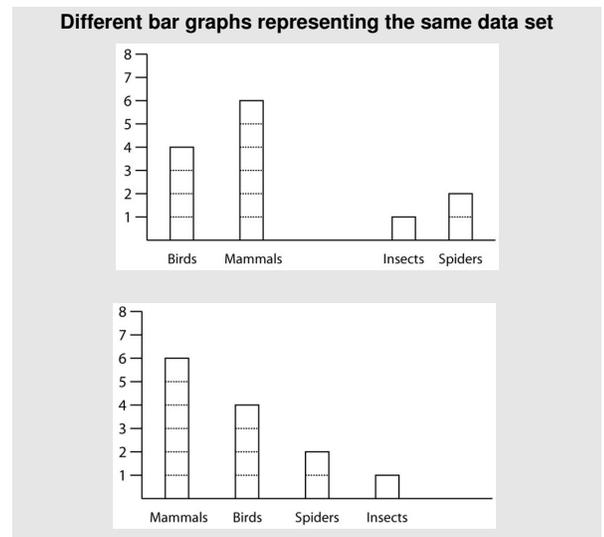
Different bar graphs representing the same data set

[2.MD.10]Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

tops of the bars should reach the respective gridlines of the appropriate tick marks. When drawing picture graphs on grid paper, the pictures representing the objects should be drawn in the squares of the grid paper.

Students could discuss ways in which bar orientation (horizontal or vertical), order, thickness, spacing, shading, colors, and so forth make the bar graphs easier or more difficult to interpret. By middle grades, students could make thoughtful design choices about data displays, rather than just accepting the defaults in a software program (MP.5).

Using information from a bar graph

How many more birds than spiders?

The count scale in a bar graph is a number line diagram with only whole numbers.
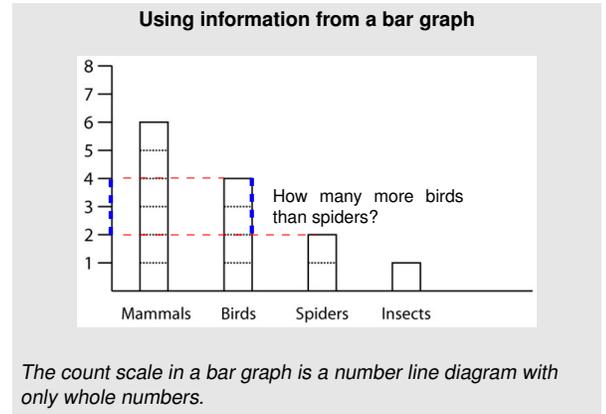
Grade 3

In Grade 3, the most important development in data representation for categorical data is that students now draw picture graphs in which each picture represents more than one object, and they draw bar graphs in which the height of a given bar in tick marks must be multiplied by the scale factor in order to yield the number of objects in the given category. These developments connect with the emphasis on multiplication in this grade.

At the end of Grade 3, students can draw a scaled picture graph or a scaled bar graph to represent a data set with several categories (six or fewer categories).[3.MD.3] They can solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.[3.OA.3,3.OA.8] See the examples in the margin, one of which involves categories of shapes.[3.G.1] As in Grade 2, category counts might be recorded efficiently in the form of a table.

Students can gather categorical data in authentic contexts, including contexts arising in their study of science, history, health, and so on. Of course, students do not have to generate the data every time they work on making bar graphs and picture graphs. That would be too time-consuming. After some experiences in generating the data, most work in producing bar graphs and picture graphs can be done by providing students with data sets. The Standards in Grades 1–3 do not require students to gather categorical data.

Where this progression is heading

Students' work with categorical data in early grades will develop into later work with bivariate categorical data and two-way tables in eighth grade (see the 6–8 Statistics and Probability Progression).

"Bivariate categorical data" are data that are categorized according to two attributes. For example, if there is an outbreak of stomach illness on a cruise ship, then passengers might be sorted in two different ways: by determining who got sick and who didn't, and by determining who ate the shellfish and who didn't. This double categorization—normally shown in the form of a two-way table—might show a strong positive or negative association, in which case it might used to support or contest (but not prove or disprove) a claim about whether the shellfish was the cause of the illness.[8.SP.4]

A problem about interpreting a scaled picture graph

The picture graph shows how many trees of each kind were in the arboretum. How many more evergreen trees than maple trees are there? (Fir trees and spruce trees are evergreen trees.)

[3.MD.3]Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs.

[3.OA.3]Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities. . . .

[3.OA.8]Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. . . .

Problems involving categorical data

1. How many shapes above belong to each of the following categories: (a) rectangles; (b) circles; (c) triangles.
2. When you are sure your counts are correct, show them as a picture graph. Use a scale with 1 picture = 5 shapes.
3. Alex added circles to the picture until the number of circles was the same as the number of polygons. How many circles did Alex add?

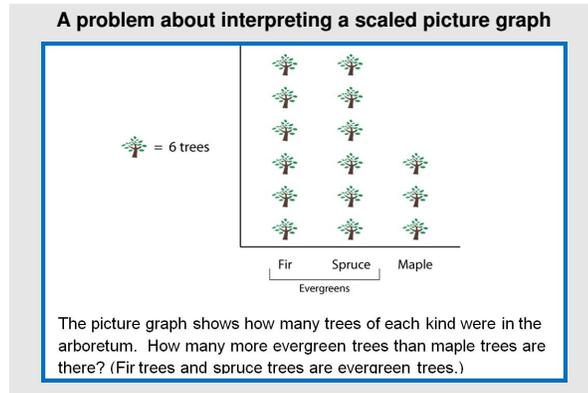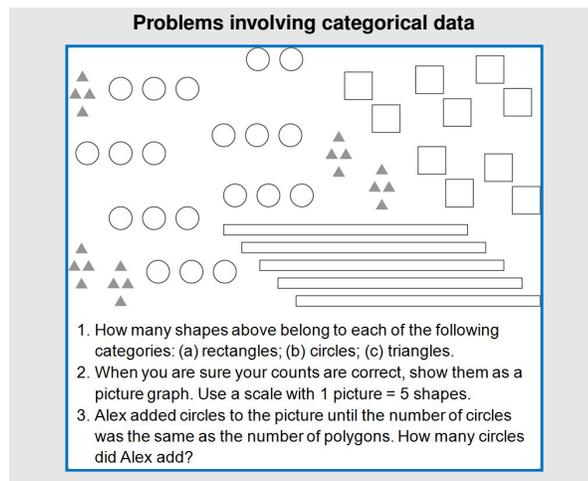[3.G.1]Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals. . . .

[8.SP.4]Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables.

Measurement Data, 2–5

Grade 2

Students in Grade 2 measure lengths to generate a set of measurement data.[2.MD.1] For example, each student might measure the length of his or her arm in centimeters, or every student might measure the height of a statue in inches. (Students might also generate their own ideas about what to measure.) The resulting data set will be a list of observations, for example as shown in the margin on the following page for the scenario of 28 students each measuring the height of a statue. (This is a larger data set than students would normally be expected to work with in elementary grades.)

How might one summarize this data set or display it visually? Because students in Grade 2 are already familiar with categorical data and bar graphs, a student might find it natural to summarize this data set by viewing it in terms of categories—the categories in question being the six distinct height values which appear in the data (63 inches, 64 inches, 65 inches, 66 inches, 67 inches, and 69 inches). For example, the student might want to say that there are four observations in the "category" of 67 inches. However, it is important to recognize that 64 inches is not a category like "spiders." Unlike "spiders," 63 inches is a numerical value with a measurement unit. That difference is why the data in this table are called measurement data.

A display of measurement data must present the measured values with their appropriate magnitudes and spacing on the measurement scale in question (length, temperature, liquid capacity, etc.). One method for doing this is to make a *line plot*. This activity connects with other work students are doing in measurement in Grade 2: representing whole numbers on number line diagrams, and representing sums and differences on such diagrams.[2.MD.5,2.MD.6]

To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: 63 inches and 69 inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale.

Note that the value 68 inches, which was not present in the data set, has been written in proper position midway between 67 inches and 69 inches. (This need to fill in gaps does not exist for a categorical data set; there no "gap" between categories such as fish and spiders!)

[2.MD.1] Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.

Students' initials and measurements of a statue in inches

initials	measurement
W.B.	64
D.W.	65
H.D.	65
G.W.	65
V.Y.	67
T.T.	66
D.F.	67
B.H.	65
H.H.	63
V.H.	64
I.O.	64
W.N.	65
B.P.	69
V.A.	65
H.L.	66
O.M.	64
L.E.	65
M.J.	66
T.D.	66
K.P.	64
H.N.	65
W.M.	67
C.Z.	64
J.I.	66
M.S.	66
T.C.	65
G.V.	67
O.F.	65

[2.MD.5] Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

[2.MD.6] Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

A scale for a line plot of the statue data

Height of the statue (inches)

Having drawn the number line diagram, the student can pro-
ceed through the data set recording each observation by drawing a
symbol, such as a dot, above the proper tick mark. If a particular
data value appears many times in the data set, dots will "pile up"
above that value. There is no need to sort the observations, or to
do any counting of them, before producing the line plot. (In fact, one
could even assemble the line plot as the data are being collected,
at the expense of having a record of who made what measurement.
Students might discuss whether such a record is valuable and why.)

Students might enjoy discussing and interpreting visual features
of line plots, such as the "outlier" value of 69 inches in this line plot.
(Did student #13 make a serious error in measuring the statue's
height? Or in fact is student #13 the only person in the class who
measured the height correctly?) However, in Grade 2 the only re-
quirement of the Standards dealing with measurement data is that
students generate measurement data and build line plots to display
the resulting data sets. (Students do not have to generate the data
every time they work on making line plots. That would be too time-
consuming. After some experiences in generating the data, most
work in producing line plots can be done by providing students with
data sets.)

Grid paper might not be as useful for drawing line plots as it
is for bar graphs, because the count scale on a line plot is seldom
shown for the small data sets encountered in the elementary grades.
Additionally, grid paper is usually based on a square grid, but the
count scale and the measurement scale of a line plot are conceptu-
ally distinct, and there is no need for the measurement unit on the
measurement scale to be drawn the same size as the counting unit
on the count scale.

A line plot of the statue data

Height of the statue (inches)
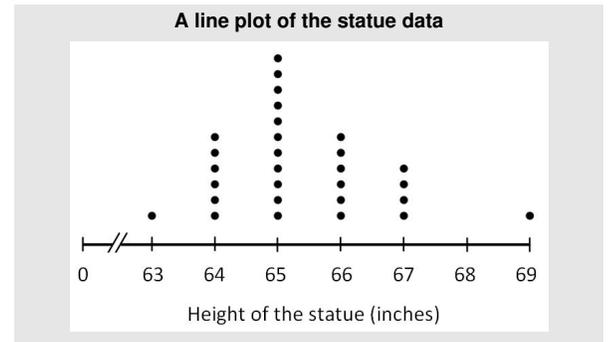
Grade 3

In Grade 3, students are beginning to learn fraction concepts (see the Number and Operations—Fractions Progression). They understand fraction equivalence in simple cases, and they use diagrams to represent and order fractions. Grade 3 students also measure lengths using rulers marked with halves and fourths of an inch. They use their developing knowledge of fractions and number lines to extend their work from the previous grade by working with measurement data involving fractional measurement values.

For example, every student in the class might measure the height of a bamboo shoot growing in the classroom, leading to the data set shown in the table. (Again, this illustration shows a larger data set than students would normally work with in elementary grades.)

To make a line plot from the data in the table, the student can ascertain the greatest and least values in the data: $13\frac{1}{2}$ inches and $14\frac{3}{4}$ inches. The student can draw a segment of a number line diagram that includes these extremes, with tick marks indicating specific values on the measurement scale. The tick marks are just like part of the scale on a ruler.

Having drawn the number line diagram, the student can proceed through the data set recording each observation by drawing a symbol, such as a dot, above the proper tick mark. As with Grade 2 line plots, if a particular data value appears many times in the data set, dots will "pile up" above that value. There is no need to sort the observations, or to do any counting of them, before producing the line plot.

Students can pose questions about data presented in line plots, such as how many students obtained measurements larger than $14\frac{1}{4}$ inches.

Students' initials and bamboo shoot measurements (inches)

initials	measurement
W.B.	13 3/4
D.W.	14 1/2
H.D.	14 1/4
G.W.	14 3/4
V.Y.	14 1/4
T.T.	14 1/2
D.F.	14
B.H.	13 1/2
H.H.	14 1/4
V.H.	14 1/4
I.O.	14 1/4
W.N.	14
B.P.	14 1/2
V.A.	13 3/4
H.L.	14
O.M.	13 3/4
L.E.	14 1/4
M.J.	13 3/4
T.D.	14 1/4
K.P.	14
H.N.	14
W.M.	14
C.Z.	13 3/4
J.I.	14
M.S.	14 1/4
T.C.	14
G.V.	14
O.F.	14 1/4

A scale for a line plot of the bamboo shoot data

Height of the Bamboo Shoot (inches)

A line plot of the bamboo shoot data

Height of the Bamboo Shoot (inches)
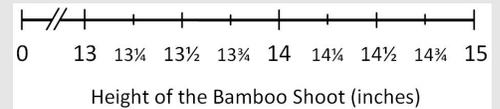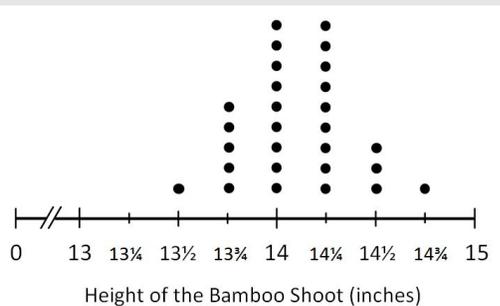
Grades 4 and 5

As in Grade 3, expectations in the domain of measurement and data for Grades 4 and 5 are coordinated with grade–level expectations for work with fractions.

Grade 4 students learn elements of fraction equivalence[4.NF.1] and arithmetic, including multiplying a fraction by a whole number[4.NF.4] and adding and subtracting fractions with like denominators.[4.NF.3] Students can use these skills to solve problems, including problems that arise from analyzing line plots. For example, with reference to the line plot above, students might find the difference between the greatest and least values in the data.

In solving such problems, students may need to label the mea–surement scale in eighths so as to produce like denominators. Mea–surements expressed in decimal notation can also be used in this grade,[4.NF.6] but computations in this notation are not expected until Grade 5.[5.NBT.7]

Grade 5 students grow in their skill and understanding of fraction arithmetic, including:

- adding and subtracting fractions with unlike denominators.[5.NF.1]

- multiplying a fraction by a fraction.[5.NF.4]

- dividing a unit fraction by a whole number or a whole number by a unit fraction.[5.NF.7]

Students can use these skills to solve problems,[5.NF.2,5.NF.6,5.NF.7c] including problems that arise from analyzing line plots. For example, given five graduated cylinders with different measures of liquid in each, students might find the amount of liquid each cylinder would contain if the total amount in all the cylinders were redistributed equally. (Students in Grade 6 will view the answer to this question as the mean value for the data set in question.)

As in earlier grades, students should work with data in science and other subjects. Grade 5 students working in these contexts should be able to give deeper interpretations of data than in earlier grades, such as interpretations that involve informal recognition of pronounced differences in populations. This prefigures the work they will do in middle grades involving distributions, comparisons of populations, and inference.

[4.NF.1]Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

[4.NF.4]Apply and extend previous understandings of multiplica-tion to multiply a fraction by a whole number.

[4.NF.3]Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

[4.NF.6]Use decimal notation for fractions with denominators 10 or 100.

[5.NBT.7]Add, subtract, multiply, and divide decimals to hun-dredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

[5.NF.1]Add and subtract fractions with unlike denominators (in-cluding mixed numbers) by replacing given fractions with equiva-lent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

[5.NF.4]Apply and extend previous understandings of multiplica-tion to multiply a fraction or whole number by a fraction.

[5.NF.7]Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

[5.NF.2]Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reason-ableness of answers.

[5.NF.6]Solve real world problems involving multiplication of frac-tions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

[5.NF.7]Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

  c Solve real world problems involving division of unit frac-tions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

Where this progression is heading

By the end of Grade 5, students should be comfortable making line plots for measurement data and analyzing data shown in the form of a line plot. In Grade 6, students will take an important step toward statistical reasoning per se when they approach line plots as pictures of distributions with features such as clustering and outliers.

Students' work with line plots during the elementary grades develops in two distinct ways during middle grades. The first development comes in sixth grade,[6.SP.4] when *histograms* are used. Like line plots, histograms have a measurement scale and a count scale; thus, a histogram is a natural evolution of a line plot and is used for similar kinds of data (univariate measurement data, the kind of data discussed above).

The other evolution of line plots in middle grades is arguably more important. It involves the graphing of bivariate measurement data.[8.SP.1-3] "Bivariate measurement data" are data that represent two measurements. For example, if you take a temperature reading every ten minutes, then every data point is a measurement of temperature as well as a measurement of time. Representing two measurements requires two measurement scales—or in other words, a coordinate plane in which the two axes are each marked in the relevant measurement units. Representations of bivariate measurement data in the coordinate plane are called *scatter plots*. In the case where one axis is a time scale, they are called *time graphs* or *line graphs*. Time graphs can be used to visualize trends over time, and scatter plots can be used to discover associations between measured variables in general. See the 6–8 Statistics and Probability Progression.

The Standards do not explicitly require students to create time graphs. However, it might be considered valuable to expose students to time series data and to time graphs as part of their work in the Number System.[6.NS.8] For example, students could create time graphs of temperature measured each hour over a 24-hour period, where, to ensure a strong connection to rational numbers, temperature values might cross from positive to negative during the night and back to positive the next day. It is traditional to connect ordered pairs with line segments in such graphs, not in order to make any claims about the actual temperature value at unmeasured times, but simply to aid the eye in perceiving trends.

[6.SP.4]Display numerical data in plots on a number line, including dot plots, histograms, and box plots.

[8.SP.1]Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.

[8.SP.2]Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

[8.SP.3]Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept.

[6.NS.8]Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

Appendix. Data work examples

These examples show some rich possibilities for data work in K–8. The examples are not shown by grade level because each includes some aspects that go beyond the expectations stated in the Standards.

Example 1. Comparing bar graphs

Are younger students lighter sleepers than older students? To study this question a class first agreed on definitions for light, medium and heavy sleepers and then collected data from first and fifth grade students on their sleeping habits. The results are shown in the margin.

How do the patterns differ? What is the typical value for first graders? What is the typical value for fifth graders? Which of these groups appears to be the heavier sleepers?
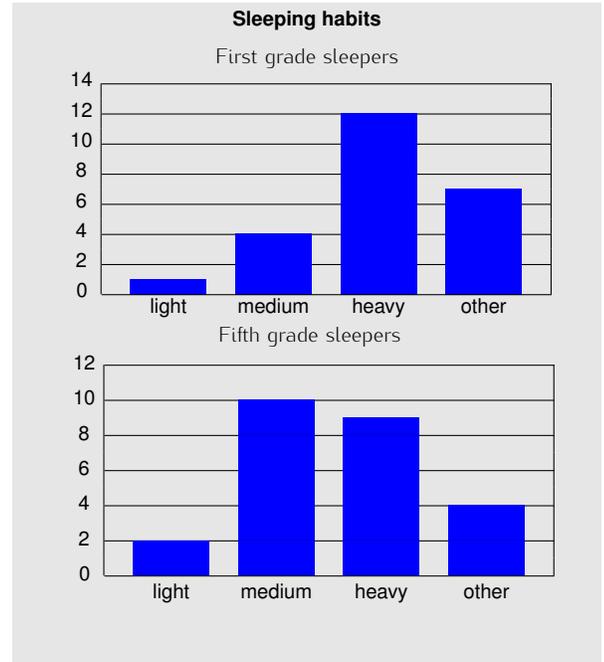
Example 2. Comparing line plots

Fourth grade students interested in seeing how heights of students change for kids around their age measured the heights of a sample of eight-year-olds and a sample of ten-year-olds. Their data are plotted in the margin.

Describe the key differences between the heights of these two age groups. What would you choose as the typical height of an eight-year-old? A ten-year-old? What would you say is the typical number of inches of growth from age eight to age ten?
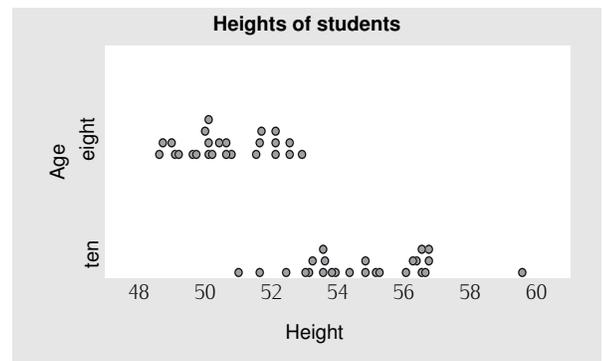
Example 3. Fair share averaging

Ten students decide to have a pizza party and each is asked to bring his or her favorite pizza. The amount paid (in dollars) for each pizza is shown in the plot to the right.

Each of the ten is asked to contribute an equal amount (his or her fair share) to the cost of the pizza. Where does that fair share amount lie on the plot? Is it closer to the smaller values or the large one? Now, two more students show up for the party and they have contributed no pizza. Plot their values on the graph and calculate a new fair share. Where does it lie on the plot? How many more students without pizza would have to show up to bring the fair share cost below $8.00?

Sleeping habits

Heights of students

Pizza party
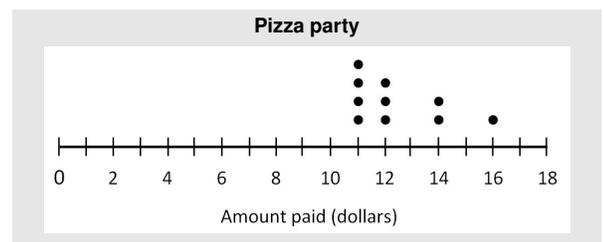
Geometric Measurement, K–5

Overview

Geometric measurement connects the two most critical domains of early mathematics, geometry and number, with each providing conceptual support to the other. Measurement is central to mathematics, to other areas of mathematics (e.g., laying a sensory and conceptual foundation for arithmetic with fractions), to other subject matter domains, especially science, and to activities in everyday life. For these reasons, measurement is a core component of the mathematics curriculum.

Measurement is the process of assigning a number to a magnitude of some attribute shared by some class of objects, such as length, relative to a unit. Length is a *continuous* attribute—a length can always be subdivided in smaller lengths. In contrast, we can count 4 apples exactly—cardinality is a discrete attribute. We can add the 4 apples to 5 other apples and know that the result is exactly 9 apples. However, the *weight* of those apples is a continuous attribute, and scientific measurement with tools gives only an approximate measurement—to the nearest pound (or, better, kilogram), or the nearest tenth or hundredth of a pound, but always with some error.•

• The Standards do not differentiate between weight and mass. Technically, mass is the amount of matter in an object. Weight is the force exerted on the body by gravity. On the Earth's surface, the distinction is not important (on the Moon, an object would have the same mass, but would weigh less due to the lower gravity).

Before learning to measure attributes, children need to recognize them, distinguishing them from other attributes. That is, the attribute to be measured has to "stand out" for the student and be discriminated from the undifferentiated sense of amount that young children often have, labeling greater lengths, areas, volumes, and so forth, as "big" or "bigger."

Students then can become increasingly competent at *direct comparison*—comparing the amount of an attribute in two objects without measurement. For example, two students may stand back to back to directly compare their heights. In many circumstances, such direct comparison is impossible or unwieldy. Sometimes, a third object can be used as an intermediary, allowing *indirect comparison*. For example, if we know that Aleisha is taller than Barbara and that Barbara is taller than Callie, then we know (due to the transitivity

of "taller than") that Aleisha is taller than Callie, even if Aleisha and Callie never stand back to back.•

The purpose of measurement is to allow indirect comparisons of objects' amount of an attribute using numbers. An attribute of an object is measured (i.e., assigned a number) by comparing it to an amount of that attribute held by another object. One measures length with length, mass with mass, torque with torque, and so on. In geometric measurement, a unit is chosen and the object is subdivided or partitioned by copies of that unit and, to the necessary degree of precision, units subordinate to the chosen unit, to determine the number of units and subordinate units in the partition.

Personal benchmarks, such as "tall as a doorway" build students' intuitions for amounts of a quantity and help them use measurements to solve practical problems. A combination of internalized units and measurement processes allows students to develop increasing accurate estimation competencies.

Both in measurement and in estimation, the concept of *unit* is crucial. The concept of basic (as opposed to subordinate) unit just discussed is one aspect of this concept. The basic unit can be informal (e.g., about a car length) or standard (e.g., a meter). The distinction and relationship between the notion of discrete "1" (e.g., one apple) and the continuous "1" (e.g., one inch) is important mathematically and is important in understanding number line diagrams (e.g., see Grade 2) and fractions (e.g., see Grade 3). However, there are also superordinate units or "units of units." A simple example is a kilometer consisting of 1,000 meters. Of course, this parallels the number concepts students must learn, as understanding that tens and hundreds are, respectively, "units of units" and "units of units of units" (i.e., students should learn that 100 can be simultaneously considered as 1 hundred, 10 tens, and 100 ones).

Students' understanding of an attribute that is measured with derived units is dependent upon their understanding that attribute as entailing other attributes simultaneously. For example,

- Area as entailing two lengths, simultaneously;

- Volume as entailing area and length (and thereby three lengths), simultaneously.

Scientists measure many types of attributes, from hardness of minerals to speed. This progression emphasizes the geometric attributes of length, area, and volume. Nongeometric attributes such as weight, mass, capacity, time, and color, are often taught effectively in science and social studies curricula and thus are not extensively discussed here. Attributes derived from two different attributes, such as speed (derived from distance and time), are discussed in the 6–7 Ratios and Proportional Relationships Progression and in the high school Quantity Progression.

Length is a characteristic of an object found by quantifying how far it is between the endpoints of the object. "Distance" is often used

• "Transitivity" abbreviates the Transitivity Principle for Indirect Measurement stated in the Standards as:

> If the length of object A is greater than the length of object B, and the length of object B is greater than the length of object C, then the length of object A is greater than the length of object C. This principle applies to measurement of other quantities as well.

Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term.

similarly to quantify how far it is between any two points in space. Measuring length or distance consists of two aspects, choosing a unit of measure and *subdividing* (mentally and physically) the object by that unit, placing that unit end to end (*iterating*) alongside the object. The length of the object is the number of units required to iterate from one end of the object to the other, without gaps or overlaps.

Length is a core concept for several reasons. It is the basic geometric measurement. It is also involved in area and volume measurement, especially once formulas are used. Length and unit iteration are critical in understanding and using the number line in Grade 3 and beyond (see the Number and Operations—Fractions Progression and the Number System Progression). Length is also one of the most prevalent metaphors for quantity and number, e.g., as the master metaphor for magnitude (e.g., vectors, see the Quantity Progression). Thus, length plays a special role in this progression.

Area is an amount of two-dimensional surface that is contained within a plane figure. Area measurement assumes that congruent figures enclose equal areas, and that area is *additive*, i.e., the area of the union of two regions that overlap only at their boundaries is the sum of their areas. Area is measured by tiling a region with a two-dimensional unit (such as a square) and parts of the unit, without gaps or overlaps. Understanding how to spatially structure a two-dimensional region is an important aspect of the progression in learning about area.

Volume is an amount of three-dimensional space that is contained within a three-dimensional shape. Volume measurement assumes that congruent shapes enclose equal volumes, and that volume is *additive*, i.e., the volume of the union of two regions that overlap only at their boundaries is the sum of their volumes. Volume is measured by packing (or tiling, or tessellating) a region with a three-dimensional unit (such as a cube) and parts of the unit, without gaps or overlaps. Volume not only introduces a third dimension and thus an even more challenging spatial structuring, but also complexity in the nature of the materials measured. That is, solid physical volume-units might be "packed," such as cubes in a three-dimensional array or cubic meters of coal, whereas liquids and solids that can be poured "fill" three-dimensional regions, taking the shape of a container, and are often measured in units such as liters● or quarts.

● One liter is one cubic decimeter, i.e., the volume of a cube with edge length $\frac{1}{10}$ meter, so a milliliter is one cubic centimeter.

A final, distinct, geometric attribute is *angle measure*. The size of an angle is the amount of rotation between the two rays that form the angle, sometimes called the sides of the angles.

Finally, although the attributes that we measure differ as just described, it is important to note: *central characteristics of measurement are the same for all of these attributes*. As one more testament to these similarities, consider the following side-by-side comparison of the Standards for measurement of area in Grades 3 and 5 and the measurement of volume in Grades 5 and 6.

Area: Grades 3 and 5	Volume: Grades 5 and 6

Understand concepts of area and relate area to multiplication and to addition.

3.MD.5. Recognize area as an attribute of plane figures and understand concepts of area measurement.

   a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

   b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

3.MD.6. Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

3.MD.7. Relate area to the operations of multiplication and addition.

   a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

   b. Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

   c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

   d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions

5.NF.4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

   b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Understand concepts of volume and relate volume to multiplication and to addition.

5.MD.3 Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

   a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume.

   b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.

5.MD.4 Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

5.MD.5 Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

   a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

   b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

   c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Solve real-world and mathematical problems involving area, surface area, and volume

6.G.2. Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

Unit square and unit cube. Note that "unit square" may refer to a physical area-unit, a unit of measurement, or a square on a diagram with or without units of measurement. At Grade 3, students work with unit squares with side lengths that are basic units (e.g., 1 centimeter or 1). At Grade 5, unit square side lengths include subordinate units (e.g., $\frac{1}{7}$ centimeter or $\frac{1}{7}$). Similarly, a "unit cube" may be a physical volume-unit, a unit of measurement, or a cube shown on a diagram. Edge lengths are basic units in Grade 5 and include subordinate units in Grade 6.

Formulas. Formulas can be expressed in many ways. The formula for the area of a rectangle can be expressed as "the area is the same as would be found by multiplying the side lengths" (as in 3.MD.7 and 5.NF.4) or as "$A = l \times w$" (implicit in 5.MD.5 and 6.G.2). What is important is that the referents of terms or symbols are clear (MP.6). For example, the formula for the volume of a right rectangular prism can be expressed as "the volume is the product of the base and the height," or as "$V = b \times h$," or as "$V = B \times h$." The referent of "base" or, respectively, "b" or "B" is "area of the base in square units." The units in which the base and height are expressed determine the units in which the volume is expressed. Also, note discussion of "apply the formula" on p. 106.

Kindergarten

Describe and compare measurable attributes   Students often initially hold undifferentiated views of measurable attributes, saying that one object is "bigger" than another whether it is longer, or greater in area, or greater in volume, and so forth. For example, two students might both claim their block building is "the biggest." Conversations about how they are comparing—one building may be taller (greater in length) and another may have a larger base (greater in area)—help students learn to discriminate and name these measurable attributes. As they discuss these situations and compare objects using different attributes, they learn to distinguish, label, and describe several measurable attributes of a single object.[K.MD.1] Thus, teachers listen for and extend conversations about things that are "big," or "small," as well as "long," "tall," or "high," and name, discuss, and demonstrate with gestures the attribute being discussed (length as extension in one dimension is most common, but area, volume, or even weight in others).

Length. Of course, such conversations often occur in comparison situations ("He has more than me!"). Kindergartners easily directly compare lengths in simple situations, such as comparing people's heights, because standing next to each other automatically aligns one endpoint.[K.MD.2] However, in other situations they may initially compare only one endpoint of objects to say which is longer. Discussing such situations (e.g., when a child claims that he is "tallest" because he is standing on a chair) can help students resolve and coordinate perceptual and conceptual information when it conflicts. Teachers can reinforce these understandings, for example, by holding two pencils in their hand showing only one end of each, with the longer pencil protruding less. After asking if they can tell which pencil is longer, they reveal the pencils and discuss whether children were "fooled." The necessity of aligning endpoints can be explicitly addressed and then re-introduced in the many situations throughout the day that call for such comparisons. Students can also make such comparisons by moving shapes together to see which has a longer side.

Even when students seem to understand length in such activities, they may not conserve length. That is, they may believe that if one of two sticks of equal lengths is vertical, it is then longer than the other, horizontal, stick. Or, they may believe that a string, when bent or curved, is now shorter (due to its endpoints being closer to each other). Both informal and structured experiences, including demonstrations and discussions, can clarify how length is maintained, or conserved, in such situations. For example, teachers and students might rotate shapes to see its sides in different orientations. As with number, learning and using language such as "It looks longer, but it really isn't longer" is helpful.

Students who have these competencies can engage in experiences that lay the groundwork for later learning. Many can begin

K.MD.1 Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.

K.MD.2 Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference.

Sticks whose endpoints are not aligned

A   B   C
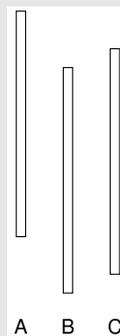
When shown this figure and asked which is "the longest stick," students may point to A because it "sticks out the farthest." Similarly, they may recognize a 12-inch vertical line as "tall" and a 12-inch horizontal line as "long" but not recognize that the two are the same length.

to learn to compare the lengths of two objects using a third object, order lengths, and connect number to length. For example, informal experiences such as making a road "10 blocks long" help students build a foundation for measuring length in the elementary grades. See the Grade 1 section on length for information about these important developments.

Area and volume. Although area and volume experiences are not instructional foci for Kindergarten, they are attended to, at least to distinguish these attributes from length, as previously described. Further, certain common activities can help build students' experiential foundations for measurement in later grades. Understanding area requires understanding this attribute as the amount of two-dimensional space that is contained within a boundary. Kindergartners might informally notice and compare areas associated with everyday activities, such as laying two pieces of paper on top of each other to find out which will allow a "bigger drawing." Spatial structuring activities described in the K–6 Geometry Progression, in which designs are made with squares covering rectilinear shapes also help to create a foundation for understanding area.

Similarly, kindergartners might compare the capacities of containers informally by pouring (water, sand, etc.) from one to the other. They can try to find out which holds the most, recording that, for example, the container labeled "J" holds more than the container labeled "D" because when J was poured into D it overflowed. Finally, in play, kindergartners might make buildings that have layers of rectangular arrays. Teachers aware of the connections of such activities to later mathematics can support students' growth in multiple domains (e.g., development of self-regulation, social-emotional, spatial, and mathematics competencies) simultaneously, with each domain supporting the other.

Grade 1

Length comparisons   First graders should continue to use direct comparison—carefully, considering all endpoints—when that is appropriate. In situations where direct comparison is not possible or convenient, they should be able to use indirect comparison and explanations that draw on transitivity (MP.3). Once they can compare lengths of objects by direct comparison, they could compare several items to a single item, such as finding all the objects in the classroom the same length as (or longer than, or shorter than) their forearm.[1.MD.1] Ideas of transitivity can then be discussed as they use a string to represent their forearm's length. As another example, students can figure out that one path from the teachers' desk to the door is longer than another because the first path is longer than a length of string laid along the path, but the other path is shorter than that string. Transitivity can then be explicitly discussed: If *A* is longer than *B* and *B* is longer than *C*, then *A* must be longer than *C* as well.

[1.MD.1] Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Seriation   Another important set of skills and understandings is ordering a set of objects by length.[1.MD.1] Such sequencing requires multiple comparisons. Initially, students find it difficult to seriate a large set of objects (e.g., more than 6 objects) that differ only slightly in length. They tend to order groups of two or three objects, but they cannot correctly combine these groups while putting the objects in order. Completing this task efficiently requires a systematic strategy, such as moving each new object "down the line" to see where it fits. Students need to understand that each object in a seriation is larger than those that come before it, and shorter than those that come after. Again, reasoning that draws on transitivity is relevant.

[1.MD.1] Order three objects by length; compare the lengths of two objects indirectly by using a third object.

Such seriation and other processes associated with the measurement and data standards are important in themselves, but also play a fundamental role in students' development. The general reasoning processes of seriation, conservation (of length and number), and classification (which lies at the heart of the standards discussed in the K–3 Categorical Data Progression) predict success in early childhood as well as later schooling.

Measure lengths indirectly and by iterating length units   Directly comparing objects, indirectly comparing objects, and ordering objects by length are important practically and mathematically, but they are not length measurement, which involves assigning a number to a length. Students learn to lay physical units such as centimeter or inch manipulatives end-to-end and count them to measure a length.[1.MD.2] Such a procedure may seem to adults to be straightforward, however, students may initially iterate a unit leaving gaps between subsequent units or overlapping adjacent units. For such students, measuring may be an activity of placing units along a

[1.MD.2] Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.
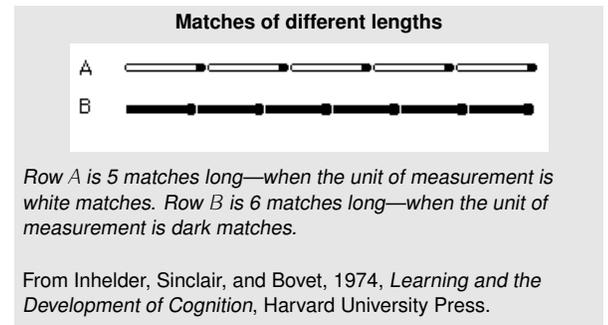
path in some manner, rather than the activity of covering a region or length with no gaps.

Also, students, especially if they lack explicit experience with continuous attributes, may make their initial measurement judgments based upon experiences counting discrete objects. For example, researchers showed children two rows of matches. The matches in each row were of different lengths, but there was a different number of matches in each so that the rows were the same length. Although, from the adult perspective, the lengths of the rows were the same, many children argued that the row with 6 matches was longer because it had more matches. They counted units (matches), assigning a number to a *discrete* attribute (cardinality). In measuring *continuous* attributes, the sizes of the units (white and dark matches) must be considered. First grade students can learn that objects used as basic units of measurement (e.g., "match-length") must be the same size.

As with transitive reasoning tasks, using comparison tasks and asking children to compare results can help reveal the limitations of such procedures and promote more accurate measuring. However, students also need to see agreements. For example, understanding that the results of measurement and direct comparison have the same results encourages children to use measurement strategies.

Another important issue concerns the use of standard or nonstandard units of length. Many curricula or other instructional guides advise a sequence of instruction in which students compare lengths, measure with nonstandard units (e.g., paper clips), incorporate the use of manipulative standard units (e.g., inch cubes), and measure with a ruler. This approach is probably intended to help students see the need for standardization. However, the use of a variety of different length units, *before students understand the concepts, procedures, and usefulness of measurement*, may actually deter students' development. Instead, students might learn to measure correctly with standard units, and even learn to use rulers, before they can successfully use nonstandard units and understand relationships between different units of measurement. To realize that arbitrary (and especially mixed-size) units result in the same length being described by different numbers, a student must reconcile the varying lengths and numbers of arbitrary units. Emphasizing nonstandard units too early may defeat the purpose it is intended to achieve. Early use of many nonstandard units may actually interfere with students' development of basic measurement concepts required to understand the need for standard units. In contrast, using manipulative standard units, or even standard rulers, is less demanding and appears to be a more interesting and meaningful real-world activity for young students.

Thus, an instructional progression based on this finding would start by ensuring that students can perform direct comparisons. Then, children should engage in experiences that allow them to connect number to length, using manipulative units that have a stan-

Matches of different lengths

Row A is 5 matches long—when the unit of measurement is white matches. Row B is 6 matches long—when the unit of measurement is dark matches.

From Inhelder, Sinclair, and Bovet, 1974, *Learning and the Development of Cognition*, Harvard University Press.

dard unit of length, such as centimeter cubes. These can be labeled "length–units" with the students. Students learn to lay such physical units end–to–end and count them to measure a length. They compare the results of measuring to direct and indirect comparisons.

As they measure with these manipulative units, students discuss the concepts and skills involved (e.g., as previously discussed, not leaving space between successive length–units). As another example, students initially may not extend the unit past the endpoint of the object they are measuring. If students make procedural errors such as these, they can be asked to tell in a precise and elaborate manner *what* the problem is, *why* it leads to incorrect measurements, and *how* to fix it and measure accurately.

Measurement activities can also develop other areas of mathematics, including reasoning and logic. In one class, first graders were studying mathematics mainly through measurement, rather than counting discrete objects. They described and represented relationships among and between lengths (MP.2, MP.3), such as comparing two sticks and symbolizing the lengths as "$A < B$." This enabled them to reason about relationships. For example, after seeing the following statements recorded on the board, if $V > M$, then $M \neq V$, $V \neq M$, and $M < V$, one first grader noted, "If it's an inequality, then you can write four statements. If it's equal, you can only write two" (MP.8).

This indicates that with high-quality experiences (such as those described in the Grade 2 section on length), many first graders can also learn to use reasoning, connecting this to direct comparison, and to measurement performed by laying physical units end-to-end.

Area and volume: Foundations   As in Kindergarten, area and volume are not instructional foci for first grade, but some everyday activities can form an experiential foundation for later instruction in these topics. For example, in later grades, understanding area requires seeing how to decompose shapes into parts and how to move and recombine the parts to make simpler shapes whose areas are already known (MP.7). First graders learn the foundations of such procedures both in composing and decomposing shapes, discussed in the K–6 Geometry Progression, and in comparing areas in specific contexts. For example, paper-folding activities lend themselves not just to explorations of symmetry but also to equal-area congruent parts. Some students can compare the area of two pieces of paper by cutting and overlaying them. Such experiences provide only initial development of area concepts, but these key foundations are important for later learning.

Volume can involve liquids or solids. This leads to two ways to measure volume, illustrated by "packing" a space such as a three-dimensional array with unit cubes and "filling" with iterations of a fluid unit that takes the shape of the container (called "liquid volume"). Many first graders initially perceive filling as having a one-

dimensional unit structure. For example, students may simply "read off" the measurement on a graduated cylinder. Thus, in a science or "free time" activity, students might compare the volumes of two containers in at least two ways. They might pour the contents of each into a graduated cylinder to compare the measurements. Or they might practice indirect comparison using transitive reasoning by using a third container to compare the volumes of the two containers. By packing cubes into containers into which cubes fit readily, students also can lay a foundation for later "packing" volume.

Grade 2

Measure and estimate lengths in standard units   Second graders learn to measure length with a variety of tools, such as rulers, meter sticks, and measuring tapes.[2.MD.1] Although this appears to some adults to be relatively simple, there are many conceptual and procedural issues to address. For example, students may begin counting at the numeral "1" on a ruler. The numerals on a ruler may signify to students when to start counting, rather than the amount of space that has already been covered. It is vital that students learn that "one" represents the space from the beginning of the ruler to the hash mark, not the hash mark itself. Again, students may not understand that units must be of equal size. They will even measure with tools subdivided into units of different sizes and conclude that quantities with more units are larger.

To learn measurement concepts and skills, students might use both simple rulers (e.g., having only whole units such as centimeters or inches) and physical units (e.g., manipulatives that are centimeter or inch lengths). As described for Grade 1, teachers and students can call these "length-units." Initially, students lay multiple copies of the same physical unit end-to-end along the ruler. They can also progress to iterating with one physical unit (i.e., repeatedly marking off its endpoint, then moving it to the next position), even though this is more difficult physically and conceptually. To help them make the transition to this more sophisticated understanding of measurement, students might draw length unit marks along sides of geometric shapes or other lengths to see the unit lengths. As they measure with these tools, students with the help of the teacher discuss the concepts and skills involved, such as the following.

- *length-unit iteration.* E.g., not leaving space between successive length-units;

- *accumulation of distance.* Understanding that the counting "eight" when placing the last length-unit means the space covered by 8 length-units, rather then just the eighth length-unit. Note the connection to cardinality;[K.CC.4]

- *alignment of zero-point.* Correct alignment of the zero-point on a ruler as the beginning of the total length, including the case in which the 0 of the ruler is not at the edge of the physical ruler;

- *meaning of numerals on the ruler.* The numerals indicate the number of length units so far;

- *connecting measurement with physical units and with a ruler.* Measuring by laying physical units end-to-end or iterating a physical unit and measuring with a ruler both focus on finding the total number of unit lengths.
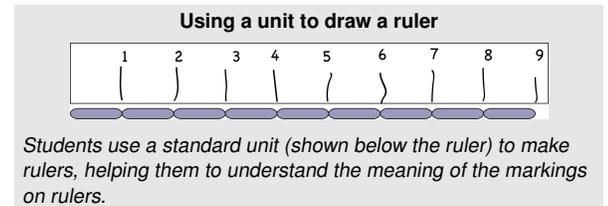
Students also can learn accurate procedures and concepts by drawing simple unit rulers. Using copies of a single length-unit such as inch-long manipulatives, they mark off length-units on strips of paper, explicitly connecting measurement with the ruler to measurement by iterating physical units. Thus, students' first rulers should be simply ways to help count the iteration of length-units. Frequently comparing results of measuring the same object with manipulative standard units and with these rulers helps students connect their experiences and ideas. As they build and use these tools, they develop the ideas of length-unit iteration, correct alignment (with a ruler), and the zero-point concept (the idea that the zero of the ruler indicates one endpoint of a length). These are reinforced as children compare the results of measuring to compare to objects with the results of directly comparing these objects.

Similarly, discussions might frequently focus on "*What* are you counting?" with the answer being "length-units" or "centimeters" or the like. This is especially important because counting discrete items often convinces students that the size of things counted does not matter (there could be exactly 10 toys, even if they are different sizes). In contrast, for measurement, unit size is critical, so teachers are advised to plan experiences and reflections on the use of other units and length-units in various discrete counting and measurement contexts. Given that counting discrete items often correctly teaches students that the length-unit size does not matter, so teachers are advised to plan experiences and reflections on the use of units in various discrete counting and measurement contexts. For example, a teacher might challenge students to consider a fictitious student's measurement in which he lined up three large and four small blocks and claimed a path was "seven blocks long." Students can discuss whether he is correct or not.

Second graders also learn the concept of the inverse relationship between the size of the unit of length and the number of units required to cover a specific length or distance.[2.MD.2] For example, it will take more centimeter lengths to cover a certain distance than inch lengths because inches are the larger unit. Initially, students may not appreciate the need for identical units. Previously described work with manipulative units of standard measure (e.g., 1 inch or 1 cm), along with related use of rulers and consistent discussion, will help children learn both the concepts and procedures of linear measurement. Thus, second grade students can learn that the larger the unit, the fewer number of units in a given measurement (as was illustrated on p. 93). That is, for measurements of a given length there is an inverse relationship between the size of the unit of measure and the number of those units. This is the time that measuring and reflecting on measuring the same object with different units, both standard and nonstandard, is likely to be most productive (see the discussion of this issue in the Grade 1 section on length). Results of measuring with different nonstandard length-units can be explicitly compared. Students also can use the concept of unit to make

Using a unit to draw a ruler

Students use a standard unit (shown below the ruler) to make rulers, helping them to understand the meaning of the markings on rulers.

[2.MD.2] Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

inferences about the relative sizes of objects; for example, if object *A* is 10 regular paperclips long and object *B* is 10 jumbo paperclips long, the number of units is the same, but the units have different sizes, so the lengths of *A* and *B* are different.

Second graders also learn to combine and compare lengths using arithmetic operations. That is, they can add two lengths to obtain the length of the whole and subtract one length from another to find out the difference in lengths.[2.MD.4] For example, they can use a simple unit ruler or put a length of connecting cubes together to measure first one modeling clay "snake," then another, to find the total of their lengths. The snakes can be laid along a line, allowing students to compare the measurement of that length with the sum of the two measurements. Second graders also begin to apply the concept of length in less obvious cases, such as the width of a circle, the length and width of a rectangle, the diagonal of a quadrilateral, or the height of a pyramid. As an arithmetic example, students might measure all the sides of a table with unmarked (foot) rulers to measure how much ribbon they would need to decorate the perimeter of the table.[2.MD.5] They learn to measure two objects and subtract the smaller measurement from the larger to find how much longer one object is than the other.

Second graders can also learn to represent and solve numerical problems about length using tape or number-bond diagrams. (See p. 25 of the Operations and Algebraic Thinking Progression for discussion of when and how these diagrams are used in Grade 1.) Students might solve two-step numerical problems at different levels of sophistication (see p. 27 of the Operations and Algebraic Thinking Progression for similar two-step problems involving discrete objects). Conversely, "missing measurements" problems about length may be presented with diagrams.

These understandings are essential in supporting work with number line diagrams.[2.MD.6] That is, to use a number line diagram to understand number and number operations, students need to understand that number line diagrams have specific conventions: the use of a single position to represent a whole number and the use of marks to indicate those positions. They need to understand that a number line diagram is like a ruler in that consecutive whole numbers are 1 unit apart, thus they need to consider the distances between positions and segments when identifying missing numbers. These understandings underlie students' successful use of number line diagrams. Students think of a number line diagram in terms of length measurement and use strategies relating to distance, proximity of numbers, and reference points.

After experience with measuring, second graders learn to estimate lengths.[2.MD.3] Real-world applications of length often involve estimation. Skilled estimators move fluently back and forth between written or verbal length measurements and representations of their corresponding magnitudes on a *mental ruler* (also called the "mental number line"). Although having real-world "benchmarks" is useful

2.MD.4 Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

2.MD.5 Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.

Missing measurements problems

Different solution methods for "A girl had a 43 cm section of a necklace and another section that was 8 cm shorter than the first. How long the necklace would be if she combined the two sections?"[2.MD.5]

Missing measurements problems

What are the missing lengths of the third and fourth sides of the rectangle?

What are the missing lengths of each step and the bottom of the stairway?

These problems might be presented in the context of turtle geometry. Students work on paper to figure out how far the Logo turtle would have to travel to finish drawing the house (the remainder of the right side, and the bottom). They then type in Logo commands (e.g., for the rectangle, forward 40 right 90 fd 100 rt 90 fd 20 fd 20 rt 90 fd 100) to check their calculations (MP.5).

2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, ..., and represent whole-number sums and differences within 100 on a number line diagram.

2.MD.3 Estimate lengths using units of inches, feet, centimeters, and meters.

(e.g., a meter is about the distance from the floor to the top of a door-knob), instruction should also help children build understandings of scales and concepts of measurement into their estimation competencies. Although "guess and check" experiences can be useful, research suggests explicit teaching of estimation strategies (such as iteration of a mental image of the unit or comparison with a known measurement) and prompting students to learn reference or benchmark lengths (e.g., an inch-long piece of gum, a 6-inch dollar bill), order points along a continuum, and build up mental rulers.

Length measurement should also be used in other domains of mathematics, as well as in other subjects, such as science, and connections should be made where possible. For example, a line plot scale is just a ruler, usually with a non-standard unit of length. Teachers can ask students to discuss relationships they see between rulers and line plot scales. Data using length measures might be graphed (see example on p. 80 of the Measurement and Data Progression). Students could also graph the results of many students measuring the same object as precisely as possible (even involving halves or fourths of a unit) and discuss what the "real" measurement of the object might be. Emphasis on students solving real measurement problems, and, in so doing, building and iterating units, as well as units of units, helps students development strong concepts and skills. When conducted in this way, measurement tools and procedures become tools for mathematics and tools for thinking about mathematics.

Area and volume: Foundations   To learn area (and, later, volume) concepts and skills meaningfully in later grades, students need to develop the ability known as *spatial structuring*. Students need to be able to see a rectangular region as decomposable into rows and columns of squares. This competence is discussed in detail in the K–6 Geometry Progression, but is mentioned here for two reasons. First, such spatial structuring precedes meaningful mathematical use of the structures, such as determining area or volume. Second, Grade 2 work in multiplication involves work with rectangular arrays,[2.G.2] and this work is an ideal context in which to simultaneously develop both arithmetical and spatial structuring foundations for later work with area.

[2.G.2] Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

[3.MD.8] Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Grade 3

Perimeter   Third graders focus on solving real-world and mathe-
matical problems involving perimeters of polygons.[3.MD.8] A perime-
ter is the boundary of a two-dimensional shape. For a polygon,
the length of the perimeter is the sum of the lengths of the sides.
Initially, it is useful to have sides marked with unit length marks,
allowing students to count the unit lengths. Later, the lengths of
the sides can be labeled with numerals. As with all length tasks,
students need to count the length-units and not the endpoints. Next,
students learn to mark off unit lengths with a ruler and label the
length of each side of the polygon. For rectangles, parallelograms,
and regular polygons, students can discuss and justify faster ways to
find the perimeter length than just adding all of the lengths (MP.3).
Rectangles and parallelograms have opposite sides of equal length,
so students can double the lengths of adjacent sides and add those
numbers or add lengths of two adjacent sides and double that num-
ber. A regular polygon has all sides of equal length, so its perimeter
length is the product of one side length and the number of sides.

Perimeter problems for rectangles and parallelograms often give
only the lengths of two adjacent sides or only show numbers for
these sides in a drawing of the shape. The common error is to add
just those two numbers. Having students first label the lengths of
the other two sides as a reminder is helpful.

Students then find unknown side lengths in more difficult "miss-
ing measurements" problems and other types of perimeter prob-
lems.[3.MD.8]

Children learn to subdivide length-units. Making one's own ruler
and marking halves and other partitions of the unit may be helpful
in this regard. For example, children could fold a unit in halves,
mark the fold as a half, and then continue to do so, to build fourths
and eighths, discussing issues that arise. Such activities relate to
fractions on the number line.[3.NF.2] Labeling all of the fractions can
help students understand rulers marked with halves and fourths but
not labeled with these fractions. Students also measure lengths
using rulers marked with halves and fourths of an inch.[3.MD.4] They
show these data by making a line plot, where the horizontal scale is
marked off in appropriate units—whole numbers, halves, or quarters
(see the Measurement and Data Progression, p. 82).

**Understand concepts of area and relate area to multiplication and
to addition**   Third graders focus on learning area. Students learn
formulas to compute area, with those formulas based on, and sum-
marizing, a firm conceptual foundation about what area is. Stu-
dents need to learn to conceptualize area as the amount of two-
dimensional space in a bounded region and to measure it by choos-
ing a unit of area, often a square. A two-dimensional geometric
figure that is covered by a certain number of squares without gaps

[3.MD.8]Solve real world and mathematical problems involving
perimeters of polygons, including finding the perimeter given the
side lengths, finding an unknown side length, and exhibiting rect-
angles with the same perimeter and different areas or with the
same area and different perimeters.

Missing measurements and other perimeter problems

21

40

The perimeter of this rectangle
is 168 length units. What are
the lengths of the three unla-
beled sides?
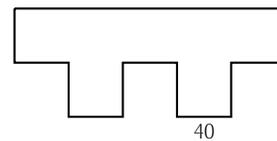
Assume all short segments are
the same length and all angles
are right.

*Compare these problems with the "missing measurements"
problems of Grade 2.*

*Another type of perimeter problem is to draw a robot on
squared grid paper that meets specific criteria. All the robot's
body parts must be rectangles. The perimeter of the head might
be 36 length-units, the body, 72; each arm, 24; and each leg, 72.
Students are asked to provide a convincing argument that their
robots meet these criteria (MP.3). Next, students are asked to
figure out the area of each of their body parts (in square units).
These are discussed, with students led to reflect on the different
areas that may be produced with rectangles of the same
perimeter. These types of problems can be also presented as
turtle geometry problems. Students create the commands on
paper and then give their commands to the Logo turtle to check
their calculations. For turtle length units, the perimeter of the
head might be 300 length-units, the body, 600; each arm, 400;
and each leg, 640.*

[3.NF.2]Understand a fraction as a number on the number line;
represent fractions on a number line diagram.

[3.MD.4]Generate measurement data by measuring lengths using
rulers marked with halves and fourths of an inch. Show the data
by making a line plot, where the horizontal scale is marked off in
appropriate units—whole numbers, halves, or quarters.

or overlaps can be said to have an area of that number of square units.[3.MD.5]

Activities such as those in the K–6 Geometry Progression teach students to compose and decompose geometric regions. To begin an explicit focus on area, teachers might then ask students which of three rectangles covers the most area. Students may first solve the problem with decomposition (cutting and/or folding) and re-composition, and eventually analyses with area-units, by covering each with unit squares (tiles).[3.MD.5, 3.MD.6] Discussions should clearly distinguish the attribute of area from other attributes, notably length.

Students might then find the areas of other rectangles. As previously stated, students can be taught to multiply length measurements to find the area of a rectangular region. But, in order that they make sense of these quantities (MP.2), they first learn to interpret measurement of rectangular regions as a multiplicative relationship of the number of square units in a row and the number of rows.[3.MD.7a] This relies on the development of spatial structuring (MP.7, see the K–6 Geometry Progression). To build from spatial structuring to understanding the number of area-units as the product of number of units in a row and number of rows, students might draw rectangular arrays of squares and learn to determine the number of squares in each row with increasingly sophisticated strategies, such as skip-counting the number in each row and eventually multiplying the number in each row by the number of rows (MP.8). They learn to partition a rectangle into identical squares by anticipating the final structure and forming the array by drawing line segments to form rows and columns. They use skip counting and multiplication to determine the number of squares in the array.

Many activities that involve seeing and making arrays of squares to form a rectangle might be needed to build robust conceptions of a rectangular area structured into squares. One such activity is illustrated in the margin. In this progression, less sophisticated activities of this sort were suggested for earlier grades so that Grade 3 students begin with some experience.

Students learn to understand and explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior (MP.3).[3.MD.7a] For example, students might explain that one length tells how many unit squares in a row and the other length tells how many rows there are.

Students might then solve numerous problems that involve rectangles of different dimensions (e.g., designing a house with rooms that fit specific area criteria) to practice using multiplication to compute areas.[3.MD.7b] The areas involved should not all be rectangular, but decomposable into rectangles (e.g., an "L-shaped" room).[3.MD.7d]

Students also might solve problems such as finding all the rectangular regions with whole-number side lengths that have an area of 12 area-units, doing this later for larger rectangles (e.g., enclosing

[3.MD.5]Recognize area as an attribute of plane figures and understand concepts of area measurement.

**Which rectangle covers the most area?**
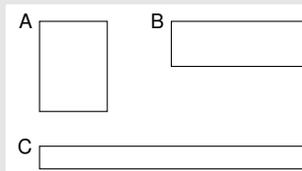
These rectangles are formed from unit squares (tiles students have used) although students are not informed of this or the rectangle's dimensions: A 4 by 3; B 2 by 6; C 1 row of 12.

Activity from Lehrer et al., 1998, "Developing understanding of geometry and space in the primary grades," in R. Lehrer & D. Chazan (Eds.), *Designing Learning Environments for Developing Understanding of Geometry and Space*, Lawrence Erlbaum Associates.

[3.MD.6]Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).

Incomplete array
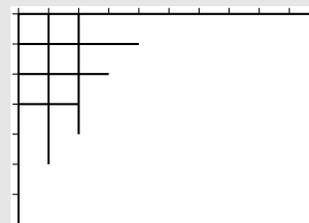
To determine the area of this rectangular region, students might be encouraged to construct a row, corresponding to the indicated positions, then repeating that row to fill the region. Cutouts of strips of rows can help the needed spatial structuring and reduce the time needed to show a rectangle as rows or columns of squares. Drawing all of the squares can also be helpful, but it is slow for larger rectangles. Drawing the unit lengths on the opposite sides can help students see that joining opposite unit endpoints will create the needed unit square grid.

[3.MD.7]Relate area to the operations of multiplication and addition.

a Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

b Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning.

d Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems.

24, 48, or 72 area-units), making sketches rather than drawing each square. They learn to justify their belief they have found all possible solutions (MP.3).

Similarly using concrete objects or drawings, and their competence with composition and decomposition of shapes, spatial structuring, and addition of area measurements, students learn to investigate arithmetic properties using area models. For example, they learn to rotate rectangular arrays physically and mentally, understanding that their areas are preserved under rotation, and thus, for example, $4 \times 7 = 7 \times 4$, illustrating the commutative property of multiplication.[3.MD.7c] They also learn to understand and explain that the area of a rectangular region of, for example, 12 length–units by 5 length–units can be found either by multiplying 12×5, or by adding two products, e.g., 10×5 and 2×5, illustrating the distributive property.

[3.MD.7]Relate area to the operations of multiplication and addition.

   c Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning.

Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures With strong and distinct concepts of both perimeter and area established, students can work on problems to differentiate their measures. For example, they can find and sketch rectangles with the same perimeter and different areas or with the same area and different perimeters and justify their claims (MP.3).[3.MD.8] Differentiating perimeter from area is facilitated by having students draw congruent rectangles and measure, mark off, and label the unit lengths all around the perimeter on one rectangle, then do the same on the other rectangle but also draw the square units. This enables students to see the units involved in length and area and find patterns in finding the lengths and areas of non–square and square rectangles (MP.7). Students can continue to describe and show the units involved in perimeter and area after they no longer need these supportive drawings.

[3.MD.8]Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Problem solving involving measurement and estimation of intervals of time, volumes, and masses of objects Students in Grade 3 learn to solve a variety of problems involving measurement and such attributes as length and area, volume, mass, and time.[3.MD.1, 3.MD.2]• Many such problems support the Grade 3 emphasis on multiplication (see the table on the next page) and the mathematical practices of making sense of problems (MP.1) and representing them with equations, drawings, or diagrams (MP.4). Such work will involve units of mass such as the kilogram.

[3.MD.1]Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.

[3.MD.2]Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l).[6] Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.[7]

[6]Excludes compound units such as cm^3 and finding the geometric volume of a container.

[7]Excludes multiplicative comparison problems (problems involving notions of "times as much"); see Glossary, Table 2.

• "Liquid volume" (see p. 94) and "geometric volume" refer to differences in methods of measurement, not in the attribute measured.

Table 4. Multiplication and division situations for measurement

	$A \times B = \square$	$A \times \square = C$ and $C \div A = \square$	$\square \times B = C$ and $C \div B = \square$
Grouped Objects (Units of Units)	You need A lengths of string, each B inches long. How much string will you need altogether?	You have C inches of string, which you will cut into A equal pieces. How long will each piece of string be?	You have C inches of string, which you will cut into pieces that are B inches long. How many pieces of string will you have?
Arrays of Objects (Spatial Structuring)	What is the area of a A cm by B cm rectangle?	A rectangle has area C square centimeters. If one side is A cm long, how long is a side next to it?	A rectangle has area C square centimeters. If one side is B cm long, how long is a side next to it?
Compare	A rubber band is B cm long. How long will the rubber band be when it is stretched to be A times as long?	A rubber band is stretched to be C cm long and that is A times as long as it was at first. How long was the rubber band at first?	A rubber band was B cm long at first. Now it is stretched to be C cm long. How many times as long is the rubber band now as it was at first?

Adapted from the *Common Core State Standards for Mathematics*, p. 89. Grade 3 work does not include Compare problems with "times as much," see Table 3 and discussion of problem types in the Grades 3 and 4 sections of the Operations and Algebraic Thinking Progression.

In the second column, division problems of the form $A \times \square = C$ are about finding an unknown multiplicand. For Equal Groups and Compare situations, these involve what is called the sharing, partitive, how-many-in-each-group, or what-is-the-unit interpretation of division. As discussed in the Operations and Algebraic Thinking Progression, Array situations can be seen as Equal Groups situations, thus, the Array situations in this column can also be seen as examples of the sharing interpretation of division.

In the third column, division problems of the form $\square \times B = C$ are about finding an unknown multiplier. For Equal Groups and Compare situations, these involve what is called the measurement, quotitive, how-many-groups, or how-many-units interpretation of division. As discussed in the Operations and Algebraic Thinking Progression, Array situations can be seen as Equal Groups situations, thus, the Array situations in this column can also be seen as examples of the measurement interpretation of division.

A few words on volume are relevant. Compared to the work in area, volume introduces more complexity, not only in adding a third dimension and thus presenting a significant challenge to students' spatial structuring, but also in the materials whose volumes are measured. These materials may be solid or fluid, so their volumes are generally measured with one of two methods, e.g., "packing" a right rectangular prism with cubic units or "filling" a shape such as a right circular cylinder.

Liquid measurement, for many third graders, may be limited to a one-dimensional unit structure (i.e., simple iterative counting of height that is not processed as three-dimensional). Thus, third graders can learn to measure liquid volumes and to solve problems requiring the use of the four arithmetic operations, when the measurements are given in the same units throughout each problem. Because measurements of liquid volumes can be represented with one-dimensional scales, problems may be presented with drawings or diagrams, such as measurements on a beaker with a measurement scale in milliliters.

Grade 4

In Grade 4, students build on competencies in measurement and in building and relating units and units of units that they have developed in number, geometry, and geometric measurement.

Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit   Fourth graders learn the relative sizes of measurement units within a system of measurement[4.MD.1] including:

> *length*: meter (m), kilometer (km), centimeter (cm), millimeter (mm); volume: liter (l), milliliter (ml, 1 cubic centimeter of water; a liter, then, is 1000 ml);

> *mass*: gram (g, about the weight of a cc of water), kilogram (kg);

> *time*: hour (hr), minute (min), second (sec).

For example, students develop benchmarks and mental images about a meter (e.g., about the height of a tall chair) and a kilometer (e.g., the length of 10 football fields including the end zones, or the distance a person might walk in about 12 minutes), and they also understand that "kilo" means a thousand, so 3000 m is equivalent to 3 km.

Expressing larger measurements in smaller units within the metric system is an opportunity to reinforce notions of place value. There are prefixes for multiples of the basic unit (meter or gram), although only a few (kilo–, centi–, and milli–) are in common use. Tables such as the one in the margin indicate the meanings of the prefixes by showing them in terms of the basic unit (in this case, meters). Such tables are an opportunity to develop or reinforce place value concepts and skills in measurement activities.

Relating units within the metric system is another opportunity to think about place value. For example, students might make a table that shows measurements of the same lengths in centimeters and meters.

Relating units within the traditional system provides an opportunity to engage in mathematical practices, especially "look for and make use of structure" (MP.7) and "look for and express regularity in repeated reasoning" (MP.8). For example, students might make a table that shows measurements of the same lengths in feet and inches.

Students also combine competencies from different domains as they solve measurement problems using all four arithmetic operations, addition, subtraction, multiplication, and division (see examples on next page and p. 103).[4.MD.2] For example, "How many liters of juice does the class need to have at least 35 cups if each cup takes 225 ml?" Students may use tape or number line diagrams for solving such problems (MP.1).

[4.MD.1]Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

Measurement unit equivalences

super- or subordinate unit	basic unit equivalent
kilometer	10^3 or 1000 meters
hectometer	10^2 or 100 meters
decameter	10^1 or 10 meters
meter	1 meter
decimeter	$\frac{1}{10}$ meter
centimeter	$\frac{1}{100}$ meter
millimeter	$\frac{1}{1000}$ meter

Note the similarity to the structure of base-ten units and U.S. currency (see illustrations on p. 64 of the Number and Operations in Base Ten Progression).
  Use of whole-number exponents to denote powers of 10 *is expected by the end of Grade 5.*

Measurement equivalences

cm	m
100	1
200	2
300	3
500	
1000	

feet	inches
0	0
1	12
2	24
3	

Expectations for conversion of measurements parallel expectations for multiplication with whole numbers and fractions.
 *In Grade 4, the emphasis is on "times as much" or "times as many," conversions that involve viewing a larger unit as superordinate to a smaller unit and multiplying the number of larger units by a whole number to find the number of smaller units.*
  In Grade 5, conversions also involve viewing a smaller unit as subordinate to a larger one (e.g., an inch is $\frac{1}{12}$ foot, so a measurement in feet is $\frac{1}{12}$ times what it is in inches) and conversions can involve multiplication by a fraction.

[4.MD.2]Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

Using number line diagrams to solve word problems

Juan spent $\frac{1}{4}$ of his money on a game. The game cost \$20. How much money did he have at first?
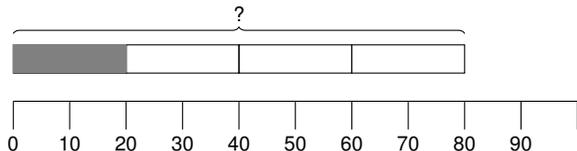
What time does Marla have to leave to be at her friend's house by a quarter after 3 if the trip takes 90 minutes?
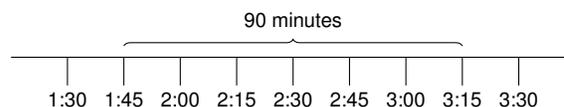
Using a number line diagram to represent time is easier if students think of digital clocks rather than round clocks. In the latter case, placing the numbers on the number line involves considering movements of the hour and minute hands.

Students learn to consider perimeter and area of rectangles, begun in Grade 3, more abstractly (MP.2). Based on work in previous grades with multiplication, spatially structuring arrays, and area, they abstract the formula for the area of a rectangle $A = l \times w$.•

Students generate and discuss advantages and disadvantages of various formulas for the perimeter length of a rectangle that is l units by w units. For example, $P = 2 \times l + 2 \times w$ has two multiplications and one addition, but $P = 2 \times (l + w)$, which has one addition and one multiplication, involves fewer calculations. The latter formula is also useful when generating all possible rectangles with a given perimeter. The length and width vary across all possible pairs whose sum is half of the perimeter (e.g., for a perimeter of 20, the length and width are all of the pairs of numbers with sum 10).

Giving verbal summaries of these formulas is also helpful. For example, a verbal summary of the basic formula, $P = l + w + l + w$, is "add the lengths of all four sides." Specific numerical instances of other formulas or mental calculations for the perimeter of a rectangle can be seen as examples of the properties of operations, e.g., $2 \times l + 2 \times w = 2 \times (l + w)$ is an example of the distributive property.

Perimeter problems often give only one length and one width, thus remembering the basic formula can help to prevent the usual error of only adding one length and one width. The formula $P = 2 \times (l + w)$ emphasizes the step of multiplying the total of the given lengths by 2. Students can make a transition from showing all length units along the sides of a rectangle or all area units within (as in Grade 3, p. 102) by drawing a rectangle showing just parts of these as a reminder of which kind of unit is being used. Writing all of the lengths around a rectangle can also be useful. Discussions of formulas such as $P = 2 \times l + 2 \times w$ can note that unlike area formulas, perimeter formulas combine length measurements to yield a length measurement.

Such abstraction and use of formulas underscores the importance of distinguishing between area and perimeter in Grade 3[3.MD.8] and maintaining the distinction in Grade 4 and later grades, where

• The formula is a generalization of the understanding, that, given a unit of length, a rectangle whose sides have length w units and l units, can be partitioned into w rows of unit squares with l squares in each row. The product $l \times w$ gives the number of unit squares in the partition, thus the area measurement is $l \times w$ square units. These square units are derived from the length unit.

Using tape diagrams to solve word problems

Lisa put two flavors of soda in a glass. There were 80 ml of soda in all. She put three times as much orange drink as strawberry. How many ml of orange did she put in?
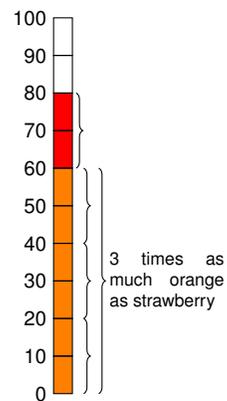
In this diagram, quantities are represented on a measurement scale.

[3.MD.8]Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

rectangle perimeter and area problems may get more complex and problem solving can benefit from knowing or being able to rapidly remind oneself of how to find an area or perimeter. By repeatedly reasoning about how to calculate areas and perimeters of rectangles, students can come to see area and perimeter formulas as summaries of all such calculations (MP.8).

Students learn to apply these understandings and formulas to the solution of real-world and mathematical problems.[4.MD.3] For example, they might be asked, "A rectangular garden has as an area of 80 square feet. It is 5 feet wide. How long is the garden?" Here, specifying the area and the width, creates an unknown factor problem (see p. 103). Similarly, students could solve perimeter problems that give the perimeter and the length of one side and ask the length of the adjacent side. Students could be challenged to solve multi-step problems such as the following. "A plan for a house includes rectangular room with an area of 60 square meters and a perimeter of 32 meters. What are the length and the width of the room?"

In Grade 4 and beyond, the mental visual images for perimeter and area from Grade 3 can support students in problem solving with these concepts. When engaging in the mathematical practice of reasoning abstractly and quantitatively (MP.2) in work with area and perimeter, students think of the situation and perhaps make a drawing. Then they recreate the "formula" with specific numbers and one unknown number as a situation equation for this particular numerical situation.• "Apply the formula" does not mean write down a memorized formula and put in known values because at Grade 4 students do not evaluate expressions (they begin this type of work in Grade 6). In Grade 4, working with perimeter and area of rectangles is still grounded in specific visualizations and numbers. These numbers can now be any of the numbers used in Grade 4 (for addition and subtraction for perimeter and for multiplication and division for area).[4.NBT.4, 4.NF.3d, 4.OA.4] By repeatedly reasoning about constructing situation equations for perimeter and area involving specific numbers and an unknown number, students will build a foundation for applying area, perimeter, and other formulas by substituting specific values for the variables in later grades.

[4.MD.3]Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

• "Situation equation" refers to the idea that the student constructs an equation as a representation of a situation rather than identifying the situation as an example of a familiar equation.

[4.NBT.4]Fluently add and subtract multi-digit whole numbers using the standard algorithm.

[4.NF.3d]Solve word problems involving addition and subtraction of fractions referring to the same whole and having like denominators, e.g., by using visual fraction models and equations to represent the problem.

[4.OA.4]Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.

Understand concepts of angle and measure angles   Angle measure is a "turning point" in the study of geometry. Students often find angles and angle measure to be difficult concepts to learn, but that learning allows them to engage in interesting and important mathematics. An *angle* is the union of two rays, *a* and *b*, with the same initial point *P*. The rays can be made to coincide by rotating one to the other about *P*; this rotation determines the size of the angle between *a* and *b*. The rays are sometimes called the *sides* of the angles.

Another way of saying this is that each ray determines a direction and the angle size measures the change from one direction to the other. (This illustrates how angle measure is related to the concepts of parallel and perpendicular lines in Grade 4 geometry.) A clockwise rotation is considered positive in surveying or turtle geometry; but a counterclockwise rotation is considered positive in Euclidean geometry. Angles are measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and degrees are the unit used to measure angles in elementary school. A full rotation is thus 360°.

Two angles are called *complementary* if their measurements have the sum of 90°. Two angles are called *supplementary* if their measurements have the sum of 180°. Two angles with the same vertex that overlap only at a boundary (i.e., share a side) are called *adjacent angles*.

Like length, area, and volume, angle measure is additive: The sum of the measurements of *adjacent angles* is the measurement of the angle formed by their union. This leads to other important properties. If a right angle is decomposed into two adjacent angles, the sum is 90°, thus they are complementary. Two adjacent angles that compose a "straight angle" of 180° must be supplementary. In some situations (see margin), such properties allow logical progressions of statements (MP.3).

As with all measureable attributes, students must first recognize the attribute of angle measure, and distinguish it from other attributes. This may not appear too difficult, as the measure of angles and rotations appears to knowledgeable adults as quite different than attributes such as length and area. However, the unique nature of angle size leads many students to initially confuse angle measure with other, more familiar, attributes. Even in contexts designed to evoke a dynamic image of turning, such as hinges or doors, many students use the length between the endpoints, thus teachers find it useful to repeatedly discuss such cognitive "traps."

As with other concepts (e.g., see the K–6 Geometry Progression), students need varied examples and explicit discussions to avoid learning limited ideas about measuring angles (e.g., misconceptions that a right angle is an angle that points to the right, or two right angles represented with different orientations are not equal in
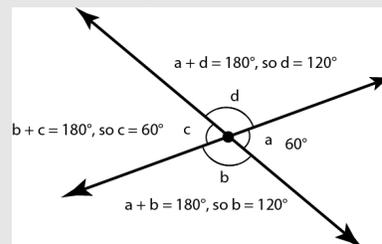
Angle terminology
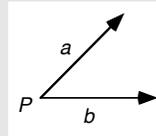
P is called the vertex *of the angle and the rays a and b are called the* arms.

Types of angles

name	measurement
right angle	90°
straight angle	180°
acute angle	between 0 and 90°
obtuse angle	between 90° and 180°
reflex angle	between 180° and 360°

Angles created by the intersection of two lines

$a + d = 180°$, so $d = 120°$

$b + c = 180°$, so $c = 60°$

$a + b = 180°$, so $b = 120°$

a 60°

When two lines intersect, they form four angles. If the measurement of one is known (e.g., angle a is 60°), the measurement of the other three can be determined.

Two representations of three angles
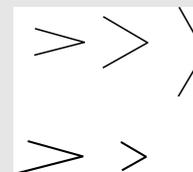
Initially, some students may correctly compare angle sizes only if all the line segments are the same length (as shown in the top row). If the lengths of the line segments are different (as shown in the bottom row), these students base their judgments on the lengths of the segments, the distances between their endpoints, or even the area of the triangles determined by the drawn arms. They believe that the angles in the bottom row decrease in size from left to right, although they have, respectively, the same angle measurements as those in the top row.

measure). If examples and tasks are not varied, students can develop incomplete and inaccurate notions. For example, some come to associate all slanted lines with 45° measures and horizontal and vertical lines with measures of 90°. Others believe angles can be "read off" a protractor in "standard" position, that is, a base is horizontal, even if neither arm of the angle is horizontal. Measuring and then sketching many angles with no horizontal or vertical arms,[4.MD.6] perhaps initially using circular 360° protractors, can help students avoid such limited conceptions.

As with length, area, and volume, children need to understand equal partitioning and unit iteration to understand angle and turn measure. Whether defined as more statically as the measure of the figure formed by the intersection of two rays or as turning, having a given angle measure involves a relationship between components of plane figures and therefore is a *property* (see the Overview in the K–6 Geometry Progression).[4.G.2]

Given the complexity of angles and angle measure, it is unsurprising that students in the early and elementary grades often form separate concepts of angles as figures and turns, and may have separate notions for different turn contexts (e.g., unlimited rotation as a fan vs. a hinge) and for various "bends."

However, students can develop more accurate and useful angle and angle measure concepts if presented with angles in a variety of situations. They learn to find the common features of superficially different situations such as turns in navigation, slopes, bends, corners, and openings. With guidance, they learn to represent an angle in any of these contexts as two rays, even when both rays are not explicitly represented in the context; for example, the horizontal or vertical in situations that involve slope (e.g., roads or ramps), or the angle determined by looking up from the horizon to a tree- or mountain-top. Eventually they abstract the common attributes of the situations as angles (which are represented with rays and a vertex, MP.4) and angle measurements (MP.2). To accomplish the latter, students integrate turns, and a general, dynamic understanding of angle measure-as-rotation, into their understandings of angles-as-objects. Computer manipulatives and tools can help children bring such a dynamic concept of angle measure to an explicit level of awareness. For example, dynamic geometry environments can provide multiple linked representations, such as a screen drawing that students can "drag" which is connected to a numerical representation of angle size. Games based on similar notions are particularly effective when students manipulate not the arms of the angle itself, but a representation of rotation (a small circular diagram with radii that, when manipulated, change the size of the target angle turned).

Students with an accurate conception of angle can recognize that angle measure is *additive*.[4.MD.7] As with length, area, and volume, when an angle is decomposed into non–overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Students can then solve interesting and challenging addition and

subtraction problems to find the measurements of unknown angles on a diagram in real–world and mathematical problems. For exam-ple, they can find the measurements of angles formed a pair of inter-secting lines, as illustrated above, or given a diagram showing the measurement of one angle, find the measurement of its complement. They can use a protractor to check, not to check their reasoning, but to ensure that they develop full understanding of the mathematics and mental images for important benchmark angles (e.g., 30°, 45°, 60°, and 90°).

Such reasoning can be challenged with many situations as il-lustrated in the margin.

Similar activities can be done with drawings of shapes using right angles and half of a right angle to develop the important bench-marks of 90° and 45°.

Missing measurements can also be done in the turtle geometry con-text, building on the previous work. Note that unguided use of Logo's tur-tle geometry does not necessary de-velop strong angle concepts. How-ever, if teachers emphasize mathe-matical tasks and, within those tasks, the difference between the angle of rotation the turtle makes (in a poly-gon, the external angle) and the an-gle formed (internal angle) and inte-grates the two, students can develop accurate and comprehensive under-standings of angle measure.   For example, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many de-grees would the turtle turn? What is the measure of the resulting an-gle? Such questions help to connect what are often initially isolated ideas about angle conceptions.

These understandings support students in finding all the missing length and angle measurements in situations such as the examples in the margin (compare to the missing measurements problems for Grade 2 and Grade 3).

Determining angle measurements

Students might be asked to determine the measurements of:
∠BOD, ∠BOF, ∠ODE, ∠CDE, ∠CDJ, ∠BHG.

Missing measurements: Length (top) and length and angle (turn)

Assume equilateral triangle

Assume regular hexagon

Assume all segments in "steps" are the same length.

Students are asked to determine the missing lengths. They might first work on paper to figure out how far the Logo turtle would have to travel to finish drawing the house, then type in Logo commands to verify their reasoning and calculations.

Grade 5

Convert like measurement units within a given measurement system   In Grade 5, students extend their abilities from Grade 4 to express measurements in larger or smaller units within a measurement system.[4.MD.1, 5.MD.1] This is an excellent opportunity to reinforce notions of place value for whole numbers and decimals, and connection between fractions and decimals (e.g., $2\frac{1}{2}$ meters can be expressed as 2.5 meters or 250 centimeters). For example, building on the table from Grade 4, Grade 5 students might complete a table of equivalent measurements in feet and inches.

Grade 5 students also learn and use such conversions in solving multi-step, real-world problems (see example in the margin).

Understand concepts of volume and relate volume to multiplication and to addition   The major emphasis for measurement in Grade 5 is volume. Volume not only introduces a third dimension and thus a significant challenge to students' spatial structuring, but also complexity in the nature of the materials measured. That is, solid units are "packed," such as cubes in a three-dimensional array, whereas a liquid "fills" three-dimensional space, taking the shape of the container. As noted earlier (see Overview, also Grades 1 and 3), the unit structure for liquid measurement may be psychologically one-dimensional for some students.

"Packing" volume is more difficult than iterating a unit to measure length and measuring area by tiling. Students learn about a unit of volume, such as a cube with a side length of 1 unit, called a unit cube.[5.MD.3] They pack cubes (without gaps) into right rectangular prisms and count the cubes to determine the volume or build right rectangular prisms from cubes and see the layers as they build.[5.MD.4] They can use the results to compare the volume of right rectangular prisms that have different dimensions. Such experiences enable students to extend their spatial structuring from two to three dimensions (see the K–6 Geometry Progression). That is, they learn to both mentally decompose and recompose a right rectangular prism built from cubes into layers, each of which is composed of rows and columns. That is, given the prism, they have to be able to decompose it, understanding that it can be partitioned into layers, and each layer partitioned into rows, and each row into cubes. They also have to be able to compose such as structure, multiplicatively, back into higher units. That is, they eventually learn to conceptualize a layer as a unit that itself is composed of units of units—rows, each row composed of individual cubes—and they iterate that structure. Thus, they might predict the number of cubes that will be needed to fill a box given the net of the box (see margin on p. 111).

Another complexity of volume is connecting measurements obtained by "packing" and measurements obtained by "filling." Often, for example, students will respond that a box can be filled with 24

[4.MD.1] Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

[5.MD.1] Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Measurement conversions

feet	0				
inches	0	1	2	3	

In Grade 6, this table can be discussed in terms of ratios and, in Grade 7, proportional relationships (see the Ratios and Proportional Relationships Progression). In Grade 5, however, the main focus is on arriving at the measurements that generate the table.

Multi-step problem with unit conversion

Kumi spent a fifth of her money on lunch. She then spent half of what remained. She bought a card game for $3, a book for $8.50, and candy for 90 cents. How much money did she have at first?

$$3.00 + 8.50 + 0.90 = 12.40$$

$31

6.20   12.40   12.40
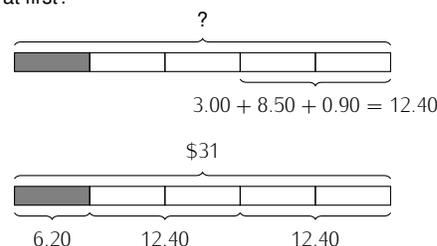
Students can use tape diagrams to represent problems that involve conversion of units, drawing diagrams of important features and relationships (MP.1).

[5.MD.3] Recognize volume as an attribute of solid figures and understand concepts of volume measurement.

[5.MD.4] Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.

centimeter cubes, or build a structure of 24 cubes, and still think of the 24 as individual, often discrete, not necessarily *units of volume.* They may, for example, not respond confidently and correctly when asked to fill a graduated cylinder marked in cubic centimeters with the amount of liquid that would fill the box. That is, they have not yet connected their ideas about filling volume with those concerning packing volume. Students learn to move between their conceptions of filling and of packing, e.g., using the same container, both filling (from a graduated cylinder marked in ml or cc) and packing (with cubes that are each 1 cubic centimeter). Comparing and discussing the volume units and what they represent• can help students learn a general, complete, and interconnected conceptualization of volume as filling three–dimensional space.

Students then learn to determine the volumes of several right rectangular prisms, using cubic centimeters, cubic inches, and cubic feet. With guidance, they learn to increasingly apply multiplicative reasoning to determine volumes, looking for and making use of structure (MP.7). That is, they understand that multiplying the length times the width of a right rectangular prism can be viewed as determining how many cubes would be in each layer if the prism were packed with or built up from unit cubes.[5.MD.5a] They also learn that the height of the prism tells how many layers would fit in the prism. That is, they understand that volume is a derived attribute that, once a length unit is specified, can be computed as the product of three length measurements or as the product of one area and one length measurement.

Then, students can learn the formulas $V = l \times w \times h$ and $V = b \times h$ for right rectangular prisms as efficient methods for computing volume, maintaining the connection between these methods and their previous work with computing the number of unit cubes that pack a right rectangular prism.[5.MD.5b] They use these competencies to find the volumes of right rectangular prisms with edges whose lengths are whole numbers and solve real–world and mathematical problems involving such prisms.

Students also recognize that volume is additive (see Overview) and they find the total volume of solid figures composed of two right rectangular prisms.[5.MD.5c] For example, students might design a science station for the ocean floor that is composed of several rooms that are right rectangular prisms and that meet a set criterion specifying the total volume of the station. They draw their station (e.g., using an isometric grid, MP.7) and justify how their design meets the criterion (MP.1).

• For example, cc abbreviates cubic centimeters, whether it refers to measurements made using a graduated cylinder marked in cc or to measurements made by packing with centimeter cubes.

Net for five faces of a right rectangular prism
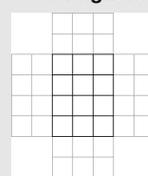
Students are given a net and asked to predict the number of cubes required to fill the container formed by the net. In such tasks, students may initially count single cubes or repeatedly add the number of cubes in a row to determine the number in each layer, and repeatedly add the number in each layer to find the total number of unit cubes. In folding the net to make the shape, students can see how the side rectangles fit together and determine the number of layers.

[5.MD.5] Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

   a Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

   b Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole-number edge lengths in the context of solving real world and mathematical problems.

   c Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real world problems.

Where this progression is heading

In Grade 6, students build on their understanding of length, area, and volume measurement, learning how to compute areas of right triangles and other special figures and volumes of right rectangular prisms that do not have measurements given in whole numbers. To do this, they use dissection arguments. These rely on the understanding that area and volume measures are additive, together with decomposition of plane and solid shapes (see the K–6 Geometry Progression) into shapes whose measurements students already know how to compute (MP.1, MP.7). In Grade 7, they use their understanding of length and area in learning and using formulas for the circumference and area of circles. In Grade 8, they use their understanding of volume in learning and using formulas for the volumes of cones, cylinders, and spheres. In high school, students learn formulas for volumes of pyramids and revisit the formulas from Grades 7 and 8, explaining them with dissection arguments, Cavalieri's Principle, and informal limit arguments. (See the Geometry Progression for Grades 7–8 and high school.)

In Grade 6, understanding of length units and spatial structuring comes into play as students learn to plot points in the coordinate plane. (See the Number System Progression.)

Students use their knowledge of measurement and units of measurement in Grades 6–8, coming to see conversions between two units of measurement as describing proportional relationships. (See the Ratios and Proportional Relationships Progression.)

Geometry, K–6

Overview

Like core knowledge of number, core geometrical knowledge appears to be a universal capability of the human mind. Geometric and spatial thinking are important in and of themselves, because they connect mathematics with the physical world, and play an important role in modeling phenomena whose origins are not necessarily physical, for example, as networks or graphs. They are also important because they support the development of number and arithmetic concepts and skills. Thus, geometry is essential for all grade levels for many reasons: its mathematical content, its roles in physical sciences, engineering, and many other subjects, and its strong aesthetic connections.

This progression discusses the most important goals for elementary geometry according to three categories.

- Geometric shapes, their components (e.g., sides, angles, faces), their properties, and their categorization based on those properties.

- Composing and decomposing geometric shapes.

- Spatial relations and spatial structuring.

Geometric shapes, components, and properties. Students develop through a series of levels of geometric and spatial thinking. As with all of the domains discussed in the Progressions, this development depends on instructional experiences. Initially, students cannot reliably distinguish between examples and nonexamples of categories of shapes, such as triangles, rectangles, and squares.● With experience, they progress to the next level of thinking, recognizing shapes in ways that are visual or syncretic (a fusion of differing systems). At this level, students can recognize shapes as wholes, but cannot form mathematically-constrained mental images of them. A given figure is a rectangle, for example, because "it looks like a door." They do not explicitly think about the components or

● In formal mathematics, a geometric shape is a boundary of a region, e.g., "circle" is the boundary of a disk. This distinction is not expected in elementary school.

about the defining attributes, or properties, of shapes. Students then move to a descriptive level in which they can think about the components of shapes, such as triangles having three sides. For example, kindergartners can decide whether all of the sides of a shape are straight and they can count the sides. They also can discuss if the shape is closed• and thus convince themselves that a three-sided shape is a triangle even if it is "very skinny" (e.g., an isosceles triangle with large obtuse angle).

At the analytic level, students recognize and characterize shapes by their *properties*. For instance, a student might think of a square as a figure that has four equal sides and four right angles. Different components of shapes are the focus at different grades, for instance, second graders measure lengths and fourth graders measure angles (see the Geometric Measurement Progression). Students find that some combinations of properties signal certain classes of figures and some do not; thus the seeds of geometric implication are planted. However, only at the next level, abstraction, do students see relationships between classes of figures (e.g., understand that a square is a rectangle because it has all the properties of rectangles).• Competence at this level affords the learning of higher-level geometry, including deductive arguments and proof.

Thus, learning geometry cannot progress in the same way as learning number, where the size of the numbers is gradually increased and new kinds of numbers are considered later. In learning about shapes, it is important to vary the examples in many ways so that students do not learn limited concepts that they must later unlearn. From Kindergarten on, students experience all of the properties of shapes that they will study in Grades K–7, recognizing and working with these properties in increasingly sophisticated ways. The Standards describe particular aspects on which students at that grade level work systematically, deeply, and extensively, building on related experiences in previous years. This progression describes a curricular pathway that illustrates possibilities for work at each grade level, and how it differs from and extends work in earlier grades.

Composing and decomposing. As with their learning of shapes, components, and properties, students follow a progression to learn about the composition and decomposition of shapes. Initially, they lack competence in composing geometric shapes. With experience, they gain abilities to combine shapes into pictures—first, through trial and error, then gradually using attributes. Finally, they are able to synthesize combinations of shapes into new shapes.•

Students compose new shapes by putting two or more shapes together and discuss the shapes involved as the parts and the totals. They decompose shapes in two ways. They take away a part by covering the total with a part (for example, covering the "top" of a triangle with a smaller triangle to make a trapezoid). And they take shapes apart by building a copy beside the original shape to see what shapes that shape can be decomposed into (initially, they may

• A two-dimensional shape with straight sides is closed if exactly two sides meet at every vertex, every side meets exactly two other sides, and no sides cross each other.

Levels of geometric thinking

Visual/syncretic. Students recognize shapes, e.g., a rectangle "looks like a door."

Descriptive. Students perceive properties of shapes, e.g., a rectangle has four sides, all its sides are straight, opposite sides have equal length.

Analytic. Students characterize shapes by their properties, e.g., a rectangle has opposite sides of equal length and four right angles.

Abstract. Students understand that a rectangle is a parallelogram because it has all the properties of parallelograms.

• Note that in the U.S. the term "trapezoid" may have two different meanings. In their study *The Classification of Quadrilaterals* (Information Age Publishing, 2008), Usiskin et al. call these the exclusive and inclusive definitions:

- T(E): a trapezoid is a quadrilateral with exactly one pair of parallel sides.

- T(I): a trapezoid is a quadrilateral with at least one pair of parallel sides.

These different meanings result in different classifications at the analytic level. According to T(E), a parallelogram is not a trapezoid; according to T(I), a parallelogram is a trapezoid. At the analytic level, the question of whether a parallelogram is a trapezoid may arise, just as the question of whether a square is a rectangle may arise. At the visual or descriptive levels, the different definitions are unlikely to affect students or curriculum.

Both definitions are legitimate. However, Usiskin et al. conclude, "The preponderance of advantages to the inclusive definition of trapezoid has caused all the articles we could find on the subject, and most college-bound geometry books, to favor the inclusive definition."

• **A note about research** The ability to describe, use, and visualize the effects of composing and decomposing geometric regions is significant in that the concepts and actions of creating and then iterating units and higher-order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis. Additionally, there is suggestive evidence that this type of composition corresponds with, and may support, children's ability to compose and decompose numbers.

need to make the decomposition on top of the total shape). With experience, students are able to use a composed shape as a new unit in making other shapes. Grade 1 students make and use such a unit of units (for example, making a square or a rectangle from two identical right triangles, then making pictures or patterns with such squares or rectangles). Grade 2 students make and use three levels of units (making an isosceles triangle from two 1″ by 2″ right triangles, then making a rhombus from two of such isosceles triangles, and then using such a rhombus with other shapes to make a picture or a pattern). Grade 2 students also compose with two such units of units (for example, making adjacent strips from a shorter parallelogram made from a 1″ by 2″ rectangle and two right triangles and a longer parallelogram made from a 1″ by 3″ parallelogram and the same two right triangles). Grade 1 students also rearrange a composite shape to make a related shape, for example, they change a 1″ by 2″ rectangle made from two right triangles into an isosceles triangle by flipping one right triangle. They explore such rearrangements of the two right triangles more systematically by matching the short right angle side (a tall isosceles triangle and a parallelogram with a "little slant"), then the long right angle sides (a short isosceles triangle and a parallelogram with a "long slant").• Grade 2 students rearrange more complex shapes, for example, changing a parallelogram made from a rectangle and two right triangles into a trapezoid by flipping one of the right triangles to make a longer and a shorter parallel side.

Composing and decomposing requires and thus builds experience with properties such as having equal lengths or equal angles.

Spatial structuring and spatial relations. Early composition and decomposition of shape is a foundation for spatial structuring, an important case of geometric composition and decomposition. Students need to conceptually structure an array to understand two–dimensional objects and sets of such objects in two–dimensional space as truly two–dimensional. Such spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because it takes previously abstracted items as content and integrates them to form new structures. For two–dimensional arrays, students must see a composite of squares (iterated units) and as a composite of rows or columns (units of units). Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane. Spatial relations such as above/below and right/left are understood within such spatial structures. These understandings begin informally, later becoming more formal.

The ability to structure a two–dimensional rectangular region into rows and columns of squares requires extended experiences with shapes derived from squares (e.g., squares, rectangles, and right

• Students are not expected to learn terms such as "isosceles" at these grades. Between Kindergarten and Grade 3, expectations for use of specific terms and identification of shapes are:

- K.G.1 Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*. [*Note.* The cluster heading for this standard provides guidance about the shapes intended: squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres.]

- 1.G.3 Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. . . .

- 2.G.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. . . .

- 3.G.1 . . . Recognize rhombuses, rectangles, and squares as examples of quadrilaterals. . . .

- 3.G.2 Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.*

triangles) and with arrays of contiguous squares that form patterns. Development of this ability benefits from experience with compositions, decompositions, and iterations of the two, but it requires extensive experience with arrays.

Students make pictures from shapes whose sides or points touch, and they fill in outline puzzles. These gradually become more elaborate, and students build mental visualizations that enable them to move from trial and error rotating of a shape to planning the orientation and moving the shape as it moves toward the target location. Rows and columns are important units of units within square arrays for the initial study of area, and squares of 1 by 1, 1 by 10, and 10 by 10 are the units, units of units, and units of units of units used in area models of two-digit multiplication in Grade 4. Layers of three-dimensional shapes are central for studying volume in Grade 5. Each layer of a right rectangular prism can also be structured in rows and columns, such layers can also be viewed as units of units of units. That is, as 1000 is a unit (one thousand) of units (one hundred) of units (tens) of units (singletons), a right rectangular prism can be considered a unit (solid, or three-dimensional array) of units (layers) of units (rows) of units (unit cubes).

Summary. The Standards for Kindergarten, Grade 1, and Grade 2 focus on three major aspects of geometry. Students build understandings of shapes and their properties, becoming able to do and discuss increasingly elaborate compositions, decompositions, and iterations of the two, as well as spatial structures and relations. In Grade 2, students begin the formal study of measurement, learning to use units of length and use and understand rulers. Measurement of angles and parallelism are a focus in Grades 3, 4, and 5. At Grade 3, students begin to consider relationships of shape categories, considering two levels of subcategories (e.g., rectangles are parallelograms and squares are rectangles). They complete this categorization in Grade 5 with all necessary levels of categories and with the understanding that any property of a category also applies to all shapes in any of its subcategories. They understand that some categories overlap (e.g., not all parallelograms are rectangles) and some are disjoint (e.g., no square is a triangle), and they connect these with their understanding of categories and subcategories. Spatial structuring for two- and three-dimensional regions is used to understand what it means to measure area and volume of the simplest shapes in those dimensions: rectangles with whole-number side lengths at Grade 3 and right rectangular prisms with whole-number edge lengths at Grade 5 (see the Geometric Measurement Progression). Students extend these understandings to regions with fractional side and edge lengths in more abstract settings—rectangles in Grade 5 (see the Number and Operations—Fractions Progression) and right rectangular prisms in Grade 6 (see this progression).

Kindergarten

Understanding and describing shapes and space is one of the two critical areas of Kindergarten mathematics. Students develop geometric concepts and spatial reasoning from experience with two perspectives on space: the shapes of objects and the relative positions of objects.

In the domain of shape, students learn to match two-dimensional shapes even when the shapes have different orientations.[K.G.4] They learn to name shapes such as circles, triangles, and squares, whose names occur in everyday language, and distinguish them from nonexamples of these categories, often based initially on visual prototypes. For example, they can distinguish the most typical examples of triangles from the obvious nonexamples.

From experiences with varied examples of these shapes (e.g., the variants shown in the margin), students extend their initial intuitions to increasingly comprehensive and accurate intuitive concept images of each shape category.[•] These richer concept images support students' ability to perceive a variety of shapes in their environments (MP.7, "Mathematically proficient students look closely to discern a . . . structure") and describe these shapes in their own words. This includes recognizing and informally naming three-dimensional shapes, e.g., "balls," "boxes," "cans." Such learning might also occur in the context of solving problems that arise in construction of block buildings and in drawing pictures, simple maps, and so forth.

Students then refine their informal language by learning mathematical concepts and vocabulary so as to increasingly describe their physical world from geometric perspectives, e.g., shape, orientation, spatial relations (MP.4). They increase their knowledge of a variety of shapes, including circles, triangles, squares, rectangles, and special cases of other shapes such as regular hexagons, and trapezoids with unequal bases and non-parallel sides of equal length. They learn to sort shapes according to these categories (MP.7, "Young students, for example, . . . may sort a collection of shapes according to how many sides the shapes have"). The need to explain their decisions about shape names or classifications prompts students to attend to and describe certain features of the shapes.[K.G.4] That is, concept images and names they have learned for the shapes are the raw material from which they can abstract common features (MP.2, "Mathematically proficient students have the ability to abstract a given situation"). This also supports their learning to represent shapes informally with drawings and by building them from components (e.g., manipulatives such as sticks).[K.G.5] With repeated experiences such as these, students become more precise (MP.6). They begin to attend to attributes, such as being a triangle, square, or rectangle, and being *closed* figures with *straight* sides. Similarly, they attend to the lengths of sides and, in simple situations, the size of angles when comparing shapes.

Students also begin to name and describe three-dimensional

[K.G.4]Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

Exemplars are the typical visual prototypes of the shape category.

Variants are other examples of the shape category.

Palpable distractors are nonexamples with little or no overall resemblance to the exemplars.

Difficult distractors are visually similar to examples but lack at least one defining attribute.

[•] Tall and Vinner describe *concept image* as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." (See "Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity," *Educational Studies in Mathematics*, *12*, pp. 151–169.) This term was formulated by Shlomo Vinner in 1980.

[K.G.1]Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

[K.G.5]Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.

shapes with mathematical vocabulary, such as "sphere," "cube," "cylinder," and "cone."[K.G.1] They identify faces of three-dimensional shapes as two-dimensional geometric figures[K.G.4] and explicitly identify shapes as two-dimensional ("flat" or lying in a plane) or three-dimensional ("solid").[K.G.3]
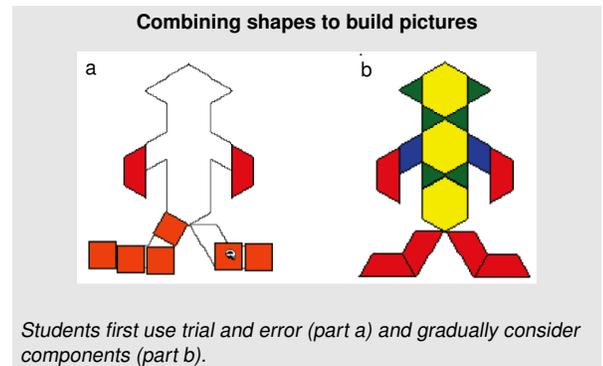
A second important area for kindergartners is the composition of geometric figures. Students not only build shapes from components, but also compose shapes to build pictures and designs. Initially lacking competence in composing geometric shapes, they gain abilities to combine shapes—first by trial and error and gradually by considering components—into pictures. At first, side length is the only component considered. Later experience brings an intuitive appreciation of angle size.

Students combine two-dimensional shapes and solve problems such as deciding which piece will fit into a space in a puzzle, intuitively using geometric motions (slides, flips, and turns, the informal names for translations, reflections, and rotations, respectively). They can construct their own outline puzzles and exchange them, solving each other's.

Finally, in the domain of spatial reasoning, students discuss not only shape and orientation, but also the relative positions of objects, using terms such as "above," "below," "next to," "behind," "in front of," and "beside."[K.G.1] They use these spatial reasoning competencies, along with their growing knowledge of three-dimensional shapes and their ability to compose them, to model objects in their environment, e.g., building a simple representation of the classroom using unit blocks and/or other solids (MP.4).

[K.G.4]Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).

[K.G.3]Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").

Combining shapes to build pictures

Students first use trial and error (part a) and gradually consider components (part b).

[K.G.1]Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

Grade 1

In Grade 1, students reason about shapes. They describe and classify shapes, including drawings, manipulatives, and physical–world objects, in terms of their geometric *attributes*. That is, based on early work recognizing, naming, sorting, and building shapes from components, they describe in their own words why a shape belongs to a given category, such as squares, triangles, circles, rectangles, rhombuses, (regular) hexagons, and trapezoids (with bases of different lengths and nonparallel sides of the same length). In doing so, they differentiate between geometrically defining attributes (e.g., "hexagons have six straight sides")• and nondefining attributes (e.g., color, overall size, or orientation).[1.G.1] For example, they might say of this shape, "This has to go with the squares, because all four sides are the same, and these are square corners. It doesn't matter which way it's turned" (MP.3, MP.7). They explain why the variants shown earlier (p. 117) are members of familiar shape categories and why the difficult distractors are not, and they draw examples and nonexamples of the shape categories. Students learn to sort shapes accurately and exhaustively based on these attributes, describing the similarities and differences of these familiar shapes and shape categories (MP.7, MP.8).

From the early beginnings of informally matching shapes and solving simple shape puzzles, students learn to intentionally compose and decompose plane and solid figures (e.g., putting two congruent isosceles triangles together with the explicit purpose of making a rhombus•),[1.G.2] building understanding of part-whole relationships as well as the properties of the original and composite shapes. In this way, they learn to perceive a combination of shapes as a single new shape (e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and simultaneously seeing the rhombus and the two triangles). Thus, they develop competencies that include solving shape puzzles and constructing designs with shapes, creating and maintaining a shape as a unit, and combining shapes to create composite shapes that are conceptualized as independent entities (MP.2). They then learn to substitute one composite shape for another congruent composite composed of different parts.

Students build these competencies, often more slowly, in the domain of three–dimensional shapes. For example, students may intentionally combine two right triangular prisms to create a right rectangular prism, and recognize that each triangular prism is half of the rectangular prism as well as the two–dimensional version (each triangular face is half of the rectangular face that they compose[1.G.3]). They also show recognition of the composite shape of "arch," e.g., recognize arches in composites of blocks like the one in the margin. (Note that the process of combining shapes to create a composite shape is much like combining 10 ones to make 1 ten.) Even simple compositions, such as building a floor or wall of rectangular prisms, build a foundation for later mathematics.
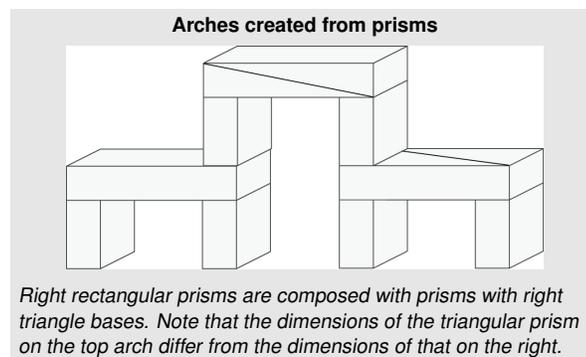
• A given type of shape may have more than one defining attribute. For example, one defining attribute for a rectangle is "two pairs of opposite sides of equal length and four right angles." Another is "four straight sides and four right angles."

[1.G.1] Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.

• Students do not study congruence until Grade 8 and need not use the term "congruent" in early grades. They might describe the triangles as "same size and same shape" or say "they match exactly."

[1.G.2] Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. [4]

[4] Students do not need to learn formal names such as "right rectangular prism."

Arches created from prisms

Right rectangular prisms are composed with prisms with right triangle bases. Note that the dimensions of the triangular prism on the top arch differ from the dimensions of that on the right.

[1.G.3] Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

As students combine shapes, they continue to develop their sophistication in describing geometric attributes and properties and determining how shapes are alike and different, building foundations for measurement and initial understandings of properties such as congruence and symmetry. Students can learn to use their intuitive understandings of measurement, congruence, and symmetry to guide their work on tasks such as solving puzzles and making simple origami constructions by folding paper to make a given two- or three-dimensional shape (MP.1).•

• For example, students might fold a square of paper once to make a triangle or nonsquare rectangle. For examples of other simple two- and three-dimensional origami constructions, see http://www.origami-instructions.com/simple-origami.html.

Grade 2

In Kindergarten, students learned to identify shapes by their names,[K.G.1] but were not expected to characterize shapes by defining attributes (see the footnote on p. 114). In Grade 2, students learn to name and describe defining attributes of categories of two-dimensional shapes, including circles, triangles, squares, rectangles, rhombuses, trapezoids, and the general category of quadrilateral. They describe pentagons, hexagons, septagons, octagons, and other polygons by the number of sides, for example, describing a septagon as either a "seven-gon" or simply "seven-sided shape" (MP.2). Because they have developed both verbal descriptions of these categories and their defining attributes and a rich store of associated mental images, they are able to draw shapes with specified attributes, such as a shape with five sides or a shape with six angles.[2.G.1] They can represent these shapes' attributes accurately (within the constraints of fine motor skills). They use length to identify the properties of shapes (e.g., a specific figure is a rhombus because all four of its sides have equal length). They recognize right angles, and can explain the distinction between a rectangle and a parallelogram without right angles and with sides of different lengths.

Students learn to combine their composition and decomposition competencies to build and operate on composite units (units of units), intentionally substituting arrangements or composites of smaller shapes or substituting several larger shapes for many smaller shapes, using geometric knowledge and spatial reasoning to develop foundations for area, fraction, and proportion. For example, they build the same shape from different parts, e.g., making with pattern blocks, a regular hexagon from two trapezoids, three rhombuses, or six equilateral triangles. They recognize that the hexagonal faces of these constructions have equal area, that each trapezoid has half of that area, and each rhombus has a third of that area.[2.G.3]

This example emphasizes the fraction concepts that are developed; students can build and recognize more difficult composite shapes and solve puzzles with numerous pieces. For example, a tangram is a special set of seven shapes which compose an isosceles right triangle. The tangram pieces can be used to make many different configurations and tangram puzzles are often posed by showing pictures of these configurations as silhouettes or outlines. These pictures often are made more difficult by orienting the shapes so that the sides of right angles are not parallel to the edges of the page on which they are displayed. Such pictures often do not show a grid that shows the composing shapes and are generally not preceded by analysis of the composing shapes.

Students also explore decompositions of shapes into regions that are congruent or have equal area.[2.G.3] For example, two squares can be partitioned into fourths in different ways. Any of these fourths represents an equal share of the shape (e.g., "the same amount of cake") even though they have different shapes.

[K.G.1]Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as *above*, *below*, *beside*, *in front of*, *behind*, and *next to*.

[2.G.1]Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.[5] Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

[5]Sizes are compared directly or visually, not compared by measuring.

Different pattern blocks compose a regular hexagon

[2.G.3]Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Squares partitioned into fourths

These different partitions of a square afford the opportunity for students to identify correspondences between the differently-shaped fourths (MP.1), and to explain how one of the fourths on the left can be transformed into one of the fourths on the right (MP.7).

Another type of composition and decomposition is essential to students' mathematical development—*spatial structuring.* Students need to conceptually structure an array to understand two-dimensional regions as truly two-dimensional. This involves more learning than is sometimes assumed. Students need to understand how a rectangle can be tiled with squares lined up in rows and columns.[2.G.2] At the lowest level of thinking, students draw or place shapes inside the rectangle, but do not cover the entire region. Only at the later levels do all the squares align vertically and horizontally, as the students learn to compose this two-dimensional shape as a collection of rows of squares and as a collection of columns of squares (MP.7).

Spatial structuring is thus the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Spatial structuring builds on previous shape composition, because previously abstracted items (e.g., squares, including composites made up of squares) are used as the content of new mental structures. Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units. At first, students might tile a rectangle with identical squares or draw such arrays and then count the number of squares one-by-one. In the lowest levels of the progression, they may even lose count of or double-count some squares. As the mental structuring process helps them organize their counting, they become more systematic, using the array structure to guide the quantification. Eventually, they begin to use repeated addition of the number in each row or each column. Such spatial structuring precedes meaningful mathematical use of the structures, including multiplication and, later, area, volume, and the coordinate plane.

Foundational activities, such as forming arrays by tiling a rectangle with identical squares (as in building a floor or wall from blocks) should have developed students' basic spatial structuring competencies before second grade—if not, teachers should ensure that their students learn these skills. Spatial structuring can be further developed with several activities with grids. Games such as "battleship" can be useful in this regard.

Another useful type of instructional activity is copying and creating designs on grids. Students can copy designs drawn on grid paper by placing manipulative squares and right triangles onto other copies of the grid. They can also create their own designs, draw their creations on grid paper, and exchange them, copying each others' designs.

Another, more complex, activity designing tessellations by iterating a "core square." Students design a unit composed of smaller units: a core square composed of a 2 by 2 array of squares filled with square or right triangular regions. They then create the tessellation ("quilt") by iterating that core in the plane. This builds

[2.G.2]Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Levels of thinking in spatial structuring
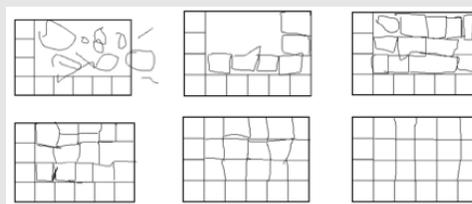
Levels of thinking portrayed by different students as they attempted to complete a drawing of an array of squares, given one column and row. This was an assessment, not an instructional task.

Copying and creating designs on grid paper
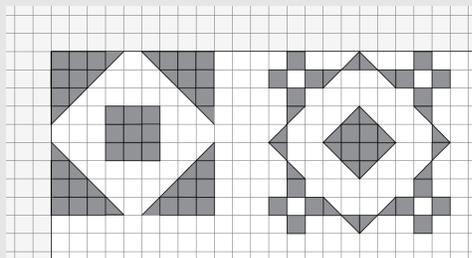
Students can copy designs such as these, using only squares (all of the same size) and isosceles right triangles (half of the square) as manipulatives, creating their copies on paper with grid squares of the same size as the manipulative square.

spatial structuring because students are iterating "units of units" and reflecting on the resulting structures. Computer software can aid in this iteration.

These various types of composition and decomposition experiences simultaneously develop students' visualization skills, including recognizing, applying, and anticipating (MP.1) the effects of applying rigid motions (slides, flips, and turns) to two-dimensional shapes.

"Core squares" iterated to make a tessellation
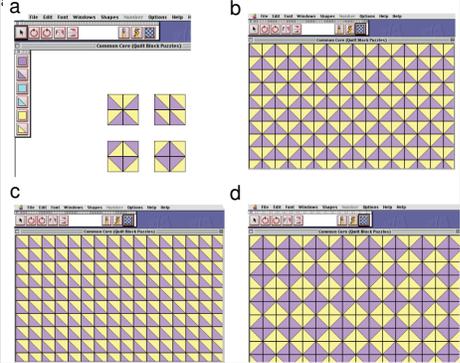
In the software environment illustrated above (Pattern Blocks and Mini-Quilts software), students need to be explicitly aware of the transformations they are using in order to use slide, flip, and turn tools. At any time, they can tessellate any one of the core squares using the "quilt" tool indicated by the rightmost icon. Part a shows four different core squares. The upper left core square produces the tessellation in part b.

Parts c and d are produced, respectively, by the upper right and lower right core squares. Interesting discussions result when the class asks which designs are mathematically different (e.g., should a rotation or flip of the core be counted as "different"?).

Grade 3

Students analyze, compare, and classify two-dimensional shapes by their properties (see the footnote on p. 114).[3.G.1] They explicitly relate and combine these classifications. Because they have built a firm foundation of several shape categories, these categories can be the raw material for thinking about the relationships between classes. For example, students can form larger, superordinate, categories, such as the class of all shapes with four sides, or quadrilaterals, and recognize that it includes other categories, such as squares, rectangles, rhombuses, parallelograms, and trapezoids. They also recognize that there are quadrilaterals that are not in any of those subcategories. A description of these categories of quadrilaterals is illustrated in the margin. The Standards do not require that such representations be constructed by Grade 3 students, but they should be able to draw examples of quadrilaterals that are not in the subcategories.

Similarly, students learn to draw shapes with prespecified attributes, without making a priori assumptions regarding their classification (MP.1, "Students . . . analyze givens, constraints, relationships, and goals"). For example, they could solve the problem of making a shape with two long sides of the same length and two short sides of the same length that is not a rectangle.
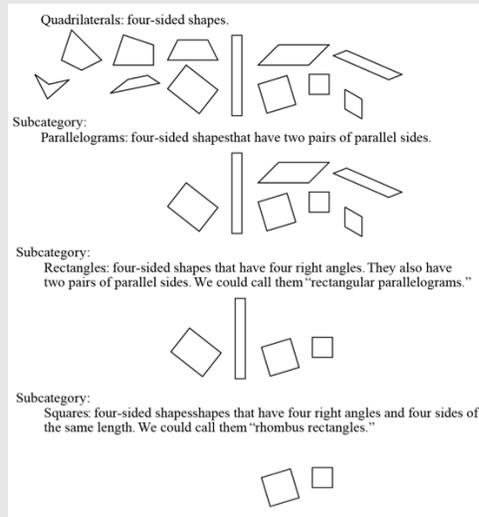
Students investigate, describe, and reason about decomposing and composing polygons to make other polygons. Problems such as finding all the possible different compositions of a set of shapes involve geometric problem solving and notions of congruence and symmetry (MP.7). They also involve the practices of making and testing conjectures (MP.1), and convincing others that conjectures are correct (or not) (MP.3). Such problems can be posed even for sets of simple shapes such as tetrominoes, four squares arranged to form a shape so that every square shares at least one side and sides coincide or share only a vertex.

More advanced paper-folding (origami) tasks afford the same mathematical practices of seeing and using structure (MP.7), conjecturing, and justifying conjectures (MP.3). Paper folding can also illustrate many geometric concepts. For example, folding a piece of paper creates a line segment. Folding a square of paper twice, horizontal edge to horizontal edge, then vertical edge to vertical edge, creates a right angle, which can be unfolded to show four right angles. Students can be challenged to find ways to fold paper into rectangles or squares and to explain why the shapes belong in those categories.

Students also develop more competence in the composition and decomposition of rectangular regions, that is, spatially structuring rectangular arrays. They learn to partition a rectangle into identical squares[3.G.2] by anticipating the final structure and thus forming the array by drawing rows and columns (see the bottom right example on p. 122; some students may still need work building or drawing

[3.G.1] Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

Quadrilaterals and some special kinds of quadrilaterals

Quadrilaterals: four-sided shapes.

Subcategory:
Parallelograms: four-sided shapes that have two pairs of parallel sides.

Subcategory:
Rectangles: four-sided shapes that have four right angles. They also have two pairs of parallel sides. We could call them "rectangular parallelograms."

Subcategory:
Squares: four-sided shapes shapes that have four right angles and four sides of the same length. We could call them "rhombus rectangles."

The representations above might be used by teachers in class. Note that the leftmost four shapes in the first section at the top left have four sides but do not have properties that would place them in any of the other categories shown (parallelograms, rectangles, squares).

Quadrilaterals that are not rectangles
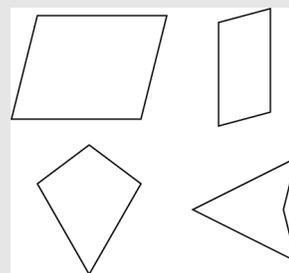
These quadrilaterals have two pairs of sides of the same length but are not rectangles.

[3.G.2] Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

squares inside the rectangle first). They count by the number of columns or rows, or use multiplication to determine the number of squares in the array. They also learn to rotate these arrays physically and mentally to view them as composed of smaller arrays, allowing illustrations of properties of operations (e.g., the commutative property of multiplication and the distributive property).

Grade 4

Students describe, analyze, compare, and classify two-dimensional figures by their properties (see the footnote on p. 114), including explicit use of angle sizes[4.G.1] and the related geometric properties of perpendicularity and parallelism.[4.G.2] They can identify these properties in two-dimensional figures. They can use side length to classify triangles as equilateral, equiangular, isosceles, or scalene; and can use angle size to classify them as acute, right, or obtuse. They then learn to cross-classify, for example, naming a shape as a right isosceles triangle. Thus, students develop explicit awareness of and vocabulary for many concepts they have been developing, including points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Such mathematical terms are useful in communicating geometric ideas, but more important is that constructing examples of these concepts, such as drawing angles and triangles that are acute, obtuse, and right,[4.G.1] help students form richer concept images connected to verbal definitions. That is, students have more complete and accurate mental images and associated vocabulary for geometric ideas (e.g., they understand that angles can be larger than 90° and their concept images for angles include many images of such obtuse angles). Similarly, students see points and lines as abstract objects: Lines are infinite in extent and points have location but no dimension. Grids are made of points and lines and do not end at the edge of the paper.

Students also learn to apply these concepts in varied contexts (MP.4). For example, they learn to represent angles that occur in various contexts as two rays, explicitly including the reference line, e.g., a horizontal or vertical line when considering slope or a "line of sight" in turn contexts. They understand the size of the angle as a rotation of a ray on the reference line to a line depicting slope or as the "line of sight" in computer environments. Students might solve problems of drawing shapes with turtle geometry.• Analyzing the shapes in order to construct them (MP.1) requires students to explicitly formulate their ideas about the shapes (MP.4, MP.6). For instance, what series of commands would produce a square? How many degrees would the turtle turn? What is the measure of the resulting angle? What would be the commands for an equilateral triangle? How many degrees would the turtle turn? What is the measure of the resulting angle? Such experiences help students connect what are often initially isolated ideas about the concept of angle.

Students might explore line segments, lengths, perpendicularity, and parallelism on different types of grids, such as rectangular and triangular (isometric) grids (MP.1, MP.2).[4.G.2, 4.G.3] Can you find a non-rectangular parallelogram on a rectangular grid? Can you find a rectangle on a triangular grid? Given a segment on a rectangular grid that is not parallel to a grid line, draw a parallel segment of the same length with a given endpoint. Given a half of a figure and

4.G.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

4.G.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

• The computer programming language Logo has a pointer, often a icon of a turtle, that draws representations of points, line segments, and shapes, with commands such as "forward 100" and "right 120."

4.G.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

a line of symmetry, can you accurately draw the other half to create a symmetric figure?

Students also learn to reason about these concepts. For example, in "guess my rule" activities, they may be shown two sets of shapes and asked where a new shape belongs (MP.1, MP.2).[4.G.2]

In an interdisciplinary lesson (that includes science and engineering ideas as well as items from mathematics), students might encounter another property that all triangles have: rigidity. If four fingers (both thumbs and index fingers) form a shape (keeping the fingers all straight), the shape of that quadrilateral can be easily changed by changing the angles. However, using three fingers (e.g., a thumb on one hand and the index and third finger of the other hand), students can see that the shape is fixed by the side lengths. Triangle rigidity explains why this shape is found so frequently in bridge, high-wire towers, amusement park rides, and other constructions where stability is sought.

[4.G.2]Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

Guess My Rule
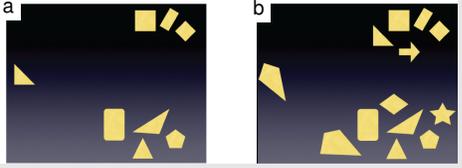
Students can be shown the two groups of shapes in part a and asked "Where does the shape on the left belong?" They might surmise that it belongs with the other triangles at the bottom. When the teacher moves it to the top, students must search for a different rule that fits all the cases.

Later (part b), students may induce the rule: "Shapes with at least one right angle are at the top." Students with rich visual images of right angles and good visualization skills would conclude that the shape at the left (even though it looks vaguely like another one already at the bottom) has one right angle, thus belongs at the top.

Grade 5

By the end of Grade 5, competencies in shape composition and de–composition, and especially the special case of spatial structuring of rectangular arrays (recall p. 122), should be highly developed (MP.7). Students need to develop these competencies because they form a foundation for understanding multiplication, area, volume, and the coordinate plane. To solve area problems, for example, the ability to decompose and compose shapes plays multiple roles. First, stu–dents understand that the area of a shape (in square units) is the number of unit squares it takes to cover the shape without gaps or overlaps. They also use decomposition in other ways. For example, to calculate the area of an "L–shaped" region, students might de–compose the region into rectangular regions, then decompose each region into an array of unit squares, spatially structuring each array into rows or columns. Students extend their spatial structuring in two ways. They learn to spatially structure in three dimensions; for example, they can decompose a right rectangular prism built from cubes into layers, seeing each layer as an array of cubes. They use this understanding to find the volumes of right rectangular prisms with edges whose lengths are whole numbers as the number of unit cubes that pack the prisms (see the Geometric Measurement Pro–gression). Second, students extend their knowledge of the coordi–nate plane, understanding the continuous nature of two–dimensional space and the role of fractions in specifying locations in that space.

Thus, spatial structuring underlies coordinates for the plane as well, and students learn both to apply it and to distinguish the ob–jects that are structured. For example, they learn to interpret the components of a rectangular grid structure as line segments or lines (rather than regions) and understand the precision of location that these lines require, rather than treating them as fuzzy boundaries or indicators of intervals. Students learn to reconstruct the levels of counting and quantification that they had already constructed in the domain of discrete objects to the coordination of (at first) two contin–uous linear measures. That is, they learn to apply their knowledge of number and length to the order and distance relationships of a coordinate grid and to coordinate this across two dimensions.[5.G.1]

Although students can often "locate a point," these understand–ings are beyond simple skills. For example, initially, students often fail to distinguish between two different ways of viewing the point $(2, 3)$, say, as instructions: "right 2, up 3"; and as the point defined by being a distance 2 from the y–axis and a distance 3 from the x–axis. In these two descriptions the 2 is first associated with the x–axis, then with the y–axis.

Students connect ordered pairs of (whole number) coordinates to points on the grid, so that these coordinate pairs constitute numer–ical objects and ultimately can be operated upon as single mathe–matical entities. They solve mathematical and real–world problems using coordinates. For example, they plan to draw a symmetric fig–
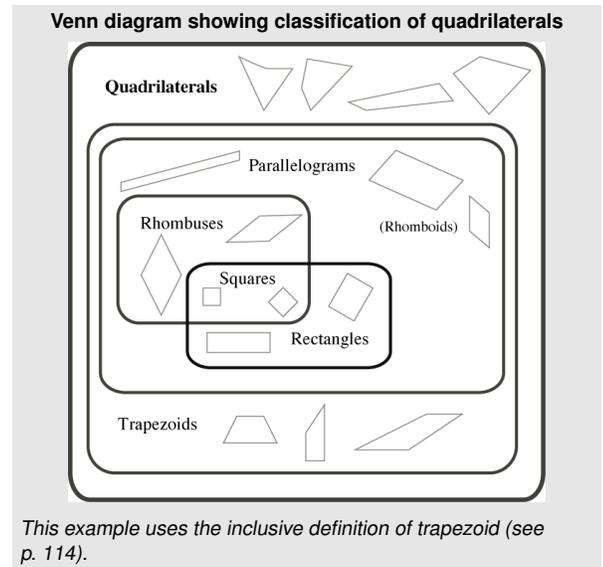
ure using computer software in which coordinates input by students are then connected by line segments.[5.G.2]

Students learn to analyze and relate categories of two–dimensional and three-dimensional shapes explicitly based on their properties.[5.G.4] Based on analysis of properties, they classify two–dimensional figures in hierarchies. For example, they conclude that all rectangles are parallelograms, because they are all quadrilaterals with two pairs of opposite, parallel, equal–length sides (MP.3). In this way, they relate certain categories of shapes as subclasses of other categories.[5.G.3] This leads to understanding propagation of properties; for example, students understand that squares possess all properties of rhombuses and of rectangles. Therefore, if they then show that rhombuses' diagonals are perpendicular bisectors of one another, they infer that squares' diagonals are perpendicular bisectors of one another as well.

[5.G.2]Represent real world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.

[5.G.4]Classify two-dimensional figures in a hierarchy based on properties.

Venn diagram showing classification of quadrilaterals

This example uses the inclusive definition of trapezoid (see p. 114).

[5.G.3]Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category.

Grade 6

Problems involving areas and volumes extend previous work and provide a context for developing and using equations. Students' competencies in shape composition and decomposition, especially with spatial structuring of rectangular arrays (recall p. 122), should be highly developed. These competencies form a foundation for understanding multiplication, formulas for area and volume, and the coordinate plane.[6.NS.6, 6.NS.8]

Using the shape composition and decomposition skills acquired in earlier grades together with the area formula for rectangles,[5.NF.4b] students learn to develop area formulas for parallelograms, then triangles. They learn how to address three different cases for triangles: a height that is a side of a right angle, a height that "lies over the base" and a height that is outside the triangle (MP.1, "Students . . . try special cases . . . of the original problem in order to gain insight into its solution").

Through such activity, students learn that that any side of a triangle can be considered as a base and the choice of base determines the height (thus, the base is not necessarily horizontal and the height is not always in the interior of the triangle). The ability to view a triangle as part of a parallelogram composed of two copies of that triangle and the understanding that area is additive (see the Geometric Measurement Progression) provides a justification (MP.3) for halving the product of the base times the height, helping students guard against the common error of forgetting to take half.

Also building on their knowledge of composition and decomposition, students decompose rectilinear polygons into rectangles, and decompose special quadrilaterals and other polygons into triangles and other shapes, using such decompositions to determine their areas, and justifying and finding relationships among the formulas for the areas of different polygons.[6.G.1]

In Grade 5, students used concepts of area measurement to see that the method they used to find areas of rectangles with whole-number side lengths in Grade 3 could be extended to rectangles with fractional side lengths.[5.NF.4b]

In Grade 6, students use the additivity of volume measurement and spatial structuring abilities developed in earlier grades to see that the method they used to find the volumes of right rectangular prisms with whole-number edge lengths[5.MD.5a] can be extended to right rectangular prisms with fractional edge lengths.

Instead of using a unit cube with an edge length of 1, sixth graders use a unit cube with an edge length that is a fractional unit to pack prisms, using their understanding that multiplying the length times the width of a right rectangular prism can be viewed as determining how many cubes would be in each layer if the prism were packed with or built up from unit cubes, and that the height of the prism tells how many layers would fit in the prism. They show that the volume is the same as would be found by multiplying the

[6.NS.6]Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

[6.NS.8]Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

[5.NF.4b]Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

[6.G.1]Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

[5.MD.5]Relate volume to the operations of multiplication and addition and solve real world and mathematical problems involving volume.

a Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

edge lengths, explain correspondences (MP.1) and write an equation to describe the situation (MP.4).[6.G.2]

For example, a $\frac{1}{4}$ by $\frac{1}{3}$ by $\frac{1}{2}$ right rectangular prism can be packed with unit cubes with edge length $\frac{1}{4\times 3}$. Taking the width and length of the prism to be $\frac{1}{4}$ and $\frac{1}{3}$, the first layer has 3×4 unit cubes and there are 6 layers, so $(3 \times 4) \times 6$ unit cubes fit in the prism. Because $12 \times 12 \times 12$ of these of these unit cubes pack a cube with edge length 1, each unit cube has volume $\frac{1}{12\times 12\times 12}$. So the volume of the $\frac{1}{4}$ by $\frac{1}{3}$ by $\frac{1}{2}$ prism is 12×6 times the volume of a unit cube:

$$\frac{12 \times 6}{12 \times 12 \times 12} = \frac{6}{12 \times 12} = \frac{1}{2 \times 12}$$

which is the product of the edge lengths.

Students can use similar reasoning to pack a cube with edge length 1 with right rectangular prisms that are not cubes. For example, because a cube with edge length 1 can be packed with 60 smaller right rectangular prisms that are each $\frac{1}{4}$ by $\frac{1}{3}$ by $\frac{1}{5}$, each of the smaller prisms has volume $\frac{1}{60}$ (see illustration in the margin). Having established the volume of a right rectangular prism that is $\frac{1}{4}$ by $\frac{1}{3}$ by $\frac{1}{5}$, students can use it in reasoning about measuring the volume of other right rectangular prisms that can be packed with these $\frac{1}{4}$ by $\frac{1}{3}$ by $\frac{1}{5}$ prisms. These examples of reasoning about packing are three-dimensional analogues (MP.1) of the tiling examples in the Number and Operations—Fractions Progression.

Students also analyze and compose and decompose polyhedral solids. They describe the shapes of the faces, as well as the number of faces, edges, and vertices. They make and use drawings of solid shapes and learn that solid shapes have an outer surface as well as an interior. They develop visualization skills connected to their mathematical concepts as they recognize the existence of, and visualize, components of three–dimensional shapes that are not visible from a given viewpoint (MP.1). They measure the attributes of these shapes, allowing them to apply area formulas to solve surface area problems (MP.7). They solve problems that require them to distinguish between units used to measure volume and units used to measure area (or length). They learn to plan the construction of complex three–dimensional compositions through the creation of corresponding two–dimensional nets (e.g., making strategic use of digital fabrication and/or graph paper, MP.5).[6.G.4] For example, they may design a living quarters (e.g., a space station) consistent with given specifications for surface area and volume (MP.2, MP.7). In this and many other contexts, students learn to apply these strategies and formulas for areas and volumes to the solution of real–world and mathematical problems.[6.G.1, 6.G.2] Problems could include those in which areas or volumes are to be found from lengths or lengths are to be found from volumes or areas and lengths.[6.EE.7]

Students extend their understanding of properties of two–dimensional shapes to use of coordinate systems.[6.G.3] For example, they

[6.G.2]Find the volume of a right rectangular prism with fractional edge lengths by packing it with unit cubes of the appropriate unit fraction edge lengths, and show that the volume is the same as would be found by multiplying the edge lengths of the prism. Apply the formulas $V = lwh$ and $V = bh$ to find volumes of right rectangular prisms with fractional edge lengths in the context of solving real-world and mathematical problems.

A note on notation

Formulas can be expressed in many ways. What is important is that the referents of terms or symbols are clear (MP.6). For example, the formula for the volume of a right rectangular prism can be expressed as "the volume is the product of the base and the height," or as "$V = b \times h$," or as "$V = B \times h$." The referent of "base" or, respectively, "b" or "B" is "area of the base in square units."

Two faces of a cube packed with prisms
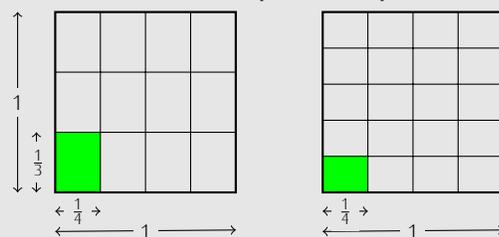
Two faces of a cube that has been packed with right rectangular prisms that are each $\frac{1}{4}$ by $\frac{1}{3}$ by $\frac{1}{5}$.

[6.G.4]Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

[6.G.1]Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

[6.EE.7]Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

[6.G.3]Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

may specify coordinates for a polygon with specific properties, justifying the attribution of those properties through reference to relationships among the coordinates (e.g., justifying that a shape is a parallelogram by computing the lengths of its pairs of horizontal and vertical sides).

As a precursor for learning to describe cross-sections of three-dimensional figures,[7.G.3] students use drawings and physical models to learn to identify parallel lines in three-dimensional shapes, as well as lines perpendicular to a plane, lines parallel to a plane, the plane passing through three given points, and the plane perpendicular to a given line at a given point.

[7.G.3]Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids.

Where this progression is heading

Composition and decomposition of shapes is used throughout geometry from Grade 6 to high school and beyond. Compositions and decompositions of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems that involve complex shapes. Decompositions are often indicated in geometric diagrams by an auxiliary line, and using the strategy of drawing an auxiliary line to solve a problem are part of looking for and making use of structure (MP.7). Recognizing the significance of an existing line in a figure is also part of looking for and making use of structure. This may involve identifying the length of an associated line segment, which in turn may rely on students' abilities to identify relationships of line segments and angles in the figure. These abilities become more sophisticated as students gain more experience in geometry. In Grade 7, this experience includes making scale drawings of geometric figures and solving problems involving angle measure, surface area, and volume (which builds on understandings described in the Geometric Measurement Progression as well as the ability to compose and decompose figures).

Number and Operations—Fractions, 3–5

Overview

The treatment of fractions in the Standards emphasizes two features: fractions, like whole numbers, are composed of units; work with fractions has many connections with previous work in other domains.

Fractions in the Standards   In the Standards, a fraction is built from unit fractions. A *unit fraction* $\frac{1}{b}$ is one part of a decomposition of the number one into b equal shares, e.g., $\frac{1}{3}$ is one part of a decomposition of 1 into 3 equal shares. A *fraction* $\frac{a}{b}$ is composed of a unit fractions, e.g., $\frac{5}{3}$ is composed of 5 unit fractions, namely, five thirds. Fractions can also be written in decimal notation ("as a decimal"), or—if greater than 1—in the form whole number followed by a number less than 1 written as a fraction ("as a mixed number"). Thus, in Grades 3–5, $\frac{7}{5}$, 1.4, and $1\frac{2}{5}$ are all considered fractions,• and, in later grades, rational numbers. Expectations for computations with fractions appear in the domains of Number and Operations—Fractions, Number and Operations in Base Ten, and the Number System.

   To achieve the expectations of the Standards, students need to be able to transform and use numerical (and later symbolic) expressions, including expressions for numbers. For example, in order to get the information they need or to understand correspondences between different approaches to the same problem or different representations for the same situation (MP.1), students may need to draw on their understanding of different representations for a given number. Transforming different expressions for the same number includes the skills traditionally labeled "conversion," "reduction," and "simplification," but these are not treated as separate topics in the Standards. Choosing a convenient form for the purpose at hand is an important skill (MP.5), as is the fundamental understanding of equivalence of forms.

• From the Standards glossary:

> **Fraction.** A number expressible in the form a/b where a is a whole number and b is a positive whole number. (The word *fraction* in these standards always refers to a non-negative number.)
>
> **Whole numbers.** The numbers 0, 1, 2, 3, . . .

Building on work in earlier grades and other domains   Students' work with fractions, visual representations of fractions, and operations on fractions builds on their earlier work in the domains of number, geometry, and measurement.

Composing and decomposing base-ten units. First and second graders work with a variety of units and "units of units." In learning about base-ten notation, first graders learn to think of a ten as a unit composed of 10 ones, and think of numbers as composed of units, e.g., "20 is 2 tens" and "34 is 3 tens and 4 ones." Second graders learn to think of a hundred as a unit composed of 10 tens as well as of 100 ones. Students decompose tens and hundreds when subtracting if they need to get more of a particular unit.

Composing and decomposing shapes. In geometry, students compose and decompose shapes. For example, first graders might put two congruent isosceles triangles together with the explicit purpose of making a rhombus. In this way, they learn to perceive a composite shape as a unit—a single new shape, e.g., recognizing that two isosceles triangles can be combined to make a rhombus, and also seeing the rhombus as decomposed into the two triangles. Working with pattern blocks, they may build the same shape, such as a regular hexagon, from different parts, two trapezoids, three rhombuses, or six equilateral triangles, and see the hexagon as decomposed into these shapes. First and second graders use fraction language to describe decompositions of simple shapes into equal shares—halves, fourths, and quarters in Grade 1, extending to thirds in Grade 2.

Decomposing wholes into units, composing fractions. In Grade 3, students are introduced to fraction notation, and their use of fractions and fraction language expands. They decompose a whole (a shape, unit of length, or line segment) into equal parts and describe one or more parts of the same whole using fraction notation as well as fraction language. For example, if a whole is decomposed into three equal parts, one part is described as $\frac{1}{3}$, two parts as $\frac{2}{3}$, three parts as $\frac{3}{3}$, four parts as $\frac{4}{3}$, and so on. In a whole decomposed into b equal parts, each part represents a *unit fraction* $\frac{1}{b}$. The *fraction* composed of a of these parts is written $\frac{a}{b}$. The number b is called the *denominator* of the fraction and the number a is called its *numerator*. Reading fraction notation aloud in fraction language (e.g., "two thirds" rather than "two over three" or "two out of three") emphasizes the idea that a fraction is composed of unit fractions, e.g., $\frac{2}{3}$ is composed of 2 thirds, just as 20 is composed of 2 tens.[3.NF.1]

Grade 3 expectations are limited to fractions with denominators 2, 3, 4, 6, and 8, allowing students to reason directly from the meaning of fraction about fractions close to or less than 1 by folding paper strips• or working with diagrams. Use of diagrams or paper strips tests and supports student understanding of crucial aspects of fractions: the parts must be the same size, the parts must use all of the whole (students sometimes just tear off part of the paper strip), and subdividing parts of the same whole makes the parts smaller.

Describing pattern block relationships at different grades
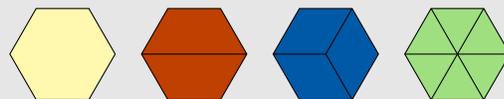
Grade 1 students might say, "A red block is half of a yellow block" and "Three blue blocks make one yellow block."
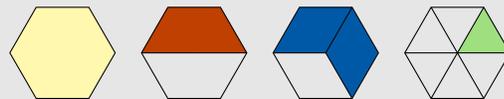
Grade 2 students might say, "A blue block is a third of a yellow block."

Grade 3 students begin to use notation such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{2}{3}$, and $\frac{3}{3}$ in describing the relationships of the pattern blocks.

Grade 4 students might use 0.5.

Grade 7 students might use $0.3333\ldots$ or $0.\overline{3}$ (see the Number System Progression).

Area representations of fractions

In Grade 3, students begin to use equal shares decompositions to represent fractions. In each of these diagrams, the whole is represented by the hexagon. Going from left to right, the diagrams represent 1, $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{1}{6}$.

[3.NF.1]Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

• These are sometimes called "fraction strips," but in the Standards "fraction strip" is used as a synonym for "tape diagram." The relationships shown as tape diagrams in this progression might instead or also be shown with paper strips, but such paper strips may become awkward or unworkable.

In Grade 4, students are introduced to decimal notation for fractions with denominators 10 and 100, and expectations extend to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

Diagrams. Initially, diagrams used in work with fractions show them as composed of unit fractions, emphasizing the idea that a fraction is composed of units just as a whole number is composed of ones. Some diagrams represent a whole as a two-dimensional region and one fraction as one or more equal parts of the region. Use of these diagrams builds on students' work in composing and decomposing geometrical shapes, e.g., seeing a square as composed of four identical rectangles. In contrast, tape diagrams and number line diagrams represent a whole in terms of length. Because they represent numbers or quantities as lengths of "tape," tape diagrams can also be interpreted in terms of area. However, tape diagrams tend to be less complex geometrically than area representations and may also have the advantage of being familiar to students from work with whole numbers in earlier grades (see the Operations and Algebraic Thinking Progression). On a number line diagram, a number is represented by a point, as well as by lengths. Tape diagrams, number line diagrams, and area models are used to represent one or more fractions as well as relationships such as equivalence, sum or difference, and product or quotient. Students' work with these diagrams is an abstraction and generalization of their work with length and area measurement.

Length measurement and number line diagrams. Number line diagrams are important representations in middle grades and beyond. But, they can be difficult for students to understand. Students often make errors because they attend to tick marks or numbers instead of lengths.• Work with length measurement, especially with rulers, can help to prepare students to understand and use number line diagrams to represent fractions in Grade 3.

In Grade 1, students learn to lay physical length-units such as centimeter or inch manipulatives end-to-end and count them to measure a length.[1.MD.2]

In Grade 2, students make measurements with physical length-units and rulers. They learn about the inverse relationship between the size of a length-unit and the number of length-units required to cover a given distance.[2.MD.2]

In learning about length measurement, they develop understandings that they will use with number line diagrams:

- *length-unit iteration.* No gaps or overlaps between successive length-units;

- *accumulation of length-units to make the total length.* E.g., counting "eight" when placing the last length-unit means the distance covered by 8 length-units, rather than just the eighth length-unit;

Fractions represented with tape diagrams
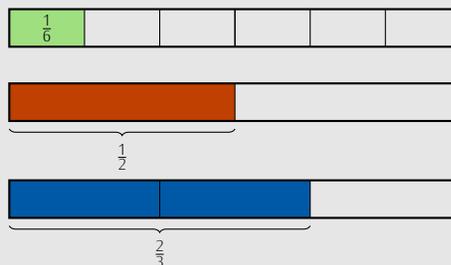
The large rectangle represents the whole. Fractions are indicated as one or more parts of an equal shares decomposition of the rectangle. Because the shares are shown as lengths of "tape," different fractions can be seen as corresponding to different areas or seen as corresponding to different lengths.

Each share represents a unit fraction. Its corresponding numeral can be written within each share, or above or below the tape (see also p. 145). Brackets can be used to show fractions composed of several unit fractions. In Grade 4, students connect this composition with addition as putting together (4.NF.3).

Fractions represented as points on a number line diagram
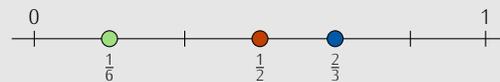
The line segment between 0 and 1 represents the whole. Points abbreviate the "lengths of tape" shown in the corresponding tape diagrams above. Tick marks indicate equal shares. Numbers appear at ends of lengths rather than next to brackets that span lengths (as in the tape diagrams above).

When learning to use these diagrams, students need to understand that the labeled points indicate lengths from 0. As discussed on p. 137, these lengths can initially be shown on or above number line diagrams as students learn to locate fractions in Grade 3. As discussed on p. 145, these lengths can again be shown as students learn to add and subtract fractions.

• Recent National Research Council reports recommend that number line diagrams not be used in Kindergarten and Grade 1. The Standards follow these recommendations. For further discussion, see the Grade 1 section of the Operations and Algebraic Thinking Progression.

[1.MD.2]Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps.

[2.MD.2]Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.

- *alignment of zero–point.* Correct alignment of the zero–point on a ruler as the beginning of the total length, including the case in which the 0 of the ruler is not at the edge of the physical ruler;

- *meaning of numerals on the ruler.* The numerals indicate the number of length units so far;

- *connecting measurement with physical units and with a ruler.* Measuring by laying physical units end–to–end or iterating a physical unit and measuring with a ruler both focus on finding the total number of unit lengths.

These correspond to analogous conventions for number line diagrams. In particular, the unit of measurement (length from 0 to 1) on a ruler corresponds to the unit of measurement (length from 0 to 1) on a number line diagram.

In their work with categorical and measurement data, second graders use bar graphs and line plots (see the Measurement and Data Progression). Bar graphs have vertical "count scales" that represent only whole numbers. For example, the 4 on a bar graph scale may represent 4 birds. Because the count scale in a bar graph is a number line diagram with only whole numbers, answering a question such as "How many more birds are there than spiders?" involves understanding differences on a number line diagram.[2.MD.10] Similarly, using a line plot to answer questions about data can involve using information from a number line diagram to find sums or differences. Line plots have horizontal "measurement scales" for length measurements. For example, the 4 on a line plot scale may represent 4 inches. In Grade 2, both types of scales are labeled only with whole numbers. However, subdivisions between numbers on measurement scales may have referents (e.g., half of an inch), but subdivisions between numbers on count scales may not (e.g., half of a bird may not make sense).

In their work with number line diagrams,[2.MD.6] second graders need to understand that these diagrams have specific conventions: the use of a single position to represent a whole number and the use of marks to indicate those positions. They need to understand that a number line diagram is like a ruler in that consecutive whole numbers are 1 unit apart, thus they need to consider the distances between positions and segments when identifying missing numbers. These understandings underlie students' successful use of number line diagrams. Students think of a number line diagram in terms of length measurement and use strategies relating to distance, proximity of numbers, and reference points.

In Grade 3, students use rulers marked with halves and fourths of an inch.[3.MD.4] Students represent fractions on number line diagrams. The interval from 0 to 1 is partitioned into b equal parts. Each part has length $\frac{1}{b}$. The unit fraction $\frac{1}{b}$ is shown on the number line diagram as the point that is distance $\frac{1}{b}$ from 0. A fraction $\frac{a}{b}$

Using information from a bar graph
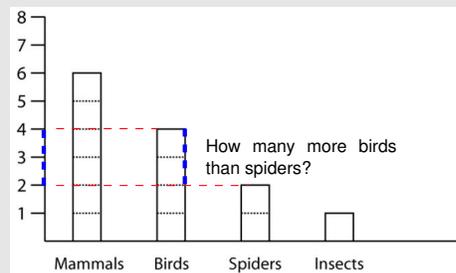
The count scale in a bar graph is a number line diagram with only whole numbers.

2.MD.10 Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put-together, take-apart, and compare problems using information presented in a bar graph.

Adding whole numbers on a number line diagram
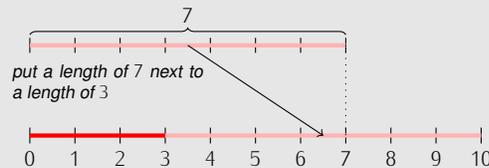
2.MD.6 Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, . . . , and represent whole-number sums and differences within 100 on a number line diagram.

Ruler markings showing halves and fourths

The interval from 0 to 1 shows the unit of measurement, e.g., 1 inch. The three marks between 0 and 1 partition this interval into four equal parts, each of length $\frac{1}{4}$ of the unit of measurement.

3.MD.4 Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

Number line diagram showing fourths
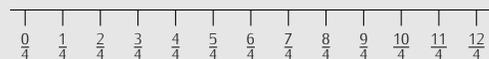
The interval from 0 to 1 has length 1. It is partitioned into four equal parts. Each part has length $\frac{1}{4}$.

is composed of a unit fractions $\frac{1}{b}$ and appears on a number line diagram as the point that is distance a lengths of $\frac{1}{b}$ from 0.[3.NF.2]

Working with lengths on the number line diagram builds on the understandings of length measurement outlined above. Typical number line diagram errors can be reduced by students or teachers focusing attention on the lengths on the diagram, for example by running a finger along the lengths as they are counted or labeled, putting a tape diagram above the diagram (see margin on this page), shading alternate lengths (see margin p. 141), or encircling the interval from 0 to $\frac{a}{b}$ to see all the lengths of $\frac{1}{b}$ that cover it without gaps or overlaps (see margin p. 145).

Area measurement and area models. Students' work with area models begins in Grade 3. These diagrams are used in Grade 3 for single-digit multiplication and division strategies (see the Operations and Algebraic Thinking Progression), to represent multi-digit multiplication and division calculations in Grade 4 (see the Number and Operations in Base Ten Progression), and in Grades 5 and 6 to represent multiplication and division of fractions (see this progression and the Number System Progression). The distributive property is central to all of these uses (see the Grade 3 section of the Operations and Algebraic Thinking Progression).

Work with area models builds on previous work with area measurement. As with length measurement, area measurement relies on several understandings:

- *area is invariant.* Congruent figures enclose regions with equal areas;

- *area is additive.* The area of the union of two regions that overlap only at their boundaries is the sum of their areas;

- *area-unit tiling.* Area is measured by tiling a region with a two-dimensional area-unit (such as a square or rectangle) and parts of the unit, without gaps or overlaps.

Perceiving a region as tiled by an area-unit relies on *spatial structuring.* For example, second graders learn to see how a rectangular region can be partitioned as an array of squares.[2.G.2] Students learn to see an object such as a row in two ways: as a composite of multiple squares and as a single entity, a row (a unit of units). Using rows or columns to cover a rectangular region is, at least implicitly, a composition of units (see the K–6 Geometry Progression).

[3.NF.2]Understand a fraction as a number on the number line; represent fractions on a number line diagram.

    a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

    b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Number line diagram with tape diagrams

Students need to see the lengths that locate fractions on a number line diagram. In this diagram, each part has size $\frac{1}{4}$. The endpoint of the part based at 0 locates the number $\frac{1}{4}$. Starting at 0 and marking off two lengths of $\frac{1}{4}$ makes an interval of size $\frac{2}{4}$. Its right endpoint locates the number $\frac{2}{4}$. Starting at 0 and marking off three lengths of $\frac{1}{4}$ makes an interval of size $\frac{3}{4}$. Its right endpoint locates the number $\frac{3}{4}$.

Fractions represented with an area model

In this diagram, the square represents the whole and has area 1. Fractions are indicated as parts of an equal shares decomposition. As in the tape diagram on p. 135, the green region is $\frac{1}{6}$ of the whole and the red region is $\frac{3}{6}$ of the whole. The remaining region is $\frac{2}{6}$ of the whole.

The equal shares decomposition can also be seen as a tiling by a $\frac{1}{3}$ by $\frac{1}{2}$ rectangle.

[2.G.2]Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.

Addition and subtraction. In Grades 4 and 5, students learn about operations on fractions, extending the meanings of the operations on whole numbers. For addition and subtraction, these meanings arise from the Add To, Take From, Put Together/Take Apart, and Compare problem types and are established before Grade 3.•

In Grade 4, students compute sums and differences, mainly of fractions and mixed numbers with like denominators. In Grade 5, students use their understanding of equivalent fractions to compute sums and differences of fractions with unlike denominators.

Multiplication. The concept of multiplication begins in Grade 3 with an entirely discrete notion of "equal groups."[3.OA.1] By Grade 4, students can also interpret a multiplication equation as a statement of comparison involving the notion "times as much."[4.OA.1] This notion easily includes continuous quantities, e.g., $3 = 4 \times \frac{3}{4}$ might describe how 3 cups of flour are 4 times as much as $\frac{3}{4}$ cup of flour.[4.NF.4,4.MD.2] By Grade 5, when students multiply fractions in general,[5.NF.4] products can be larger or smaller than either factor, and multiplication can be seen as an operation that "stretches or shrinks" by a scale factor.[5.NF.5]

Grade 3 work with whole–number multiplication and division focuses on two problem types, Equal Groups and Arrays. (For descriptions of these problem types and examples that involve discrete attributes, see the Grade 3 section of the Operations and Algebraic Thinking Progression. For examples with continuous attributes, see the Geometric Measurement Progression. Both illustrate measurement (quotitive) and sharing (partitive) interpretations of division.)

Initially, problems involve multiplicands that represent discrete attributes (e.g., cardinality). Later problems involve continuous attributes (e.g., length). For example, problems of the Equal Groups type involve situations such as:

- There are 3 bags with 4 plums in each bag. How many plums are there in all?

and, in the domain of measurement:

- You need 3 lengths of string, each 4 feet long. How much string will you need altogether?

Both of these problems are about 3 groups of four things each—3 fours—in which the group of four can be seen as a whole (1 bag or 1 length of string) or as a composite of units (4 plums or 4 feet). In the United States, the multiplication expression for 3 groups of four is usually written as 3×4, with the multiplier first. (This convention is used in this progression. However, as discussed in the Operations and Algebraic Thinking Progression, in other countries this may be written as 4×3 and it is useful to discuss the different interpretations in connection with the commutative property.)

• For descriptions and examples of these problem types, see the Overview of K–2 in the Operations and Algebraic Thinking Progression.

[3.OA.1] Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each.

[4.OA.1] Interpret a multiplication equation as a comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.

[4.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

[4.MD.2] Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as number line diagrams that feature a measurement scale.

[5.NF.4] Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

   a Interpret the product $(a/b) \times q$ as a parts of a partition of q into b equal parts; equivalently, as the result of a sequence of operations $a \times q \div b$.

   b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

   a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

   b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

In Grade 4, problem types for whole–number multiplication and division expand to include Multiplicative Compare with whole numbers. In this grade, Equal Groups and Arrays extend to include problems that involve multiplying a fraction by a whole number. For example, problems of the Equal Groups type might be:

- You need 3 lengths of string, each $\frac{1}{4}$ foot long. How much string will you need altogether?

- You need 3 lengths of string, each $\frac{5}{4}$ feet long. How much string will you need altogether?

Like the two previous problems, these two problems are about objects that can be seen as wholes (1 length of string) or in terms of units. However, instead of being feet, the units are $\frac{1}{4}$-feet.

In Grade 5, students connect fractions with division, understanding numerical instances of $\frac{a}{b} = a \div b$ for whole numbers a and b, with b not equal to zero (MP.8).[5.NF.3] With this understanding, students see, for example, that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any number by a fraction. Problem types for multiplication expand to include Multiplicative Compare with unit fraction language, e.g., "one third as much as," and students solve problems that involve multiplying by a fraction. For example, a problem of the Equal Groups type might be:

- You need $\frac{1}{3}$ of a length of string that is $2\frac{1}{4}$ feet long. How much string will you need altogether?

Measurement conversion. At Grades 4 and 5, expectations for conversion of measurements parallel expectations for multiplication by whole numbers and by fractions. In 4.MD.1, the emphasis is on "times as much" or "times as many,"• conversions that involve viewing a larger unit as a composite of smaller units and multiplying the number of larger units by a whole number to find the number of smaller units. For example, conversion from feet to inches involves viewing a foot as composed of inches (e.g., viewing a foot as 12 inches or as 12 times as long as an inch), so a measurement in inches is 12 times what it is in feet. In 5.MD.1, conversions also involve viewing a smaller unit as part of a decomposition of a larger unit (e.g., an inch is $\frac{1}{12}$ foot), so a measurement in feet is $\frac{1}{12}$ times what it is in inches and conversions require multiplication by a fraction (5.NF.4).

Division. Using their understanding of division of whole numbers and multiplication of fractions, students in Grade 5 solve problems that involve dividing a whole number by a unit fraction or a unit fraction by a whole number. In Grade 6, they extend their work to problems that involve dividing a fraction by a fraction (see the Number System Progression).

[5.NF.3] Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

• See the Grade 4 section of the Operations and Algebraic Thinking Progression for discussion of linguistic aspects of "as much" and related formulations for Multiplicative Compare problems.

[4.MD.1] Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb, oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.

[5.MD.1] Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real world problems.

Grade 3

The meaning of fractions and fraction notation   In Grades 1 and 2, students use fraction language to describe partitions of shapes into equal shares.[2.G.3] In Grade 3, they start to develop a more general concept of fraction, building on the idea of partitioning a whole into equal parts and expressing the number of parts symbolically, using fraction notation. The whole can be a shape such as a circle or rectangle, a line segment, or any one finite entity susceptible to subdivision. However, in Grade 3 the main focus is on shapes that are easier for students to draw and subdivide, e.g., lengths of "tape" rather than circles. In Grade 5, this is extended to include representing a whole that is a collection of objects as a fraction times a whole number.

Grade 3 students start with unit fractions (fractions with numerator 1), which are formed by partitioning a whole into equal parts and taking one part, e.g., if a whole is partitioned into 4 equal parts then each part is $\frac{1}{4}$ of the whole, and 4 copies of that part make the whole. Next, students build fractions from unit fractions, seeing the numerator 3 of $\frac{3}{4}$ as saying that $\frac{3}{4}$ is what you get by putting 3 of the $\frac{1}{4}$s together.[3.NF.1] They read any fraction this way. In particular there is no need to introduce "proper fractions" and "improper fractions" initially; $\frac{5}{3}$ is what you get by combining 5 parts when a whole is partitioned into 3 equal parts.

Two important aspects of fractions provide opportunities for the mathematical practice of attending to precision (MP.6):

- Specifying the whole.

- Explaining what is meant by "equal parts."

Initially, students can use an intuitive notion of congruence ("same size and same shape" or "matches exactly") to explain why the parts are equal, e.g., when they partition a square into four equal squares or four equal rectangles.[3.G.2]

Students come to understand a more precise meaning for "equal parts" as "parts with equal measurements." For example, when a ruler is partitioned into halves or fourths of an inch by unlabeled tick marks,[3.MD.4] students see that each subdivision has the same length. Giving the tick marks on these rulers numerical labels expressed as halves or fourths as shown in the margin can help students understand such rulers.

Analyzing area representations, students reason about the area of a shaded region to decide what fraction of the whole it represents (MP.3).

The goal is for students to see unit fractions as basic building blocks of fractions, in the same sense that the number 1 is the basic building block of the whole numbers. Just as every whole number can be obtained by combining ones, every fraction can be obtained by combining copies of one unit fraction.

[2.G.3] Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words *halves*, *thirds*, *half of*, *a third of*, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

[3.NF.1] Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$.

**The importance of specifying the whole**
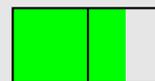
Without specifying the whole it is not reasonable to ask what fraction is represented by the shaded area. If the left square is the whole, the shaded area represents the fraction $\frac{3}{2}$; if the entire rectangle is the whole, the shaded area represents $\frac{3}{4}$.

[3.G.2] Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

[3.MD.4] Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, or quarters.

Tick marks labeled in fourths
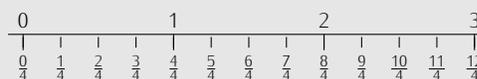
Area representations of $\frac{1}{4}$

In each representation, the square is the whole. The two squares on the left are partitioned into four parts that have the same size and shape, and so the same area. In the three squares on the right, the shaded area is $\frac{1}{4}$ of the whole area, even though it is not easily seen as one part in a partition of the square into four parts of the same shape and size.

The number line and number line diagrams On a number line diagram, the whole that is equally partitioned is the line segment between 0 and 1. This segment has length 1 and is called the *unit interval*. Iterating this whole to the right marks off the whole numbers, so that the intervals between consecutive whole numbers, from 0 to 1, 1 to 2, 2 to 3, etc., are all of the same length, as shown. Students might think of the *number line* which is represented by number line diagrams as an infinite ruler with the unit interval as the unit of measurement. (The Standards and this progression distinguish between the abstract number line and number line diagrams, but students need not make this distinction. In the classroom, number line diagrams can simply be called "number lines.")

To construct a unit fraction on a number line diagram, e.g., $\frac{1}{3}$, students partition the unit interval into 3 intervals of equal length and recognize that each has length $\frac{1}{3}$. They determine the location of the number $\frac{1}{3}$ by iterating a length-unit—marking off this length from 0—and locate other fractions with denominator 3 by marking off the number of lengths indicated by the numerator.[3.NF.2]

Although number line diagrams are important representations for students as they develop an understanding of a fraction as a number, initially they use other representations such as area representations, strips of paper, and tape diagrams. These, like number line diagrams, show a fraction as composed of like unit fractions and can be subdivided, representing important aspects of fractions.

The number line reinforces the analogy between fractions and whole numbers. Just as 5 is the point on the number line reached by marking off 5 times the length of the unit interval from 0, so $\frac{5}{3}$ is the point obtained in the same way using a different interval as the unit of measurement, namely the interval from 0 to $\frac{1}{3}$.
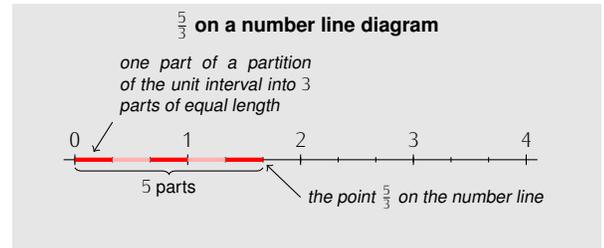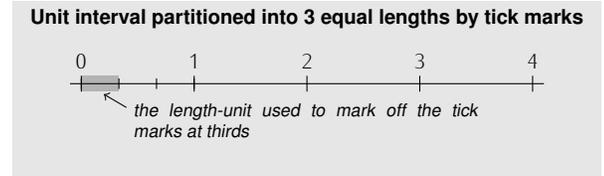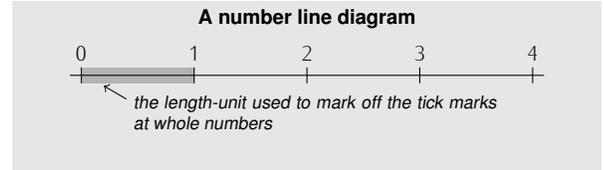
Equivalent fractions Grade 3 students do some preliminary reasoning about equivalent fractions,[3.NF.3] in preparation for work in Grade 4. As students experiment on number line diagrams they discover that many fractions label the same point on the number line, and are therefore equal; that is, they are *equivalent fractions*.[3.NF.3a] For example, the fraction $\frac{1}{2}$ is equal to $\frac{2}{4}$ and also to $\frac{3}{6}$. Students can also use tape diagrams or area representations to see fraction equivalence, seeing, for example, that the same length or area may be composed of 1 equal share of a decomposition or multiple smaller shares of a different decomposition with more parts (see margin).

In particular, students in Grade 3 see whole numbers as fractions, recognizing, for example, that the point on the number line designated by 2 is now also designated by $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, etc. so that[3.NF.3c]

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \cdots$$
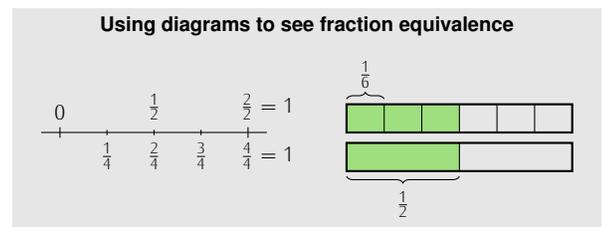
Of particular importance are the ways of writing 1 as a fraction:

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \cdots$$

A number line diagram

the length-unit used to mark off the tick marks at whole numbers

Unit interval partitioned into 3 equal lengths by tick marks

the length-unit used to mark off the tick marks at thirds

$\frac{5}{3}$ **on a number line diagram**

one part of a partition of the unit interval into 3 parts of equal length

5 parts

the point $\frac{5}{3}$ on the number line

[3.NF.2]Understand a fraction as a number on the number line; represent fractions on a number line diagram.

a Represent a fraction $1/b$ on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into b equal parts. Recognize that each part has size $1/b$ and that the endpoint of the part based at 0 locates the number $1/b$ on the number line.

b Represent a fraction a/b on a number line diagram by marking off a lengths $1/b$ from 0. Recognize that the resulting interval has size a/b and that its endpoint locates the number a/b on the number line.

Using diagrams to see fraction equivalence

0 $\frac{1}{2}$ $\frac{2}{2} = 1$

$\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4} = 1$

$\frac{1}{6}$

$\frac{1}{2}$

[3.NF.3]Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a Understand two fractions as equivalent (equal) if they are the same size, or the same point on a number line.

b Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$. Explain why the fractions are equivalent, e.g., by using a visual fraction model.

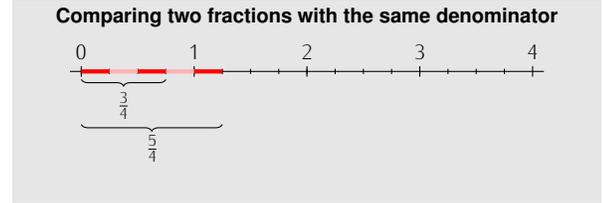c Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

Comparing fractions   Previously, in Grade 2, students compared lengths using a standard unit of measurement.[2.MD.3] In Grade 3, they build on this idea to compare fractions with the same denominator. They see that for two fractions that have the same denominator, the underlying unit fractions are the same size, so the fraction with the greater numerator is greater because it is made of more unit fractions. For example, on the number line the segment from 0 to $\frac{3}{4}$ is shorter than the segment from 0 to $\frac{5}{4}$ because it is 3 fourths long as opposed to 5 fourths long. Therefore $\frac{3}{4} < \frac{5}{4}$.[3.NF.3d]

In Grade 2, students gained experience with the inverse relationship between the size of a physical length–unit and the number of length–units required to cover a given distance.[2.MD.2] Grade 3 students see that for unit fractions, the fraction with the larger denominator is smaller, by reasoning, for example, that in order for more (identical) pieces to make the same whole, the pieces must be smaller. From this, they reason that for two fractions that have the same numerator, the fraction with the smaller denominator is greater. For example, $\frac{2}{5} > \frac{2}{7}$, because $\frac{1}{7} < \frac{1}{5}$, so 2 lengths of $\frac{1}{7}$ is less than 2 lengths of $\frac{1}{5}$. Because students have had years of comparing whole numbers, they may initially say, "$7 > 5$, so $\frac{2}{7} > \frac{2}{5}$." Work with visual representations of fractions helps students use fluently the idea that a larger number in the denominator means smaller underlying unit fractions.

As with equivalence of fractions, it is important in comparing fractions to make sure that each fraction refers to the same whole.

As students move towards understanding fractions as points on the number line, they develop an understanding of order in terms of position. Given two fractions—thus two points on the number line— the one to the left is said to be smaller and the one to its right is said to be larger. (This is opposite to the order in base-ten notation, where values increase from right to left.) Understanding order as position on the number line will become important in Grade 6 when students start working with negative numbers.
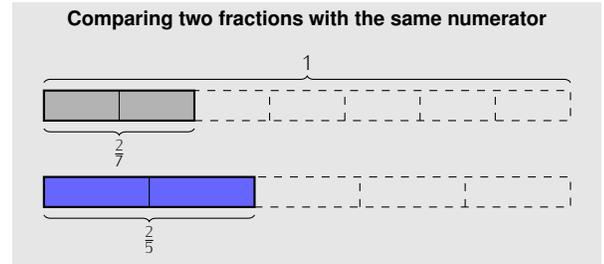
Comparing two fractions with the same denominator

Comparing two fractions with the same numerator

The importance of referring to the same whole when comparing fractions

A student might think that $\frac{1}{4} > \frac{1}{2}$, because a fourth of the pizza on the right is bigger than a half of the pizza on the left.

Grade 4

In Grade 4 students move on from the special cases discussed in Grade 3 to a fundamental property of equivalent fractions: multiplying the numerator and denominator of a fraction by the same non-zero whole number results in a fraction equal to the original fraction. This property forms the basis for important work in Grade 4, including the comparison of fractions and the introduction of finite decimals. Understanding this property can be difficult because numerators and denominators increase when multiplied but the underlying unit fractions become smaller. The numerator and denominator increase is salient in numerical expressions, but the unit fraction decrease is salient in diagrams (see examples in the margin). This is why understanding correspondences between numerical expressions and diagrams for equivalent fractions (MP.1) is important and is included in standard 4.NF.1.

Equivalent fractions   Students can use area representations, strips of paper, tape diagrams, and number line diagrams to reason about equivalence.[4.NF.1] They see that the numerical process of multiplying the numerator and denominator of a fraction by the same number, n, corresponds to partitioning each piece of the diagram into n smaller equal pieces (MP.1). Each region or length that represents a unit fraction is partitioned into n smaller regions or lengths, each of which represents a unit fraction. The whole has then been partitioned into n times as many pieces, and there are n times as many smaller unit fraction pieces as in the original fraction.

This argument, once understood for a range of examples, can be seen as a general argument, working directly from the Grade 3 understanding of a fraction (3.NF.1, 3.NF.2).

Each pair of diagrams in the margin can also be read in reverse order, viewing a partition with fewer pieces as obtained from one with more. For example, in the area representation on the left, each of the 3 partition pieces is obtained by composing 4 of the pieces from the area representation on the right. This illustrates relationships that can be expressed in terms of division:

$$\frac{8}{12} = \frac{8 \div 4}{12 \div 4} = \frac{2}{3}.$$

Because the equations $8 \div 4 = 2$ and $12 \div 4 = 3$ tell us that $8 = 4 \times 2$ and $12 = 4 \times 3$, using the symmetric property of equality we see this as an example of the fundamental property in disguise:

$$\frac{4 \times 2}{4 \times 3} = \frac{2}{3}.$$

Using the fundamental property to write a fraction without common factors in numerator and denominator is often called "simplifying the fraction." It is possible to over-emphasize the importance

4.NF.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

Using an area representation to show that $\frac{2}{3} = \frac{4 \times 2}{4 \times 3}$
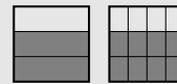
The whole is the square. On the left, the square is partitioned into 3 rectangles of equal area (3 thirds). The shaded region is 2 of these thirds, so represents $\frac{2}{3}$.

To get the figure on the right, each of the 3 rectangles has been partitioned into 4 smaller rectangles of equal area.

Viewed in terms of rows, this makes 3 rows of 4 small rectangles, so the square is now partitioned into 3×4 equal pieces (12 twelfths). The shaded area is 2×4 of these twelfths, so represents $\frac{2 \times 4}{3 \times 4}$.

Viewed in terms of columns, this makes 4 columns of 3 small rectangles, so the square is now partitioned into 4×3 equal pieces (12 twelfths). The shaded area is 4×2 of these twelfths, so represents $\frac{4 \times 2}{4 \times 3}$.

Using a tape diagram to show that $\frac{2}{3} = \frac{4 \times 2}{4 \times 3}$

The whole is the tape. In the top diagram, the tape is partitioned into 3 equal pieces, thus each piece represents $\frac{1}{3}$ and the shaded section represents $\frac{2}{3}$. Each section of the top diagram is partitioned into four equal pieces to produce the bottom diagram. In the bottom diagram, the tape is partitioned into 4×3 equal pieces, thus each piece represents $\frac{1}{4 \times 3}$ and the shaded section represents $\frac{4 \times 2}{4 \times 3}$.

Using a number line diagram to show that $\frac{4}{3} = \frac{5 \times 4}{5 \times 3}$
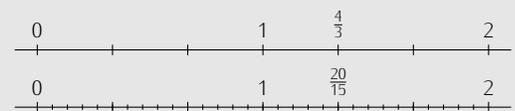
$\frac{4}{3}$ is 4 parts when each part is $\frac{1}{3}$, and we want to see that this is also 5×4 parts when each part is $\frac{1}{5 \times 3}$. Partition each interval of length $\frac{1}{3}$ into 5 parts of equal length. There are 5×3 parts of equal length in the unit interval, and $\frac{4}{3}$ is 5×4 of these. Therefore $\frac{4}{3} = \frac{5 \times 4}{5 \times 3} = \frac{20}{15}$.

of simplifying fractions. There is no mathematical reason why fractions must *always* be written in simplified form, although it may be convenient to do so in some cases, e.g., before comparing $\frac{16}{24}$ and $\frac{2}{3}$.

Grade 4 students use their understanding of equivalent fractions to compare fractions with different numerators and different denominators.[4.NF.2]• For example, to compare $\frac{5}{8}$ and $\frac{7}{12}$ they use the fundamental property to rewrite both fractions, multiplying the numerator and denominator of each fraction by the denominator of the other fraction:

$$\frac{5}{8} = \frac{12 \times 5}{12 \times 8} = \frac{60}{96} \quad \text{and} \quad \frac{7}{12} = \frac{7 \times 12}{8 \times 12} = \frac{56}{96}.$$

Because $\frac{60}{96}$ and $\frac{56}{96}$ have the same denominator, students can compare them using Grade 3 methods and see that $\frac{56}{96}$ is smaller, so

$$\frac{7}{12} < \frac{5}{8}.$$

Students can also think of a number smaller than 96 that is also a multiple of 8 and of 12, such as 24, and use that as the common denominator.• In this case, they need to figure out what multipliers to use instead of the two denominators. For example, $\frac{3}{4}$ and $\frac{5}{6}$ can be compared by rewriting as $\frac{9}{12}$ and $\frac{10}{12}$ or as $\frac{18}{24}$ and $\frac{20}{24}$. Cases where one denominator is a multiple of the other can be discussed, e.g., comparison of $\frac{2}{3}$ and $\frac{7}{9}$ leads to comparison of $\frac{6}{9}$ and $\frac{7}{9}$.

Students also reason using benchmarks such as $\frac{1}{2}$ and 1. For example, they see that $\frac{7}{8} < \frac{13}{12}$ because $\frac{7}{8}$ is less than 1 but $\frac{13}{12}$ is greater than 1. Students may express the same argument in terms of the number line: $\frac{7}{8}$ is less than 1, therefore to the left of 1; $\frac{13}{12}$ is greater than 1, therefore to the right of 1; so $\frac{7}{8}$ is to the left of $\frac{13}{12}$, which means that $\frac{7}{8}$ is less than $\frac{13}{12}$.

Grade 4 students who have learned about fraction multiplication can see equivalence as "multiplying by 1":

$$\frac{7}{9} = \frac{7}{9} \times 1 = \frac{7}{9} \times \frac{4}{4} = \frac{28}{36}.$$

However, this does not constitute a valid argument at this grade, if all students have not yet learned fraction multiplication.

Adding and subtracting fractions   The meaning of addition is the same for both fractions and whole numbers, even though algorithms for calculating their sums can be different. Just as the sum of 4 and 7 can be interpreted as the length of the segment obtained by putting together two segments of lengths 4 and 7, so the sum of $\frac{2}{3}$ and $\frac{8}{5}$ can be interpreted as the length of the segment obtained by putting together two segments of length $\frac{2}{3}$ and $\frac{8}{5}$. It is not necessary to know the value of $\frac{2}{3} + \frac{8}{5}$ in order to know what the sum means.
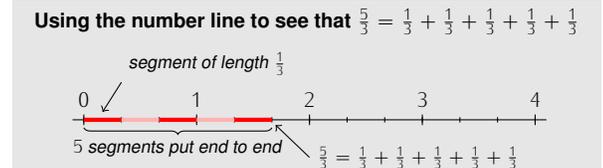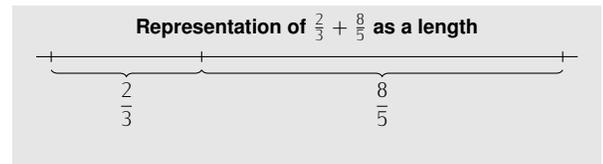
4.NF.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols >, =, or <, and justify the conclusions, e.g., by using a visual fraction model.

• Note that Grade 4 expectations are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100.

• In Grades 4 and 5, the focus is on understanding how the fundamental property can be used rather than trying to choose multipliers that will result in a least common denominator. However, when comparing approaches (MP.1), students can note when different common denominators are used.

Using number line diagrams to compare $\frac{3}{4}$ and $\frac{5}{6}$
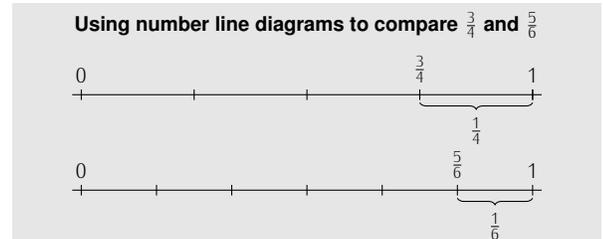
Both $\frac{3}{4}$ and $\frac{5}{6}$ are less than 1, so are to the left of 1 on the number line. The distance from $\frac{3}{4}$ to 1 is $\frac{1}{4}$. The distance from $\frac{5}{6}$ to 1 is $\frac{1}{6}$. Because $\frac{1}{4}$ is greater than $\frac{1}{6}$, $\frac{3}{4}$ is further to the left of 1 than $\frac{5}{6}$ is. That means $\frac{3}{4}$ is left of $\frac{5}{6}$ on the number line, so $\frac{3}{4}$ is less than $\frac{5}{6}$.

Representation of $\frac{2}{3} + \frac{8}{5}$ as a length

Using the number line to see that $\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

segment of length $\frac{1}{3}$

5 segments put end to end $\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

Seeing a fraction written as a sum of unit fractions can help students see that numerators are counted or added and the denominator stays the same. This can help students avoid the common "add tops and add bottoms" when adding in fraction notation.

This simple understanding of addition as putting together allows students to see in a new light the way fractions are composed of unit fractions. Number line diagrams, the same type of diagrams that students used in Grade 3 to see a fraction as a point on the number line (3.NF.2), allow them to see a fraction as a sum of unit fractions. Just as $5 = 1 + 1 + 1 + 1 + 1$, so

$$\frac{5}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

because $\frac{5}{3}$ is the total length of 5 thirds.[4.NF.3]

Armed with this insight, students decompose and compose fractions with the same denominator.[4.NF.3b] They add fractions with the same denominator.[4.NF.3c] Here, equations are used to describe approaches that might also be shown with diagrams (MP.1) because tape diagrams and number line diagrams are important in Grade 4 to support reasoning expressed symbolically.

$$\frac{3}{6} + \frac{2}{6} = \overbrace{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}}^{3\ sixths} + \overbrace{\frac{1}{6} + \frac{1}{6}}^{2\ sixths}$$

$$= \frac{\overbrace{1 + 1 + 1 + 1 + 1}^{3+2\ sixths}}{6}$$

$$= \frac{5}{6}$$

Using the understanding gained from work with whole numbers of the relationship between addition and subtraction, students also subtract fractions with the same denominator. For example, to subtract $\frac{5}{6}$ from $\frac{17}{6}$, they decompose

$$\frac{17}{6} = \frac{12}{6} + \frac{5}{6}, \quad \text{so} \quad \frac{17}{6} - \frac{5}{6} = \frac{17-5}{6} = \frac{12}{6} = 2.$$

Students also compute sums of whole numbers and fractions, by representing the whole number as an equivalent fraction with the same denominator as the fraction, e.g.

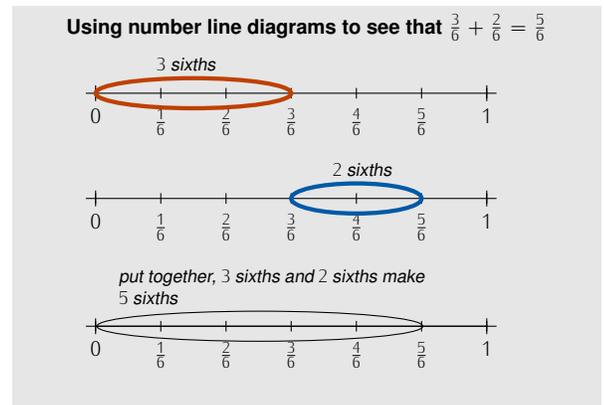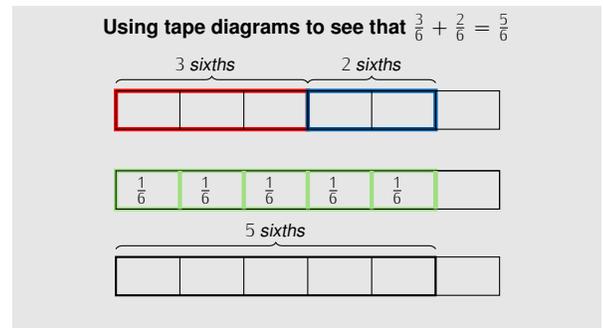$$7 + \frac{1}{5} = \frac{35}{5} + \frac{1}{5} = \frac{36}{5}.$$

Students use this method to add mixed numbers with like denominators.• Converting a mixed number to a fraction should not be viewed as a separate technique to be learned by rote, but simply as addition.

Similarly, writing an improper fraction as a mixed number is a matter of decomposing the fraction into a sum of a whole number and a number less than 1.[4.NF.3b] Students can draw on their knowledge from Grade 3 of whole numbers written in fraction notation. For example, knowing that $1 = \frac{3}{3}$, they see
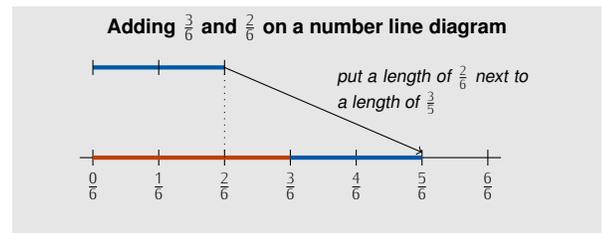
$$\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}.$$

4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

a Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

c Add and subtract mixed numbers with like denominators, e.g., by replacing each mixed number with an equivalent fraction, and/or by using properties of operations and the relationship between addition and subtraction.

Using tape diagrams to see that $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

Using number line diagrams to see that $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$

• Students need to understand that a mixed number is the sum of a whole number and a fraction less than 1, and that it is written as a whole number plus a fraction smaller than 1, without the + sign. For example, $5\frac{3}{4}$ means $5 + \frac{3}{4}$ and $7\frac{1}{5}$ means $7 + \frac{1}{5}$. It can be helpful for students to write the plus sign after the whole number to clarify the meaning.

Adding $\frac{3}{6}$ and $\frac{2}{6}$ on a number line diagram

4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.

b Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model.

Calculations with mixed numbers provide opportunities for students to compare approaches and justify steps in their computations (MP.3). Here, equations with parentheses• and diagrams are used to describe three approaches that students might take in calculating $2\frac{1}{5} - \frac{2}{5}$.

Decomposing the 2 into fifths.

$$2\frac{1}{5} - \frac{2}{5} = \left(1 + 1 + \frac{1}{5}\right) - \frac{2}{5}$$
$$= \left(\frac{5}{5} + \frac{5}{5} + \frac{1}{5}\right) - \frac{2}{5}$$
$$= \frac{5}{5} + \frac{4}{5}$$
$$= \frac{9}{5}$$

Decomposing the 2 as $1 + 1$, and using the associative and commutative properties.

$$2\frac{1}{5} - \frac{2}{5} = \left(1 + 1 + \frac{1}{5}\right) - \frac{2}{5}$$
$$= \left(1 + \frac{1}{5}\right) + \left(\frac{5}{5} - \frac{2}{5}\right)$$
$$= \left(1 + \frac{1}{5}\right) + \frac{3}{5}$$
$$= 1 + \frac{4}{5}$$
$$= 1\frac{4}{5}$$

Decomposing a one as 5 fifths.

$$2\frac{1}{5} - \frac{2}{5} = \left(1 + \frac{5}{5} + \frac{1}{5}\right) - \frac{2}{5}$$
$$= 1\frac{6}{5} - \frac{2}{5}$$
$$= 1\frac{4}{5}$$

The third approach is an analogue of what students learned when subtracting two-digit whole numbers in Grade 2: decomposing a unit of the minuend into smaller units (see the Number and Operations in Base Ten Progression). Instead of decomposing a ten into 10 ones as in Grade 2, a one has been decomposed into 5 fifths. The same approach of decomposing a one (this time into 10 tenths) could be used to compute $2\frac{1}{10} - \frac{2}{10}$:

$$2\frac{1}{10} - \frac{2}{10} = \left(1 + \frac{10}{10} + \frac{1}{10}\right) - \frac{2}{10} = 1\frac{11}{10} - \frac{2}{10} = 1\frac{9}{10}.$$

• Use of parentheses, but not necessarily fluency with parentheses, is expected in Grade 5 (see 5.OA.1). However, reading expressions with parentheses may begin earlier.

Calculating $2\frac{1}{5} - \frac{2}{5}$: Decomposing the 2 into fifths

Calculating $2\frac{1}{5} - \frac{2}{5}$: Using properties

Calculating $2\frac{1}{5} - \frac{2}{5}$: Decomposing a one as 5 fifths

$1 + \frac{5}{5} + \frac{1}{5}$

$\not{2}\frac{\not{1}4}{\not{5}}$
$-\frac{2}{5}$
$1\frac{4}{5}$

writing vertically with helping step

$1\frac{6}{5}$

$\not{2}\frac{\not{1}}{\not{5}}$
$-\frac{2}{5}$
$1\frac{4}{5}$

mentally regrouping $\frac{5}{5}$

This approach is used in Grade 5 when such computations are carried out in decimal notation.[5.NBT.7]

When solving word problems students learn to attend carefully to the underlying quantities (MP.6). In an equation of the form $A + B = C$ or $A - B = C$ for a word problem, the numbers A, B, and C must all refer to the same whole, in terms of the same units.[4.NF.3d] For example, students understand that the problem

> Bill had $\frac{2}{3}$ cup of juice. He drank half of his juice. How much juice did Bill have left?

cannot be solved by computing $\frac{2}{3} - \frac{1}{2}$. Although the $\frac{2}{3}$ and "half" both refer to the same object (the amount of juice that Bill had), the whole for $\frac{2}{3}$ is 1 cup, but the half refers to the amount of juice that Bill drank, using the $\frac{2}{3}$ cup as the whole.

Similarly, in solving

> If $\frac{1}{4}$ of a garden is planted with daffodils, $\frac{1}{3}$ with tulips, and the rest with vegetables, what fraction of the garden is planted with flowers?

students understand that the sum $\frac{1}{4} + \frac{1}{3}$ tells them the fraction of the garden that was planted with flowers, but not the number of flowers that were planted.

Multiplication of a fraction by a whole number Previously in Grade 3, students learned that 5×3 can be represented as the number of objects in 5 groups of 3 objects, describing this product as five threes. (As discussed in the Operations and Algebraic Thinking Progression, in other countries this may be described as three fives.) Third graders use multiplication to solve problems about equal groups and arrays, first about objects with discrete attributes (e.g., bags of plums), then about objects with continuous attributes (e.g., lengths of string), representing these with tape diagrams. Third graders also learn that a fraction is composed of unit fractions, e.g., $\frac{5}{3}$ is five thirds just as 50 is five tens, and represent fractions with tape diagrams. Grade 4 students combine these understandings to see

$$\frac{5}{3} \quad \text{as} \quad 5 \times \frac{1}{3}.$$

Grade 3 representations of 5×3

| 3 plums | 3 plums | 3 plums | 3 plums | 3 plums |

There are 5 bags with 3 plums in each bag. How many plums are there in all?

| 3 feet | 3 feet | 3 feet | 3 feet | 3 feet |

You need 5 lengths of string, each 3 feet long. How much string will you need altogether?

Grade 3 representation of $\frac{5}{3}$

| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

In general, they come to see a fraction as the numerator times the unit fraction with the same denominator,[4.NF.4a] e.g.,

$$\frac{7}{5} = 7 \times \frac{1}{5}, \qquad \frac{11}{3} = 11 \times \frac{1}{3}.$$

The same thinking, based on the analogy between fractions and whole numbers, allows students to give meaning to the product of a whole number and a fraction,[4.NF.4b] e.g., they see

$$3 \times \frac{2}{5} \quad \text{as} \quad \frac{3 \times 2}{5} = \frac{6}{5}.$$

Students solve word problems involving multiplication of a fraction by a whole number.[4.NF.4c]

> If a bucket holds $2\frac{3}{4}$ gallons and 3 buckets of water fill a tank, how many gallons does the tank hold?

The answer is $3 \times 2\frac{3}{4}$ which is

$$3 \times \left(2 + \frac{3}{4}\right) = 3 \times \frac{11}{4} = \frac{33}{4} = 8\frac{1}{4}.$$

Students can also use the distributive property to calculate

$$3 \times \left(2 + \frac{3}{4}\right) = 3 \times 2 + 3 \times \frac{3}{4} = 6 + \frac{9}{4} = 6 + 2\frac{1}{4} = 8\frac{1}{4}.$$

Decimal fractions and decimal notation   Fractions with denominators 10 and 100, called *decimal fractions*, arise when students express dollars as cents,[4.MD.2] and have a more fundamental importance, developed in Grade 5, in the base-ten system (see the Grade 5 section of the Number and Operations in Base Ten Progression). For example, because there are 10 dimes in a dollar, 3 dimes is $\frac{3}{10}$ of a dollar; and it is also $\frac{30}{100}$ of a dollar because it is 30 cents, and there are 100 cents in a dollar. Such reasoning provides a context for the fraction equivalence

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100}.$$

Grade 4 students learn to add decimal fractions by writing them as fractions with the same denominator:[4.NF.5]

$$\frac{3}{10} + \frac{27}{100} = \frac{30}{100} + \frac{27}{100} = \frac{57}{100}.$$

They can interpret this as saying that 3 dimes together with 27 cents make 57 cents. In Grade 5, students build on this experience to compute sums in fraction notation[5.NF.1] or in decimal notation.[5.NBT.7]

Fractions with denominators equal to 10 and 100 can be written by using a *decimal point*.[4.NF.6]• For example,

$$\frac{27}{10} \text{ can be written as } 2.7$$

$$\frac{27}{100} \text{ can be written as } 0.27.$$

The number of digits to the right of the decimal point indicates the number of zeros in the denominator, so that $2.70 = \frac{270}{100}$ and $2.7 = \frac{27}{10}$. Students use their knowledge of equivalent fractions (4.NF.1) to reason that $2.70 = 2.7$ because

$$2.70 = \frac{270}{100} = \frac{10 \times 27}{10 \times 10} = \frac{27}{10} = 2.7.$$

Reflecting these understandings, there are several ways to read decimals aloud. For example, 0.15 can be read aloud as "15 hundredths" or "1 tenth and 5 hundredths,"• reflecting

$$15 \times \frac{1}{100} = 1 \times \frac{1}{10} + 5 \times \frac{1}{100}$$

just as 1,500 can be read aloud as "15 hundred" or "1 thousand, 5 hundred," reflecting

$$15 \times 100 = 1 \times 1{,}000 + 5 \times 100.$$

Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

Students compare decimals using the meaning of a decimal as a fraction, making sure to compare fractions with the same denominator. For example, to compare 0.2 and 0.09, students think of 0.2 as its equivalent 0.20, and see that $0.20 > 0.09$ because[4.NF.7]

$$\frac{20}{100} > \frac{9}{100}.$$

In Grade 5, the argument using the meaning of a decimal as a fraction and using the fundamental property to rewrite decimals as fractions with the same denominator generalizes to work with decimals that have more than two digits.[5.NBT.3b]

Rulers, centimeter grids, and diagrams can help students to understand how small thousandths are relative to 1. A metric ruler can show millimeters as thousandths of a meter. The area of each square of a centimeter grid is a ten-thousandth of a square meter. Thousandths can also be represented as parts of a square as in Grade 4—if the square is assumed to represent $\frac{1}{10}$ (as in the margin) rather than 1. Rulers, grids, and diagrams can support understanding that $0.03 > 0.008$ and $0.2 > 0.008$, but in Grade 5 most comparisons will be done by writing or thinking of the decimals as fractions with the same denominator, e.g.,

$$\frac{30}{1000} > \frac{8}{1000} \quad \text{and} \quad \frac{200}{1000} > \frac{8}{1000}.$$

• Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five."

[4.NF.7] Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols >, =, or <, and justify the conclusions, e.g., by using a visual model.

Seeing that $0.2 > 0.09$

The large square represents 1. *The shaded region on the left shows* 0.2 *of the square, since it is two parts when the square is partitioned into 10 parts of equal area. The shaded region on the right shows* 0.09 *of the square, since it is 9 parts when the unit is partitioned into 100 parts of equal area.*

As on p. 143, students can also use the two partitions of the large square shown above to see that .2 = .20.
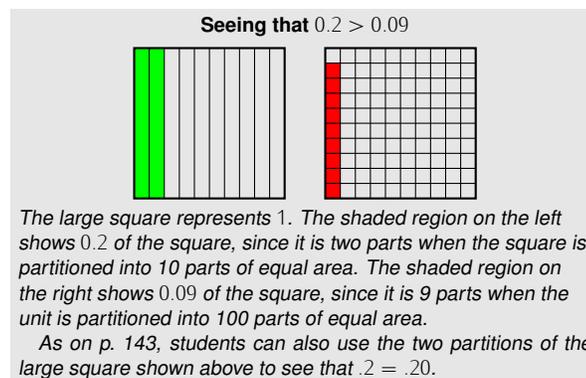
[5.NBT.3b] Compare two decimals to thousandths based on meanings of the digits in each place, using >, =, and < symbols to record the results of comparisons.

Seeing that $0.03 > 0.008$

The large square represents 0.1 *rather than* 1 *as above.*
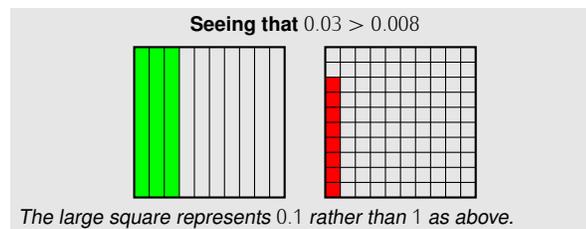
Grade 5

Adding and subtracting fractions   In Grade 4, students acquire some experience in calculating sums of fractions with different de-nominators when they work with decimals and add fractions with denominators 10 and 100, such as

$$\frac{2}{10} + \frac{7}{100} = \frac{20}{100} + \frac{7}{100} = \frac{27}{100}.$$

Note that this is a situation where one denominator is a divisor of the other, so that only one fraction has to be changed. Students might have encountered similar situations, for example using a strip of paper or a tape diagram to reason that

$$\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

They understand the process as expressing both addends in terms of the same unit fraction so that they can be added. Grade 5 stu-dents extend this reasoning to situations where it is necessary to re-express both fractions in terms of a new denominator.[5.NF.1] For example, in calculating $\frac{2}{3} + \frac{5}{4}$ they reason that if each third in $\frac{2}{3}$ is partitioned into four equal parts, and if each fourth in $\frac{5}{4}$ is parti-tioned into three equal parts, then each fraction will be a sum of unit fractions with denominator $3 \times 4 = 4 \times 3 = 12$:

$$\frac{2}{3} + \frac{5}{4} = \frac{2 \times 4}{3 \times 4} + \frac{5 \times 3}{4 \times 3} = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}.$$

In general, two fractions can be added by partitioning the unit fractions in one into the number of equal parts determined by the denominator of the other:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = \frac{a \times d + b \times c}{b \times d}.$$

It is not necessary to find a least common denominator to calculate sums of fractions, and in fact the effort of finding a least common de-nominator is a distraction from understanding algorithms for adding fractions.

Students make sense of fractional quantities when solving word problems, estimating answers mentally to see if they make sense.[5.NF.2] For example in the problem

> Ludmilla and Lazarus each have some lemons. They need a cup of lemon juice to make hummus for a party. Ludmilla squeezes $\frac{1}{2}$ cup from hers and Lazarus squeezes $\frac{2}{5}$ cup from his. How much lemon juice do they have? Is it enough?

students estimate that there is almost but not quite one cup of lemon juice, because $\frac{2}{5} < \frac{1}{2}$. They calculate $\frac{1}{2} + \frac{2}{5} = \frac{9}{10}$, and see this as $\frac{1}{10}$ less than 1, which is probably a small enough shortfall that it will not ruin the recipe. They detect an incorrect result such as $\frac{1}{2} + \frac{2}{5} = \frac{3}{7}$ by noticing that $\frac{3}{7} < \frac{1}{2}$.
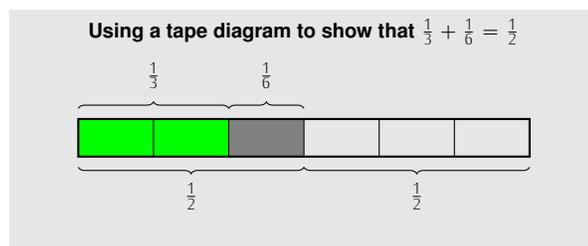
Using a tape diagram to show that $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

5.NF.1 Add and subtract fractions with unlike denominators (in-cluding mixed numbers) by replacing given fractions with equiva-lent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators.

5.NF.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reason-ableness of answers.

Multiplying and dividing fractions   In Grade 4, students connected fractions with multiplication, understanding that

$$\frac{5}{3} = 5 \times \frac{1}{3}.$$

In Grade 5, they connect fractions with division, understanding that

$$5 \div 3 = \frac{5}{3},$$

or, more generally, $a \div b = \frac{a}{b}$ for whole numbers a and b, with b not equal to zero.[5.NF.3] They can explain this connection using the sharing (partitive) interpretation of division (see figure in margin). They also create story contexts to represent problems involving division of whole numbers. For example, they see two ways of solving:

> If 9 people want to share a 50–pound sack of rice equally by weight, how many pounds of rice should each person get?

First, they might partition each pound among the 9 people, calculating $50 \times \frac{1}{9} = \frac{50}{9}$ so that each person gets $\frac{50}{9}$ pounds. Second, they might use the equation $9 \times 5 = 45$ to see that each person can be given 5 pounds, with 5 pounds remaining. Partitioning the remainder gives $5\frac{5}{9}$ pounds for each person.

Students have, since Grade 1, been using language such as "third of" to describe one part when a whole is partitioned into three equal parts. Using their new understanding of the connection between fractions and division illustrated by examples like the sharing situation in the margin, students now see that $\frac{5}{3}$ is one third of 5, which leads to the meaning of multiplication by a unit fraction:

$$\frac{1}{3} \times 5 = \frac{5}{3}.$$

This in turn extends to multiplication of any number by a fraction.[5.NF.4a]

$\frac{1}{3} \times 5$ is 1 part when 5 is partitioned in 3 equal parts

,

$\frac{2}{3} \times 5$ is 2 parts , $\frac{3}{3} \times 5$ is 3 parts ,

$\frac{4}{3} \times 5$ is 4 parts , and so on.

5 **partitioned into** 3 **equal parts:** $5 \div 3 = \frac{5}{3}$ **and** $\frac{1}{3} \times 5 = \frac{5}{3}$
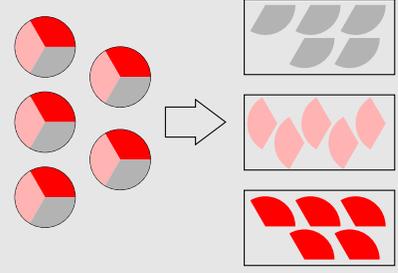
If you share 5 objects equally among 3 people, each of the 5 objects should contribute $\frac{1}{3}$ of itself to each share. Thus, each share consists of 5 pieces, each of which is $\frac{1}{3}$ of an object. Because $5 \times \frac{1}{3} = \frac{5}{3}$, each share is $\frac{5}{3}$ of an object.

Using a tape diagram to show that $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
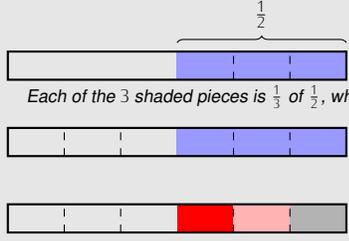
One half partitioned in 3 equal parts.

Each of the 3 shaded pieces is $\frac{1}{3}$ of $\frac{1}{2}$, which is $\frac{1}{3} \times \frac{1}{2}$.

Other half partitioned in 3 equal parts.

There are 6 equal pieces, so each is $\frac{1}{6}$, as well as $\frac{1}{3} \times \frac{1}{2}$.

Using this understanding of multiplication by a fraction, students develop the general formula for the product of two fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d},$$

for whole numbers a, b, c, d, with b, d not zero. Grade 5 students need not express the formula in this general algebraic form, but rather recognize numerical instances from reasoning repeatedly from many examples (MP.8), using strips of paper, tape diagrams, and number line diagrams.

Having established a meaning for the product of two fractions and an understanding of how to calculate such products, students use concepts of area measurement from Grade 3[3.MD.5] to see that the method that they used to find areas of rectangles with whole-number side lengths in Grade 3[3.MD.7a] can be extended to rectangles with fractional side lengths.

In Grade 3, students learned to understand and explain why multiplying the side lengths of a rectangle yields the same measurement of area as counting the number of tiles (with the same unit length) that fill the rectangle's interior (MP.3).[3.MD.7a] For example, students might have explained that the area of the rectangle is the number of rows of unit squares times the number of unit squares in each row.

In Grade 5, instead of using a unit square with a side length of 1, students use a unit square with a side length that is a unit fraction.[5.NF.4b] For example, a $\frac{5}{3}$ by $\frac{3}{4}$ rectangle can be tiled by unit squares of side length $\frac{1}{12}$. Because 12×12 of these unit squares tile a square of side length 1, each has area $\frac{1}{12 \times 12}$ (see lower left). The area of the rectangle is the number of squares times the area of each square, which is $\frac{5}{3} \times \frac{3}{4}$, the product of the side lengths.
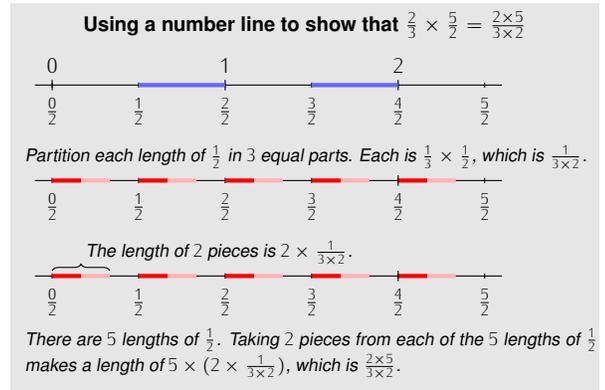
Students can use similar reasoning with other tilings of a square of side length 1. For example, when working with a rectangle that has fractional side lengths, students can see it as tiled by copies of a smaller rectangle with unit fraction side lengths (see lower right).

Using a number line to show that $\frac{2}{3} \times \frac{5}{2} = \frac{2 \times 5}{3 \times 2}$

Partition each length of $\frac{1}{2}$ in 3 equal parts. Each is $\frac{1}{3} \times \frac{1}{2}$, which is $\frac{1}{3 \times 2}$.

The length of 2 pieces is $2 \times \frac{1}{3 \times 2}$.

There are 5 lengths of $\frac{1}{2}$. Taking 2 pieces from each of the 5 lengths of $\frac{1}{2}$ makes a length of $5 \times (2 \times \frac{1}{3 \times 2})$, which is $\frac{2 \times 5}{3 \times 2}$.

3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement.

    a A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area.

    b A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.

3.MD.7 Relate area to the operations of multiplication and addition.

    a Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

    b Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.

Tiling by a unit square with unit fraction side length

Because 12×12 squares with side length $\frac{1}{12}$ tile a square with side length 1, each small square has area $\frac{1}{12 \times 12}$.

The rectangle is tiled with 5×4 rows of squares, with 3×3 squares in each row. Each square has area $\frac{1}{12 \times 12}$, so the rectangle has area $\frac{5 \times 4 \times 3 \times 3}{12 \times 12}$, which is $\frac{5}{3} \times \frac{3}{4}$.

Tiling by a rectangle with unit fraction side lengths

Because 4×3 rectangles $\frac{1}{4}$ wide and $\frac{1}{3}$ high tile a square with side length 1, each rectangle has area $\frac{1}{4 \times 3}$.

The large rectangle is tiled with 5 rows of rectangles, with 3 rectangles in each row. Each rectangle has area $\frac{1}{4 \times 3}$, so the large rectangle has area $\frac{3 \times 5}{4 \times 3}$, which is $\frac{5}{3} \times \frac{3}{4}$.
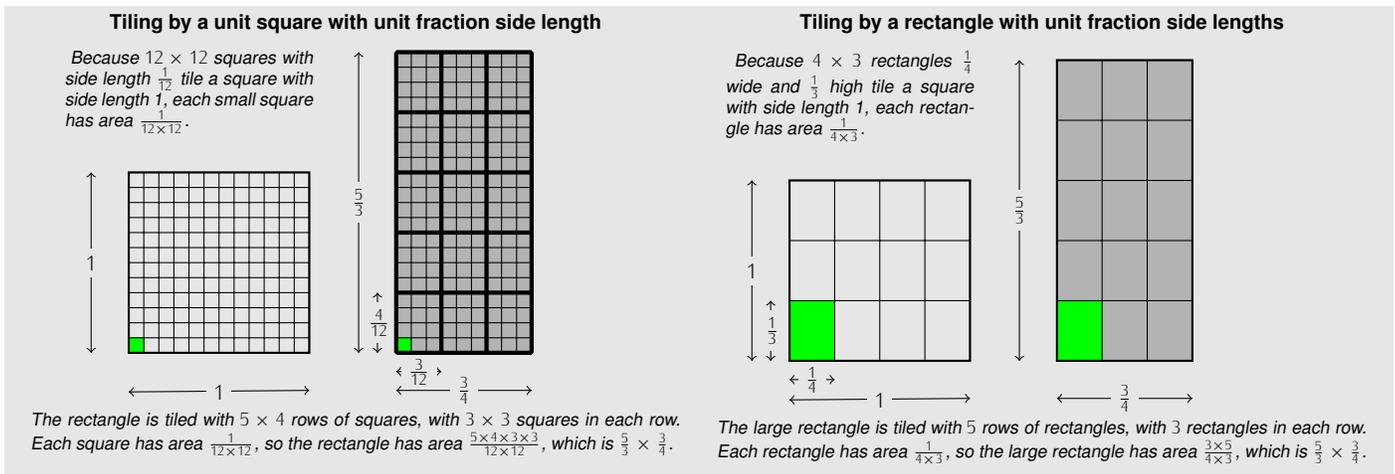
Students also understand fraction multiplication by creating story problems. For example, to explain

$$\frac{2}{3} \times 4 = \frac{8}{3},$$

they might say

> Ron and Hermione have 4 pounds of Bertie Bott's Every Flavour Beans. They decide to share them 3 ways, saving one share for Harry. How many pounds of beans do Ron and Hermione get?

In multiplication calculations, the distributive property may be shown symbolically or—because the area of a rectangle is the product of its side lengths—with an area model (see margin). Here, it is used in a variation of a word problem from the Grade 4 section.

> If a bucket holds $2\frac{3}{4}$ gallons and 43 buckets of water fill a tank, how many gallons does the tank hold?

The answer is $43 \times 2\frac{3}{4}$, which is

$$
\begin{aligned}
43 \times \left(2 + \frac{3}{4}\right) &= 43 \times 2 + 43 \times \frac{3}{4} \\
&= 86 + \left(40 \times \frac{3}{4}\right) + \left(3 \times \frac{3}{4}\right) \\
&= 86 + 30 + \frac{9}{4} \\
&= 118\frac{1}{4}.
\end{aligned}
$$

Using the relationship between division and multiplication, students start working with quotients that have unit fractions. Having seen that dividing a whole number by a whole number, e.g., $5 \div 3$, is the same as multiplying the number by a unit fraction, $\frac{1}{3} \times 5$, they now extend the same reasoning to division of a unit fraction by a whole number, seeing for example that[5.NF.7a]
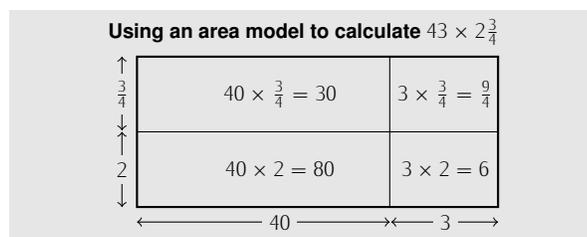
$$\frac{1}{6} \div 3 = \frac{1}{6 \times 3} = \frac{1}{18}.$$

Also, they reason that since there are 6 portions of $\frac{1}{6}$ in 1, there must be 3×6 in 3, and so[5.NF.7b]

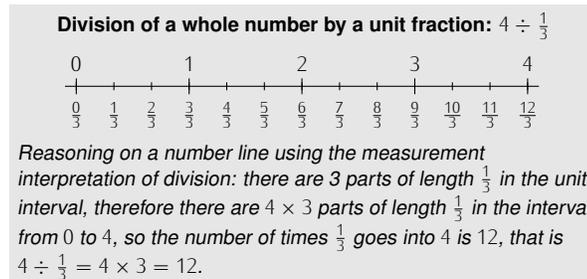$$3 \div \frac{1}{6} = 3 \times 6 = 18.$$

Students use story problems to make sense of division:[5.NF.7c]

> How much chocolate will each person get if 3 people share $\frac{1}{2}$ lb of chocolate equally? How many $\frac{1}{3}$-cup servings are in 2 cups of raisins?

Using an area model to calculate $43 \times 2\frac{3}{4}$

	$40 \times \frac{3}{4} = 30$	$3 \times \frac{3}{4} = \frac{9}{4}$
$\frac{3}{4}$		
2	$40 \times 2 = 80$	$3 \times 2 = 6$

5.NF.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

 a Interpret division of a unit fraction by a non-zero whole number, and compute such quotients.

 b Interpret division of a whole number by a unit fraction, and compute such quotients.

 c Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions, e.g., by using visual fraction models and equations to represent the problem.

Division of a unit fraction by a whole number: $\frac{1}{2} \div 3$

Reasoning with a tape diagram using the sharing interpretation of division: the tape is the whole and the shaded length is $\frac{1}{2}$ of the whole. If the shaded length is partitioned into 3 equal parts, then 2×3 of those parts compose the whole, so $\frac{1}{2} \div 3 = \frac{1}{2 \times 3} = \frac{1}{6}$.

Division of a whole number by a unit fraction: $4 \div \frac{1}{3}$

Reasoning on a number line using the measurement interpretation of division: there are 3 parts of length $\frac{1}{3}$ in the unit interval, therefore there are 4×3 parts of length $\frac{1}{3}$ in the interval from 0 to 4, so the number of times $\frac{1}{3}$ goes into 4 is 12, that is $4 \div \frac{1}{3} = 4 \times 3 = 12$.

Students attend carefully to the underlying quantities when solving problems. For example, if $\frac{1}{2}$ of a fund-raiser's funds were raised by the sixth grade, and if $\frac{1}{3}$ of the sixth grade's funds were raised by Ms. Wilkin's class, then $\frac{1}{3} \times \frac{1}{2}$ gives the fraction of the fund-raiser's funds that Ms. Wilkin's class raised, but it does not tell us how much money Ms. Wilkin's class raised.[5.NF.6]

Multiplication as scaling   In preparation for Grade 6 work with ratios and proportional relationships, students learn to see products such as 5×3 or $\frac{1}{2} \times 3$ as expressions that can be interpreted as an amount, 3, and a scaling factor, 5 or $\frac{1}{2}$. Thus, in addition to knowing that $5 \times 3 = 15$, they can also say that 5×3 is 5 times as big as 3, without evaluating the product. Likewise, they see $\frac{1}{2} \times 3$ as half the size of 3.[5.NF.5a]

The understanding of multiplication as scaling is an important opportunity for students to reason abstractly (MP.2). Work with multiplication as scaling can serve as a useful summary of how the results of multiplication and division depend on the numbers involved. Previous work with multiplication by whole numbers enables students to see multiplication by numbers bigger than 1 as producing a larger quantity, as when a price is doubled, for example. Grade 5 work with multiplying by unit fractions, and interpreting fractions in terms of division, enables students to see that multiplying a quantity by a number smaller than 1 produces a smaller quantity, as when a price is multiplied by $\frac{1}{2}$, for example.[5.NF.5b]

The special case of multiplying by 1, which leaves a quantity unchanged, can be related to fraction equivalence by expressing 1 as $\frac{n}{n}$, as explained on page 144.

[5.NF.6] Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

[5.NF.5] Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Where this progression is heading

In Grade 5, students interpreted a fraction as the number resulting from division of the numerator by the denominator, e.g., they saw that $5 \div 3 = \frac{5}{3}$. In Grade 6, students see whole numbers and fractions as part of the system of rational numbers, understanding order, magnitude, and absolute value in terms of the number line. In Grade 7, students use properties of operations and their understanding of operations on fractions to extend those operations to rational numbers. Their new understanding of division allows students to extend their use of fraction notation from non–negative rational numbers to all rational numbers, e.g., $\frac{-3}{4} = -3 \div 4$ and $\frac{\frac{2}{3}}{-\frac{1}{2}} = \frac{2}{3} \div -\frac{1}{2}$ (see the Number System Progression). Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers.

Work with fractions and multiplication is a building block for work with ratios.• In Grades 6 and 7, students use their understanding of wholes and parts, and their knowledge of multiplication within 100, to reason about ratios of two quantities, making and analyzing tables of equivalent ratios, and graphing pairs from these tables in the coordinate plane. These tables and graphs represent proportional relationships, which students see as functions in Grade 8.

Understanding of multiplication as scaling is extended in work with ratios (see the Ratios and Proportional Relationships Progression) and in work with scale drawings (see the 7–8 Geometry Progression). Students' understanding of scaling is further extended when they work with similarity and dilations of the plane, using physical models, transparencies, or geometry software in Grade 8, and using properties of dilations in high school (see the high school Geometry Progression).

• Note that in the Standards, "fraction" and "ratio" refer to different concepts and that different notation is used with each. For example, $\frac{3}{2}$ is not used to represent $3 : 2$. Equivalence for fractions is denoted with the equal sign, e.g., $\frac{3}{2} = \frac{6}{4}$, but the equal sign is not used to denote the equivalence of two pairs of numerical measurements that are in the same ratio.

Ratios and Proportional Relationships, 6–7

Overview

The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades. Ratios and proportional relationships are foundational for further study in mathematics and science and useful in everyday life. Students use ratios in geometry and in algebra when they study similar figures and slopes of lines, and later when they study sine, cosine, tangent, and other trigonometric ratios in high school. Students use ratios when they work with situations involving constant rates of change, and later in calculus when they work with average and instantaneous rates of change of functions. An understanding of ratio is essential in the sciences to make sense of quantities that involve derived attributes such as speed, acceleration, density, surface tension, electric or magnetic field strength, and to understand percentages and ratios used in describing chemical solutions. Ratios and percentages are also useful in many situations in daily life, such as in cooking and in calculating tips, miles per gallon, taxes, and discounts. They also are also involved in a variety of descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g., birth rate, per capita income, body mass index, rain fall, and crop yield) and underlie a variety of measures, for example, in finance (exchange rate), medicine (dose for a given body weight), and technology (kilobits per second).

Ratios and rates in the Standards   Ratios arise in situations in which two or more quantities are related.• Sometimes the quantities have the same units (e.g., 3 cups of apple juice and 2 cups of grape juice), other times they do not (e.g., 3 meters and 2 seconds). Some authors distinguish ratios from rates, using the term "ratio" when units are the same and "rate" when units are different; others use ratio to encompass both kinds of situations. The Standards use

• In the Standards, a quantity involves measurement of an attribute. Quantities may be discrete, e.g., 4 apples, or continuous, e.g., 4 inches. They may be measurements of physical attributes such as length, area, volume, weight, or other measurable attributes such as income. Quantities can vary with respect to another quantity. For example, the quantities "distance between the earth and the sun in miles," "distance (in meters) that Sharoya walked," or "my height in feet" vary with time.

ratio in the second sense, applying it to situations in which units are the same as well as to situations in which units are different. Relationships of two quantities in such situations may be described in terms of ratios, rates, percentages, or proportional relationships.

A *ratio* associates two or more quantities. Ratios can be indicated in words as "3 to 2" and "3 for every 2" and "3 out of every 5" and "3 parts to 2 parts." This use might include units, e.g., "3 cups of flour for every 2 eggs" or "3 meters in 2 seconds." Notation for ratios can include the use of a colon, as in 3 : 2, when referents for underlying quantities are clear (MP.6). The quotient $\frac{3}{2}$ is sometimes called the value of the ratio 3 : 2.•

Ratios of two quantities have associated *rates*. For example, the ratio 3 feet for every 2 seconds has the associated rate $\frac{3}{2}$ feet for every 1 second; the ratio 3 cups apple juice for every 2 cups grape juice has the associated rate $\frac{3}{2}$ cups apple juice for every 1 cup grape juice. In Grades 6 and 7, students describe rates in terms such as "for each 1," "for each," and "per." In the Standards, the *unit rate* is the numerical part of such rates; the "unit" in "unit rate" is often used to highlight the 1 in "for each 1" or "for every 1."

Equivalent ratios arise by multiplying each number in a ratio by the same positive number. For example, the pairs of numbers of meters and seconds in the margin are in equivalent ratios. Such pairs are also said to be in the same ratio. Proportional relationships involve collections of pairs of measurements in equivalent ratios. In contrast, a proportion is an equation stating that two ratios are equivalent. Equivalent ratios have the same unit rate.

The pairs of meters and seconds in the margin show distance and elapsed time varying together in a *proportional relationship*. This situation can be described as "distance traveled and time elapsed are proportionally related," or "distance and time are directly proportional," or simply "distance and time are proportional," or "distance is proportional to time." The proportional relationship can be represented with the equation $d = \frac{3}{2}t$. The factor $\frac{3}{2}$ is the constant unit rate associated with the different pairs of measurements in the proportional relationship; it is known as a *constant of proportionality*.

More precise definitions of the terms presented here and a framework for organizing and relating the concepts are presented in the Appendix.

• In everyday language, the word "ratio" sometimes refers to the value of a ratio, for example in the phrases "take the ratio of price to earnings" or "the ratio of circumference to diameter is π."

Representing pairs in a proportional relationship

Sharoya walks 3 meters every 2 seconds. Let d be the number of meters Sharoya has walked after t seconds. d and t are in a proportional relationship.

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

d and t *are related by the equation* $d = \frac{3}{2}t$.
Students sometimes use the equal sign incorrectly to indicate proportional relationships, for example, they might write "3 m = 2 sec" to represent the correspondence between 3 meters and 2 seconds. In fact, 3 meters is not equal to 2 seconds, so use of the equal sign would not be appropriate (MP.6). This relationship can be represented in a table or by writing "3 m → 2 sec." Note that the unit rate appears in the pair $\left(\frac{3}{2}, 1\right)$.

	ratio language	rate language	unit rate
same units	3 cups apple juice to 2 cups grape juice 3 cups apple juice for every 2 cups grape juice 3 cups apple juice per 2 cups grape juice	$\frac{3}{2}$ cups apple juice for each 1 cup grape juice 1.5 cups apple juice per cup grape juice	$\frac{3}{2}$
different units	3 meters in 2 seconds 3 meters for every 2 seconds 3 meters per 2 seconds	1.5 meters for each second $\frac{3}{2}$ meters per second	$\frac{3}{2}$

In high school, students express rates in terms of derived units, e.g., writing $\frac{3}{2}$ m/s instead of $\frac{3}{2}$ meters per second.

The word *percent* means "per 100" (*cent* is an abbreviation of the Latin *centum* "hundred"). If 35 milliliters out of every 100 milliliters in a juice mixture are orange juice, then the juice mixture is 35% orange juice (by volume). If a juice mixture is viewed as made of 100 equal parts, of which 35 are orange juice, then the juice mixture is 35% orange juice.
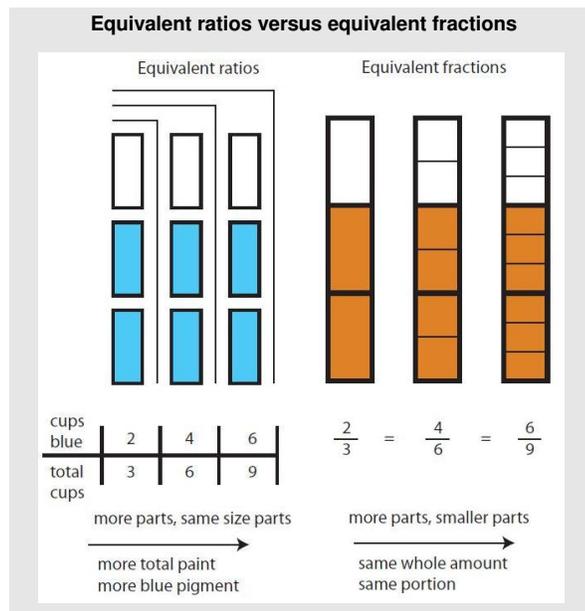
A percent is a rate per 100. One unit of the second quantity is partitioned in 100 parts and expressed as 100. The corresponding amount of the first quantity is expressed in terms of those parts. Because of this, the percent does not include units of measurement such as liters or grams.•

• This can make descriptions such as "35% orange juice" ambiguous because the orange juice could have been measured by volume or by weight. Often, this is addressed by adding descriptors such as "by volume" (as in the juice example) or "by weight" (MP.6).

Recognizing and describing ratios, rates, and proportional relationships   "For each," "for every," "per," and similar terms distinguish situations in which two quantities have a proportional relationship from other types of situations. For example, without further information "2 pounds for a dollar" is ambiguous. It may be that pounds and dollars are proportionally related and *every* two pounds costs a dollar. Or it may be that there is a discount on bulk, so weight and cost do not have a proportional relationship. Thus, recognizing ratios, rates, and proportional relationships involves looking for structure (MP.7). Describing and interpreting descriptions of ratios, rates, and proportional relationships involves precise use of language (MP.6).

Representing ratios, rates, collections of equivalent ratios, and proportional relationships   Ratio notation should be distinct from fraction notation. Using the same notation for ratios and rational numbers may suggest that computations are the same for both, but this is not the case. For example, suppose a batch of paint is a mixture of 1 cup of white paint and 2 cups of blue paint. So the ratio of white to blue is 1 cup to 2 cups. Two batches of this paint have double these amounts, making an equivalent ratio of 2 cups to 4 cups. If these ratios are represented as $\frac{1}{2}$ and $\frac{2}{4}$, then it seems that two times $\frac{1}{2}$ is $\frac{2}{4}$. Another problem arises with addition. Suppose one batch of paint is made from 2 cups of red paint and 2 cups of yellow paint, and another is made from 1 cup of red paint and 3 cups of yellow paint. So the ratio of red to yellow is 2 : 2 in the first batch and 1 : 3 in the second batch. Now suppose the two batches are combined: what is the ratio of red to yellow in the combined batches? Add 2 and 1 to get 3 cups of red paint, and 2 and 3 to get 5 cups of yellow paint, so the ratio is 3 : 5 in the combined batches. If the ratios are represented as fractions, it seems that $\frac{2}{2} + \frac{1}{3}$ is $\frac{3}{5}$.

In middle grades, students are expected to describe a rate in words rather than as a number followed by a unit, e.g., $\frac{3}{2}$ meters per second rather than $\frac{3}{2}$ m/s. Like all quantities, derived quantities such as rates can be specified by a number followed by a unit.• Understanding such derived quantities requires students to under-

Equivalent ratios versus equivalent fractions

Equivalent ratios

Equivalent fractions

| cups blue | 2 | 4 | 6 |
| total cups | 3 | 6 | 9 |

more parts, same size parts

more total paint
more blue pigment

$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$

more parts, smaller parts

same whole amount
same portion

• Although derived quantities such as area or rate are sometimes described as products or quotients of attributes, for example, rectangular area (length multiplied by length) or speed (distance divided by time), these descriptions may suggest that a derived quantity is written as a product or quotient of other quantities, e.g., 5 inches × 4 inches or $\frac{3 \text{ miles}}{2 \text{ hours}}$, or is not itself a quantity.

stand two or more quantities simultaneously (e.g., speed as entailing displacement *and* time, simultaneously).

Diagrams. Together with tables, students can also use tape diagrams and double number line diagrams to represent collections of equivalent ratios. Both types of diagrams visually depict the relative sizes of the quantities.

Tape diagrams are best used when the two quantities have the same units. They can be used to solve problems and also to highlight the multiplicative relationship between the quantities.

Double number line diagrams are best used when the quantities have different units. They can help make visible that there are many, even infinitely many, pairs in the same ratio, including those with rational number entries. As in tables, unit rates appear paired with 1 in double number line diagrams.

Graphs and equations. A collection of equivalent ratios can be graphed in the coordinate plane. The graph represents a proportional relationship. The unit rate appears in the equation and graph as the slope of the line, and in the coordinate pair with first coordinate 1.

Choosing an order. Representing a ratio or collection of equivalent ratios may require choosing an order for the quantities represented. When a ratio is indicated in words, e.g., "orange paint is made by mixing 1 cup red paint for every 3 cups yellow paint," another representation might follow the order in the description, e.g., in a table, indicating possible amounts of red paint in the leftmost column and corresponding amounts of yellow paint in the column to its right (as on p. 161), or reverse the order. In either case, the columns need to be labeled appropriately (MP.6). Similarly, when the equivalent ratios are graphed, the number of cups of red paint could be shown on the horizontal or vertical axis. The plotted values would lie on a line with slope 3 in the first case and on a line with slope $\frac{1}{3}$ in the second case. The relationship shown in each graph could be described as "the amount of yellow paint is proportional to the amount of red paint" or "the amount of red paint is proportional to the amount of yellow paint."

When there are two descriptions of a relationship, the order of quantities in one description often follows the order in another. For example, if a graph shows amount of yellow paint on the vertical axis and amount of red paint on the horizontal axis, the correspondence of graph and description may be more obvious if the relationship is described as "the amount of yellow paint is proportional to the amount of red paint" or $y = 3r$ rather than "the amount of red paint is proportional to the amount of yellow paint" or $r = \frac{1}{3}y$.

Choosing and maintaining an order affords some simplification, allowing references to *the* rate or *the* unit rate to be unambiguous. But, students need to be aware that reversals in order may occur within descriptions of a situation (see problem statement examples on p. 164).

Representing ratios with tape diagrams

apple juice:
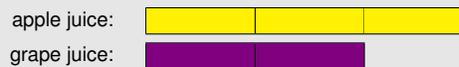
grape juice:

This diagram can be interpreted as representing any mixture of apple juice and grape juice with a ratio of 3 to 2. The total amount of juice is represented as partitioned into 5 parts of equal size, represented by 5 rectangles. For example, if the diagram represents 5 cups of juice mixture, then each of these rectangles represents 1 cup. If the total amount of juice mixture is 1 gallon, then each part represents $\frac{1}{5}$ gallon and there are $\frac{3}{5}$ gallon of apple juice and $\frac{2}{5}$ gallon of grape juice.

Representing ratios with double number line diagrams
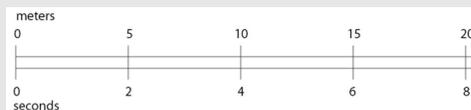
On double number line diagrams, if A and B are in the same ratio, then A and B are located at the same distance from 0 on their respective lines. Multiplying A and B by a positive number p results in a pair of numbers whose distance from 0 is p times as far. So, for example, 3 times the pair 2 and 5 results in the pair 6 and 15 which is located at 3 times the distance from 0.

Grade 6

Representing and reasoning about ratios and collections of equivalent ratios   Because the multiplication table is familiar to sixth graders, situations that give rise to columns or rows of a multiplication table can provide good initial contexts when ratios and proportional relationships are introduced. Pairs of quantities in equivalent ratios arising from whole-number measurements such as "3 lemons for every \$1" or "for every 5 cups grape juice, mix in 2 cups peach juice" lend themselves to being recorded in a table.[6.RP.3a] Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios.

As students work with tables of quantities in equivalent ratios (also called ratio tables), they should practice using and understanding ratio and rate language.[6.RP.1,6.RP.2] It is important for students to focus on the meaning of the terms "for every," "for each," "for each 1," and "per" because these equivalent ways of stating ratios and rates are at the heart of understanding the structure in these tables, providing a foundation for learning about proportional relationships in Grade 7.

Students graph the pairs of values displayed in ratio tables on coordinate axes. The graph of such a collection of equivalent ratios lies on a line through the origin, and the pattern of increases in the table can be seen in the graph as coordinated horizontal and vertical increases.[6.EE.9]

[6.RP.3]Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

   a Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

[6.RP.1]Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

[6.RP.2]Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.[1]

[1]Expectations for unit rates in this grade are limited to non-complex fractions.

[6.EE.9]Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

Showing structure in tables and graphs

In the tables, equivalent ratios are generated by repeated addition (left) and by scalar multiplication (right). Students might be asked to identify and explain correspondences between each table and the graph beneath it (MP.1).

By reasoning about ratio tables to compare ratios,[6.RP.3a] students can deepen their understanding of what a ratio describes in a context and what quantities in equivalent ratios have in common. For example, suppose Abby's orange paint is made by mixing 1 cup red paint for every 3 cups yellow paint and Zack's orange paint is made by mixing 3 cups red for every 5 cups yellow. Students could discuss that all the mixtures within a single ratio table for one of the paint mixtures are the same shade of orange because they are multiple batches of the same mixture. For example, 2 cups red and 6 cups yellow is two batches of 1 cup red and 3 cups yellow; each batch is the same color, so when the two batches are combined, the shade of orange doesn't change. Therefore, to compare the colors of the two paint mixtures, any entry within a ratio table for one mixture can be compared with any entry from the ratio table for the other mixture.

It is important for students to focus on the rows (or columns) of a ratio table as multiples of each other. If this is not done, a common error when working with ratios is to make additive comparisons. For example, students may think incorrectly that the ratios 1 : 3 and 3 : 5 of red to yellow in Abby's and Zack's paints are equivalent because the difference between the number of cups of red and yellow in both paints is the same, or because Zack's paint could be made from Abby's by adding 2 cups red and 2 cups yellow. The margin shows several ways students could reason correctly to compare the paint mixtures.

Strategies for solving problems   Although it is traditional to move students quickly to solving proportions by setting up an equation, the Standards do not require this method in Grade 6. There are a number of strategies for solving problems that involve ratios. As students become familiar with relationships among equivalent ratios, their strategies become increasingly abbreviated and efficient.

For example, suppose grape juice and peach juice are mixed in a ratio of 5 to 2 and we want to know how many cups of grape juice to mix with 12 cups of peach juice so that the mixture will still be in the same ratio. Students could make a ratio table as shown in the margin, and they could use the table to find the grape juice entry that pairs with 12 cups of peach juice in the table. This perspective allows students to begin to reason about proportions by starting with their knowledge about multiplication tables and by building on this knowledge.

As students generate equivalent ratios and record them in tables, their attention should be drawn to the important role of multiplication and division in how entries are related to each other. Initially, students may fill ratio tables with columns or rows of the multiplication table by skip counting, using only whole-number entries, and placing these entries in numerical order. Gradually, students should consider entries in ratio tables beyond those they find by skip counting, including larger entries and fraction or decimal entries. Finding

[6.RP.3a]Make tables of equivalent ratios relating quantities with whole-number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

Three ways to compare paint mixtures

After tables are used to compare ratios, the values in the two tables might be plotted on the same set of axes, allowing visual comparisons of points, e.g., $(3, 9)$ and $(3, 5)$, or steepness of associated lines.

Double number lines used for situations with different units

Double number line diagrams indicate coordinated multiplying and dividing of quantities. This can also be indicated in tables.
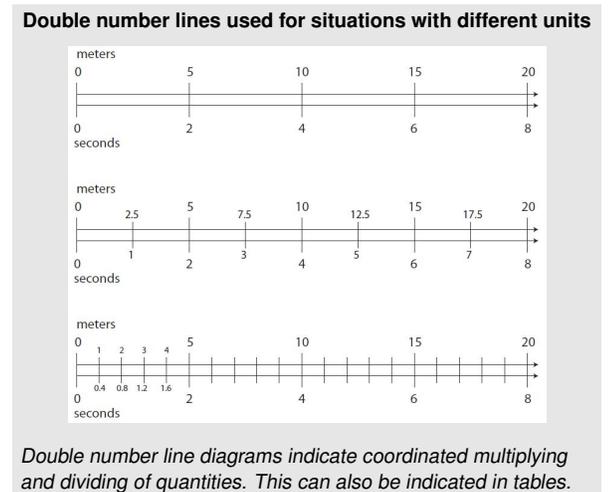
these other entries will require the explicit use of multiplication and division, not just repeated addition or skip counting. For example, if Seth runs 5 meters every 2 seconds, then Seth will run 2.5 meters in 1 second because in half the time he will go half as far. In other words, when the elapsed time is divided by 2, the distance traveled should also be divided by 2. More generally, if the elapsed time is multiplied (or divided) by N, the distance traveled should also be multiplied (or divided) by N. Double number lines can be useful in representing ratios that involve fractions and decimals.

As students become comfortable with fraction and decimal en-tries in tables of quantities in equivalent ratios, they should learn to appreciate that unit rates are especially useful for finding entries. A unit rate gives the number of units of one quantity per 1 unit of the other quantity. The amount for N units of the other quantity is then found by multiplying by N. Once students feel comfortable doing so, they may wish to work with abbreviated tables instead of working with long tables that have many values. The most abbreviated ta-bles consist of only two columns or two rows; solving a proportion is a matter of finding one unknown entry in the table.

Measurement conversion provides other opportunities for stu-dents to use relationships given by unit rates.[6.RP.3d] For example, recognizing "12 inches in a foot," "1000 grams in a kilogram," or "one kilometer is $\frac{5}{8}$ of a mile" as rates, can help to connect concepts and methods developed for other contexts with measurement conversion.

Representing a problem with a tape diagram

Slimy Gloopy mixture is made by mixing glue and liquid laundry starch in a ratio of 3 to 2. How much glue and how much starch is needed to make 85 cups of Slimy Gloopy mixture?

5 parts $\longrightarrow$ 85 cups
1 part $\longrightarrow$ $85 \div 5 = 17$ cups
3 parts $\longrightarrow$ $3 \cdot 17 = 51$ cups
2 parts $\longrightarrow$ $2 \cdot 17 = 34$ cups

51 cups glue and 34 cups starch are needed.

Tape diagrams can be useful aids for solving problems.
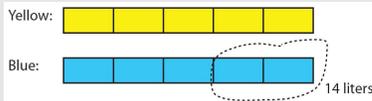

Representing a multi-step problem with two pairs of tape diagrams

Yellow and blue paint were mixed in a ratio of 5 to 3 to make green paint. After 14 liters of blue paint were added, the amount of yellow and blue paint in the mixture was equal. How much green paint was in the mixture at first?
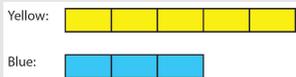
2 parts $\longrightarrow$ 14 liters
1 part $\longrightarrow$ $14 \div 2 = 7$ liters
(original total) 8 parts $\longrightarrow$ $8 \cdot 7 = 56$ liters

There was 56 liters of green paint to start with.

This problem can be very challenging for sixth or seventh graders.

A progression of strategies for solving a proportion

If 2 pounds of beans cost $5, how much will 15 pounds of beans cost?

Method 1

pounds	2	4	6	8	10	12	14	1	15
dollars	5	10	15	20	25	30	35	2.50	37.50

"I found 14 pounds costs $35 and then 1 more pound is another $2.50, so that makes $37.50 in all."

Method 2

"I found 1 pound first because if I know how much it costs for each pound then I can find any number of pounds by multiplying."

Method 3

The previous method, done in one step.

With this perspective, the second column is seen as the first column times a number. To solve the proportion one first finds this number.

[6.RP.3d] Use ratio reasoning to convert measurement units; ma-nipulate and transform units appropriately when multiplying or di-viding quantities.

Solving a percent problem

If 75% of the budget is $1200, what is the full budget?
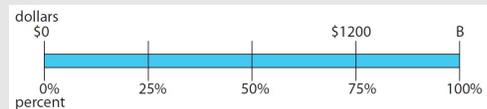
"I said   75% is 3 parts and is $1200
25% is 1 part and is $1200 \div 3 = 400
100% is 4 parts and is $4 \cdot $400 = 1600"

portion	75	3	1200
whole	100	4	1600

75% is $\frac{1200}{B}$

$\frac{75}{100} = \frac{1200}{B}$

$B = 1600$

75% of B is 1200

$\frac{75}{100} \cdot B = 1200$

In reasoning about and solving percent problems, students can use a variety of strategies. Representations such as this, which is a blend between a tape diagram and a double number line diagram, can support sense-making and reasoning about percent.

Grade 7

In Grade 7, students extend their reasoning about ratios and pro-
portional relationships in several ways. Students use ratios in cases
that involve pairs of rational number entries, and they compute as-
sociated unit rates. They identify these unit rates in representations
of proportional relationships. They work with equations in two vari-
ables to represent and analyze proportional relationships. They
also solve multi-step ratio and percent problems, such as problems
involving percent increase and decrease.

At this grade, students will also work with ratios specified by
rational numbers, such as $\frac{3}{4}$ cups flour for every $\frac{1}{2}$ stick butter.[7.RP.1]
Students continue to use ratio tables, extending this use to finding
unit rates.

Recognizing proportional relationships   Students examine situa-
tions carefully, to determine if they describe a proportional relation-
ship.[7.RP.2a] For example, if Josh is 10 and Reina is 7, how old will
Reina be when Josh is 20? We cannot solve this problem with the
proportion $\frac{10}{7} = \frac{20}{R}$ because it is not the case that for every 10 years
that Josh ages, Reina ages 7 years. Instead, when Josh has aged 10
another years, Reina will as well, and so she will be 17 when Josh
is 20.

For example, if it takes 2 people 5 hours to paint a fence, how
long will it take 4 people to paint a fence of the same size (assuming
all the people work at the same steady rate)? We cannot solve this
problem with the proportion $\frac{2}{5} = \frac{4}{H}$ because it is not the case that
for every 2 people, 5 hours of work are needed to paint the fence.
When more people work, it will take fewer hours. With twice as
many people working, it will take half as long, so it will take only
2.5 hours for 4 people to paint a fence. Students must understand the
structure of the problem, which includes looking for and understand
the roles of "for every," "for each," and "per."

Students recognize that graphs that are not lines through the
origin and tables in which there is not a constant ratio in the entries
do not represent proportional relationships. For example, consider
circular patios that could be made with a range of diameters. For
such patios, the area (and therefore the number of pavers it takes
to make the patio) is not proportional to the diameter, although the
circumference (and therefore the length of stone border it takes to
encircle the patio) is proportional to the diameter. Note that in the
case of the circumference, C, of a circle of diameter D, the constant
of proportionality in $C = \pi D$ is the number π, which is not a rational
number.

Equations for proportional relationships   As students work with
proportional relationships, they write equations of the form $y = cx$,
where c is a constant of proportionality, i.e., a unit rate.[7.RP.2c] They

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7.RP.1 Compute unit rates associated with ratios of fractions, in-
cluding ratios of lengths, areas and other quantities measured in
like or different units.

Ratio problem with rational numbers: Three approaches

To make Perfect Purple paint mix $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup
red paint. If you want to mix blue and red paint in the same
ratio to make 20 cups of Perfect Purple paint, how many cups
of blue paint and how many cups of red paint will you need?

Method 1

"I thought about making 6 batches of purple because that is a
whole number of cups of purple. To make 6 batches, I need 6
times as much blue and 6 times as much red too. That was 3
cups blue and 2 cups red and that made 5 cups purple. Then 4
times as much of each makes 20 cups purple."

Method 2

$$\frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \cdot \frac{6}{5} = \frac{6}{10} \qquad \frac{6}{10} \cdot 20 = 12$$

$$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \cdot \frac{6}{5} = \frac{6}{15} \qquad \frac{6}{15} \cdot 20 = 8$$

"I found out what fraction of the paint is blue and what fraction is
red. Then I found those fractions of 20 to find the number of
cups of blue and red in 20 cups."

Method 3

*Like Method 2, but in tabular form, and viewed as multiplicative
comparisons.*

7.RP.2 Recognize and represent proportional relationships be-
tween quantities.

  a Decide whether two quantities are in a proportional rela-
tionship, e.g., by testing for equivalent ratios in a table or
graphing on a coordinate plane and observing whether
the graph is a straight line through the origin.

see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table and they recognize the unit rate as the vertical increase in a "unit rate triangle" or "slope triangle" with horizontal side of length 1 for a graph of a proportional relationship.[7.RP.2b]

7.RP.2 Recognize and represent proportional relationships between quantities.

    b Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

Correspondence among a table, graph, and equation of a proportional relationship

For every 5 cups grape juice, mix in 2 cups peach juice.

x cups grape	y cups peach
(0)	(0)
5	2
1	$\frac{2}{5}$
2	$2 \cdot \frac{2}{5}$
3	$3 \cdot \frac{2}{5}$
4	$4 \cdot \frac{2}{5}$
x	$x \cdot \frac{2}{5}$

$y = \frac{2}{5} x$

On the graph: For each 1 unit you move to the right, move up $\frac{2}{5}$ of a unit.

When you go 2 units to the right, you go up $2 \cdot \frac{2}{5}$ units.

When you go 3 units to the right, you go up $3 \cdot \frac{2}{5}$ units.

When you go 4 units to the right, you go up $4 \cdot \frac{2}{5}$ units.

When you go x units to the right, you go up $x \cdot \frac{2}{5}$ units.

Starting from $(0, 0)$, to get to a point (x, y) on the graph, go x units to the right, so go up $x \cdot \frac{2}{5}$ units.

Therefore $y = x \cdot \frac{2}{5}$ $y = \frac{2}{5} x$

Students connect their work with equations to their work with tables and diagrams. For example, if Seth runs 5 meters every 2 seconds, then how long will it take Seth to run 100 meters at that rate? The traditional method is to formulate an equation, $\frac{5}{2} = \frac{100}{T}$, cross-multiply, and solve the resulting equation to solve the problem.

Such problems can be framed in terms of proportional relationships and the constant of proportionality or unit rate, which is obscured by the traditional method of setting up proportions. For example, if Seth runs 5 meters every 2 seconds, he runs at a rate of 2.5 meters per second, so distance d (in meters) and time t (in seconds) are related by $d = 2.5t$. If $d = 100$ then $t = \frac{100}{2.5} = 40$, so he takes 40 seconds to run 100 meters.•

A common error in setting up proportions is placing numbers in incorrect locations. This is especially easy to do when the order in which quantities are stated in the problem is switched within the problem statement. For example, the second of the following two problem statements is more difficult than the first because of the reversal.

• Here is an example of how the traditional method can be explained in terms of equivalent ratios. If $\frac{5}{2}$ and $\frac{100}{T}$ are viewed as unit rates obtained from the equivalent ratios $5 : 2$ and $100 : T$, then they must be equivalent fractions because equivalent ratios have the same unit rate. To see the rationale for cross-multiplying, note that when the fractions are given the common denominator $2T$, then the numerators become $5T$ and $2 \cdot 100$ respectively. Once the denominators are equal, the fractions are equal exactly when their numerators are equal, so $5T$ must equal $2 \cdot 100$ for the unit rates to be equal. This is why we can solve the equation $5T = 2 \cdot 100$ to find the amount of time it will take for Seth to run 100 meters.

"If a factory produces 5 cans of dog food for every 3 cans of cat food, then when the company produces 600 cans of dog food, how many cans of cat food will it produce?"

"If a factory produces 5 cans of dog food for every 3 cans of cat food, then how many cans of cat food will the company produce when it produces 600 cans of dog food?"

In the second problem, the situation is framed as "amount of dog food is proportional to amount of cat food," but the problem asks, "How many cans of cat food will the company produce?" Students might ask themselves "What is proportional to what?" in each part of the problem.

Multistep problems   Students extend their work to solving multistep ratio and percent problems.[7.RP.3] Problems involving percent increase or percent decrease require careful attention to the referent whole. For example, consider the difference in these two percent decrease and percent increase problems:

Skateboard problem 1. After a 20% discount, the price of a SuperSick skateboard is $140. What was the price before the discount?

Skateboard problem 2. A SuperSick skateboard costs $140 now, but its price will go up by 20%. What will the new price be after the increase?

The solutions to these two problems are different because the 20% refers to different wholes or 100% amounts. In the first problem, the 20% is 20% of the larger pre-discount amount, whereas in the second problem, the 20% is 20% of the smaller pre-increase amount. Notice that the distributive property is implicitly involved in working with percent decrease and increase. For example, in the first problem, if x is the original price of the skateboard (in dollars), then after the 20% discount, the new price is $x - 20\%x$. The distributive property shows that the new price is $80\%x$:

$$x - 20\%x = 100\%x - 20\%x = (100\% - 20\%)x = 80\%x$$

Percentages can also be used in making comparisons between two quantities. Students must attend closely to the wording of such problems to determine what the whole or 100% amount a percentage refers to, e.g., "25% more seventh graders than sixth graders" means that the number of extra seventh graders is the same as 25% of the sixth graders.

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders?

Skateboard problem 1

original 100% $x

| 20% | 20% | 20% | 20% | 20% |

discounted 80% $140

After a 20% discount, the price is 80% of the original price. So 80% of the original is $140.

percent	**dollars**
80%	$140
20%	$35
100%	$175

$\div 4$ (80% → $140) $\div 4$
$\div 4$ (20% → $35) $\div 4$
$\cdot 5$ or add 80%+20% (100% → $175) $\cdot 5$ or add $140+$35

"To find 20% I divided by 4. Then 80% plus 20% is 100%"

x is original price in dollars

| | **percent** | **dollars** |
| discounted original | $\frac{80}{100}$ = | $\frac{140}{x}$ |

$$80 \cdot x = 140 \cdot 100$$
$$x = \frac{140 \cdot 100}{80} = 175$$

80% of the original price is $140.

$$\frac{80}{100} \cdot x = 140$$
$$\frac{4}{5} \cdot x = 140$$
$$x = 140 \div \frac{4}{5} = 175$$

Before the discount, the price of the skateboard was $175.

Skateboard problem 2

original 100% $140

| 20% | 20% | 20% | 20% | 20% | 20% |

new, increased 120% $x

After a 20% increase, the price is 120% of the original price. So the new price is 120% of $140.

percent	**dollars**
100%	$140
20%	$28
100%	$168

$\div 5$ (100% → $140) $\div 5$
(20% → $28)
$\cdot 6$ or add 100%+20% (100% → $168) $\cdot 6$ or add $140+$28

"To find 20% I divided by 5. Then 100% plus 20% is 120%"

x is increased price in dollars

	percent	**dollars**
discounted original	$\frac{120}{100}$ =	$\frac{x}{140}$
	$\frac{12}{10}$ =	$\frac{x}{140}$

$$x = 140 \cdot \frac{12}{10} = 168$$

The new, increased price is 120% of $140.

$$x = \frac{120}{100} \cdot 140 = 168$$

The new price after the increase is $168.

Using percentages in comparisons

100% of the sixth graders

Sixth graders:

| 25% of 6th graders | 25% of 6th graders | 25% of 6th graders | 25% of 6th graders |

135 kids

Seventh graders:

same amount as 25% of the sixth graders

$\div 9$ (9 parts → 135)
1 part → $135 \div 9 = 15$
$\cdot 4$ 4 parts → $4 \cdot 15 = 60$ 60 sixth graders
$\cdot 5$ 5 parts → $5 \cdot 15 = 75$ 75 seventh graders

Connection to Geometry   One new context for proportions at Grade 7 is scale drawings.[7.G.1] To compute unknown lengths from known lengths, students can set up proportions in tables or equations, or they can reason about how lengths compare multiplicatively. Students can use two kinds of multiplicative comparisons. They can apply a scale factor that relates lengths in two different figures, or they can consider the ratio of two lengths within one figure, find a multiplicative relationship between those lengths, and apply that relationship to the ratio of the corresponding lengths in the other figure. When working with areas, students should be aware that areas do not scale by the same factor that relates lengths. (Areas scale by the square of the scale factor that relates lengths, if area is measured in the unit of measurement derived from that used for length.)

[7.G.1] Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Connection to Statistics and Probability   Another new context for proportions at Grade 7 is to drawing inferences about a population from a random sample.[7.SP.1] Because random samples can be expected to be approximately representative of the full population, one can imagine selecting many samples of that same size until the full population is exhausted, each with approximately the same characteristics. Therefore the ratio of the size of a portion having a certain characteristic to the size of the whole should have approximately the same value for samples as for the full population.

[7.SP.1] Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

Where this progression is heading

The study of proportional relationships is a foundation for the study of functions, which begins in Grade 8 and continues through high school and beyond (see the Functions Progression). Linear functions are characterized by having a constant rate of change (the change in the outputs is a constant multiple of the change in the corresponding inputs). Proportional relationships underlie a major type of linear function; those that have a positive rate of change and take 0 to 0.

In high school, students extend their understanding of quantity. They write rates concisely in terms of derived units, e.g., $\frac{3}{2}$ m/s rather than "$\frac{3}{2}$ meters in every 1 second" or "$\frac{3}{2}$ meters per second."• They encounter a wider variety of derived units and situations in which they must conceive units that measure attributes of interest (see the Quantity Progression).

• Before high school, expectations for use of derived units are limited to units of area and volume. Initially, these are written without exponents, e.g., square cm in Grade 3 and cubic cm in Grade 5. Use of whole-number exponents to denote powers of 10 is expected by the end of Grade 5 and use of whole-number exponents in numerical expressions by the end of Grade 6.

Connection to Geometry

If the two rectangles are similar, then how wide is the larger rectangle?

12 cm

4 cm

8 cm   ? cm
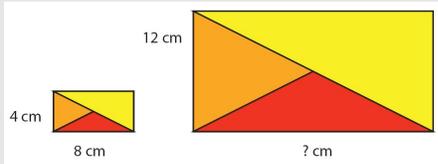
Use a scale factor: Find the scale factor from the small rectangle to the larger one:

The big rectangle is 3 times as high as the small rectangle.

• 3   12 cm

4 cm

8 cm   ? cm

• 3

	small		big
height	4	$\xrightarrow{\cdot\,3}$	12
width	8	$\xrightarrow{\cdot\,3}$	?

$3 \cdot 8 \text{ cm} = 24 \text{ cm wide}$

So the width of the big rectangle should also be 3 times the width of the small rectangle.

Use an internal comparison: Compare the width to the height in the small rectangle. The ratio of the width to height is the same in the large rectangle.

12 cm

4 cm

• 2   8 cm   • 2   ? cm

	small	big
height	4	12
width	8	?

↓ • 2   ↓ • 2

$2 \cdot 12 \text{ cm} = 24 \text{ cm wide}$

In the small rectangle, the width is 2 times the height.

So in the big rectangle, the width should also be 2 times the height.

Connection to Statistics and Probability

There are 150 tiles in a bin. Some of the tiles are blue and the rest are yellow. A random sample of 10 tiles was selected. Of the 10 tiles, 3 were yellow and 7 were blue. What are the best estimates for how many blue tiles are in the bin and how many yellow tiles are in the bin?

Student 1

yellow:	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
blue:	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
total:	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150

"I figured if you keep picking out samples of 10 they should all be about the same, so I got this ratio table. Out of 150 tiles, about 45 should be yellow and about 105 should be blue."

Student 2

yellow:	3	45
blue:	7	105
total:	10	150

• 15

"I also made a ratio table. I said that if there are 15 times as many tiles in the bin as in the sample, then there should be about 15 times as many yellow tiles and 15 times as many blue tiles. $15 \cdot 3 = 45$, so 45 yellow tiles. $15 \cdot 7 = 105$, so 105 blue tiles."

Student 3

30% yellow tiles	$30\% \cdot 150 = \dfrac{3 \cdot 10}{10 \cdot 10} \cdot 150 = \dfrac{3}{10} \cdot 15 \cdot 10 = 45$
70% blue tiles	$70\% \cdot 150 = \dfrac{7 \cdot 10}{10 \cdot 10} \cdot 150 = \dfrac{7}{10} \cdot 15 \cdot 10 = 105$

"I used percentages. 3 out of 10 is 30% yellow and 7 out of 10 is 70% blue. The percentages in the whole bin should be about the same as the percentages in the sample."

Appendix. A framework for ratio, rate, and proportional relationships

This section presents definitions of the terms ratio, rate, and proportional relationship that are consistent with the Standards and it briefly summarizes some of the essential characteristics of these concepts. It also provides an organizing framework for these concepts. Because many different authors have used ratio and rate terminology in widely differing ways, there is a need to standardize the terminology for use with the Standards and to have a common framework for curriculum developers, professional development providers, and other education professionals to discuss the concepts. This section does not describe how the concepts should be presented to students in Grades 6 and 7.

Definitions and essential characteristics

A *ratio* of two numbers is a pair of non–negative numbers, $A : B$, which are not both 0.

   When there are A units of one quantity for every B units of another quantity, a *rate* associated with the ratio $A : B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.) The associated *unit rate* is $\frac{A}{B}$. The term *unit rate* is the numerical part of the rate; the "unit" is used to highlight the 1 in "per 1 unit of the second quantity." Unit rates should not be confused with unit fractions (which have a 1 in the numerator).

   A rate is expressed in terms of a unit that is derived from the units of the two quantities (such as ᵐ⁄ₛ, which is derived from meters and seconds). In high school and beyond, a rate is usually written as

$$\frac{A}{B} \frac{\text{units}}{\textbf{\textsf{UNITS}}}$$

where the two different fonts highlight the possibility that the quantities may have different units. In practice, when working with a ratio $A : B$, the rate $\frac{A}{B}$ units per 1 **UNIT** and the rate $\frac{B}{A}$ **UNITS** per 1 unit are both useful.

   The *value* of a ratio $A : B$ is the quotient $\frac{A}{B}$ (if B is not 0). Note that the value of a ratio may be written as a decimal, percent, fraction, or mixed number. The value of the ratio $A : B$ tells how A and B compare multiplicatively; specifically, it tells how many times as big A is as B. In practice, when working with a ratio $A : B$, the value $\frac{A}{B}$ as well as the value $\frac{B}{A}$, associated with the ratio $B : A$, are both useful. These values of each ratio are viewed as unit rates in some contexts (see Perspective 1 in the next section).

   Two ratios $A : B$ and $C : D$ are *equivalent* if there is a positive number, c, such that $C = cA$ and $D = cB$. To check that two ratios are equivalent one can check that they have the same value (because

$\frac{cA}{cB} = \frac{A}{B}$), or one can "cross–multiply" and check that $AD = BC$ (because $AcB = BcA$). Equivalent ratios have the same unit rate.

A *proportional relationship* is a collection of pairs of numbers that are in equivalent ratios. A ratio $A : B$ determines a proportional relationship, namely the collection of pairs (cA, cB), for c positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k, is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with endpoint at the origin.

Two perspectives on ratios and their associated rates

Although ratios, rates, and proportional relationships can be described in purely numerical terms, these concepts are most often used with quantities.

Ratios are often described as comparisons by division, especially when focusing on an associated rate or value of the ratio. There are also two broad categories of basic ratio situations. Some division situations, notably those involving area, can fit into either category of division. Many ratio situations can be viewed profitably from within either category of ratio. For this reason, the two categories for ratio will be described as *perspectives* on ratio.

First perspective: Ratio as a composed unit or batch   Two quantities are in a ratio of A to B if for every A units present of the first quantity there are B units present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are cA units of the first quantity and cB units of the second quantity. With this perspective, the two quantities can have the same or different units.

With this perspective, a ratio is specified by a composed unit or "batch," such as "3 feet in 2 seconds," and the unit or batch can be repeated or subdivided to create new pairs of amounts that are in the same ratio. For example, 12 feet in 8 seconds is in the ratio 3 to 2 because for every 3 feet, there are 2 seconds. Also, 12 feet in 8 seconds can be viewed as a 4 repetitions of the unit "3 feet in 2 seconds." Similarly, $\frac{3}{2}$ feet in 1 second is $\frac{1}{2}$ of the unit "3 feet in 2 seconds."

With this perspective, quantities that are in a ratio A to B give rise to a rate of $\frac{A}{B}$ units of the first quantity for every 1 unit of the second quantity (as well as to the rate of $\frac{B}{A}$ units of the second quantity for every 1 unit of the first quantity). For example, the ratio 3 feet in 2 seconds gives rise to the rate $\frac{3}{2}$ feet for every 1 second.

With this perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be viewed as the numerical part of a rate associated with the ratio $A : B$.

Two perspectives on ratio

1. There are 3 cups of apple juice for every 2 cups of grape juice in the mixture.

This way uses a composed unit: 3 cups apple juice and 2 cups grape juice. Any mixture that is made from some number of the composed unit is in the ratio 3 to 2.
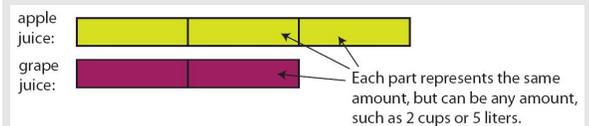

1 composed unit
1 "batch"
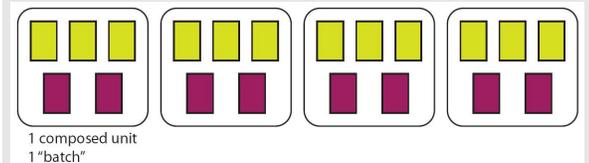
In each of these mixtures, apple juice and grape juice are mixed in a ratio of 3 to 2:

| # cups apple juice | 3 | 6 | 9 | 12 | 3/2 | 1 |
| # cups grape juice | 2 | 4 | 6 | 8 | 1 | 2/3 |

made of 2 composed units   made of 1/2 of a composed unit

2. The mixture is made from 3 parts apple juice and 2 parts grape juice, where all parts are the same size, but can be any size.

apple juice:
grape juice:

Each part represents the same amount, but can be any amount, such as 2 cups or 5 liters.

If 1 part is :	1 cup	2 cups	5 liters	3 quarts
amt of apple juice:	3 cups	6 cups	15 liters	9 quarts
amt of grape juice:	2 cups	4 cups	10 liters	6 quarts

Second perspective: Ratio as fixed numbers of parts   Two quantities which have the same units, are in a ratio of A to B if there is a part of some size such that there are A parts present of the first quantity and B parts present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are Ac units of the first quantity and Bc units of the second quantity.

With this perspective, one thinks of a ratio as two pieces. One piece is constituted of A parts, the other of B parts. To create pairs of measurements in the same ratio, one specifies an amount and fills each part with that amount. For example, in a ratio of 3 parts sand to 2 parts cement, each part could be filled with 5 cubic yards, so that there are 15 cubic yards of sand and 10 cubic yards of cement; or each part could be filled with 10 cubic meters, so that there are 30 cubic meters of sand and 20 cubic meters of cement. When describing a ratio from this perspective, the units need not be explicitly stated, as in "mix sand and cement in a ratio of 3 to 2." However, the type of quantity must be understood or stated explicitly, as in "by volume" or "by weight."

With this perspective, a ratio $A : B$ has an associated value, $\frac{A}{B}$, which describes how the two quantities are related multiplicatively. Specifically, $\frac{A}{B}$ is the factor that tells how many times as much of the first quantity there is as of the second quantity. (Similarly, the factor $\frac{B}{A}$ associated with the ratio $B : A$, tells how many times as much of the second quantity there is as of the first quantity.) For example, if sand and cement are mixed in a ratio of 3 to 2, then there is $\frac{3}{2}$ times as much sand as cement and there is $\frac{2}{3}$ times as much cement as sand.

With this second perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be considered a factor that does not have a unit.