4/2/2019

2017 Kansas Mathematics Standards

Flip Book

High School- Algebra II

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope"; 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path; if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size." In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important, but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster-Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move "toward improved instructional practice" and support "one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students" (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in High School

The State Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students Grades K-12. Below are a few examples of how these Practices may be integrated into tasks that High School students complete.

Practice	Explanation and Example
1) Make sense of problems and persevere in solving them.	Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. High School students make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. They might transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. They can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They constantly check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They understand the approaches of others to solving complex problems and identify correspondence between different approaches.
2) Reason abstractly and quantitatively.	Mathematically proficient students make sense of quantities and their relationships in problem situations. They represent a wide variety of real world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*—to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
3) Construct viable arguments and critique the reasoning of others.	Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. High School students reason inductively about data, making plausible arguments that take into account the context from which the data arose. They compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High School students determine domains to which an argument applied, they listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4) Model with mathematics.	Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. High School students might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximation to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.
5) Use appropriate tools strategically.	Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil/paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software. They are sufficiently familiar with tools appropriate for their grade to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.
6) Attend to precision.	Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussions with others and in their *own* reasoning. They state the meaning of the symbols they choose. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. High school students have learned to examine claims and make explicit use of definitions.
7) Look for and make use of structure.	Mathematically proficient students look closely to discern a pattern or structure. For example, high school students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.
8) Look for and express regularity in repeated reasoning.	Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1,2) with slope 3, they might abstract the equation $\frac{(y-2)}{(x-1)} = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Implementing Standards for Mathematical Practice

This guide was created to help educators implement these standards into their classroom instruction. These are the practices for the **students,** and the teacher can assist students in using them efficiently and effectively.

#1 – Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others' solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line..., chart..., table...?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools. (Tools may include; concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
    - a task when there is no need to have an exact answer
    - a task when there is not enough information to get an exact answer
    - a task to check if the answer from a calculation is reasonable

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value; working with properties; etc.)
- Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm. When "unit size" cannot be equally distributed, it is necessary to break down into a smaller "unit size". (example below)

$4\overline{)351}$ -32 31 -28 3	3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units. There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units. This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation. i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for Mathematics in Algebra II

Building on their work with linear, quadratic, and absolute value functions, students extend their repertoire of functions to include exponential, logarithmic, polynomial, rational, and radical functions in the Algebra II course. Students work closely with the expressions that define the functions, are facile with algebraic manipulations of expressions, and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers.

For the high school Algebra II course, instructional time should focus on four critical areas:

1. A central theme of this Algebra II course is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students explore the structural similarities between the system of polynomials and the system of integers. They draw on analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Connections are made between multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers.

2. Building on their previous work with functions they compare and contrast linear and exponential functions, distinguishing between additive and multiplicative change. They graph square root, cube root, exponential and logarithmic functions, emphasizing the inverse relationship with exponentials and showing intercepts and end behavior. Additionally, they graph polynomial functions and identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations.

3. Students synthesize and generalize what they have learned about a variety of function families. They explore the effects of transformations on graphs of diverse functions, including functions arising in an application, in order to abstract the general principle that transformations on a graph always have the same effect regardless of the type of the underlying function.

4. They identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as “the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions” is at the heart of this Algebra II course. The narrative discussion and diagram of the modeling cycle should be considered when knowledge of functions, statistics, and geometry is applied in a modeling context.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Growth Mindset

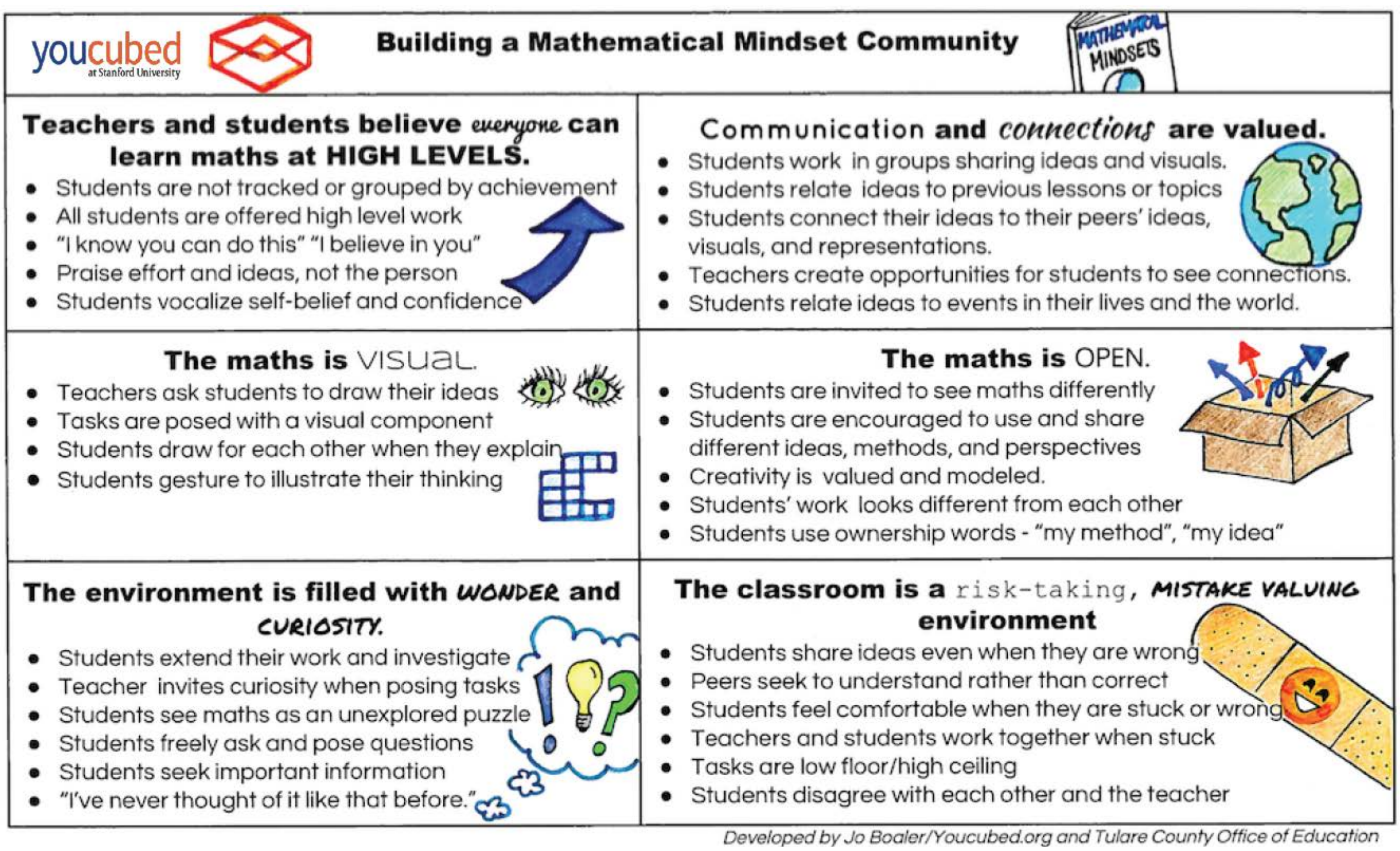
The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.


youcubed at Stanford University

Building a Mathematical Mindset Community

MATHEMATICAL MINDSETS

Teachers and students believe everyone can learn maths at HIGH LEVELS.
- Students are not tracked or grouped by achievement
- All students are offered high level work
- "I know you can do this" "I believe in you"
- Praise effort and ideas, not the person
- Students vocalize self-belief and confidence

Communication and connections are valued.
- Students work in groups sharing ideas and visuals.
- Students relate ideas to previous lessons or topics
- Students connect their ideas to their peers' ideas, visuals, and representations.
- Teachers create opportunities for students to see connections.
- Students relate ideas to events in their lives and the world.

The maths is VISUAL.
- Teachers ask students to draw their ideas
- Tasks are posed with a visual component
- Students draw for each other when they explain
- Students gesture to illustrate their thinking

The maths is OPEN.
- Students are invited to see maths differently
- Students are encouraged to use and share different ideas, methods, and perspectives
- Creativity is valued and modeled.
- Students' work looks different from each other
- Students use ownership words - "my method", "my idea"

The environment is filled with WONDER and CURIOSITY.
- Students extend their work and investigate
- Teacher invites curiosity when posing tasks
- Students see maths as an unexplored puzzle
- Students freely ask and pose questions
- Students seek important information
- "I've never thought of it like that before."

The classroom is a risk-taking, MISTAKE VALUING environment
- Students share ideas even when they are wrong
- Peers seek to understand rather than correct
- Students feel comfortable when they are stuck or wrong
- Teachers and students work together when stuck
- Tasks are low floor/high ceiling
- Students disagree with each other and the teacher

Developed by Jo Boaler/Youcubed.org and Tulare County Office of Education

High School Notation

(★) **Modeling Standards:** Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol. The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

Grade Level Classifications

To assist with the organization of high school mathematics courses, the standards have grade level classifications to identify the appropriate grade at which they should be taught. The classifications were designed with the following framework in mind:

Year of School	Traditional Course Sequence	Integrated Course Sequence
9^{th} Grade	Algebra I	Mathematics 1
10^{th} Grade	Geometry	Mathematics 2
11^{th} Grade	Algebra II	Mathematics 3

There will be variation with student placement in the courses listed above. At the present time, the “gateway” math class in Kansas for postsecondary schooling is College Algebra. The standards committee used this as a guide when identifying grade level classifications.

The grade level classifications are as follows

(9/10)	These standards are required for all students by the end of their first two years of high school math courses.
(11)	These standards are required for all students by the end of their third year math course.
(9/10/11)	These standards are required for all students in their first three years of high school math courses. These standards are often further divided to (9/10) and (11) to identify specific concepts and their appropriate grade level. *(9/10) should primarily accomplish the standards described as linear, quadratic and absolute value while (11) should primarily accomplish the standards described as logarithmic, square root, cube root, and exponential.*
(all)	These standards should be taught throughout every high school math course, and often represent over-arching themes or key features of the mathematical concept. These standards should be taught in conjunction with the appropriate grade level standards.
(+)	These standards should be taught as **extensions** to grade level standards when possible, or in a 4^{th} year math course. These standards prepare students to take advanced courses in high school such as college algebra, calculus, advanced statistics, or discrete mathematics.

High School – Algebra II Overview

High School—Modeling

High School – Number and Quantity

The Real Number System (N.RN)

A. Use properties of rational and irrational numbers.
N.RN.2 N.RN.3

Quantities (★) (N.Q)

A. Reason quantitatively and use units to solve problems.
N.Q.1 (★) N.Q.2 (★) N.Q.3 (★)

The Complex Number System (N.CN)

A. Perform arithmetic operations with complex numbers.
N.CN.1 N.CN.2 N.CN.3

C. Use complex numbers in polynomial identities and equations.
N.CN.8

Vector and Matrix Quantities (N.VM)

C. Perform operations on matrices and use matrices in applications.
N.VM.6 N.VM.7 N.VM.8

High School – Algebra

Seeing Structure in Expressions (A.SSE)

A. Interpret the structure of expressions.
A.SSE.1 (★) A.SSE.2

B. Write expressions in equivalent forms to solve problems.
A.SSE.3b - c (★)

Arithmetic with Polynomials and Rational Expressions (A.APR)

A. Perform arithmetic operations on polynomials.
A.APR.2 A.APR.3

B. Use polynomial identities to solve problems.
A.APR.4

Creating Equations (★) (A.CED)

A. Create equations that describe numbers or relationships.
A.CED.1 (★) A.CED.2 (★) A.CED.3 (★)
A.CED.4 (★)

Reasoning with Equations and Inequalities (A.REI)

A. Understand solving equations as a process of reasoning and explain the reasoning.
A.REI.1

B. Solve equations and inequalities in one variable.
A.REI.2 A.REI.3a A.REI.4
A.REI.5b - c

C. Represent and solve equations and inequalities graphically.
A.REI.8 A.REI.9 (★)

High School – Functions

Interpreting Functions (F.IF)

A. Understand the concept of a function and use function notation.
F.IF.1 F.IF.2 F.IF.3

B. Interpret functions that arise in applications in terms of the context.
F.IF.4 (★) F.IF.5 (★) F.IF.6 (★)

C. Analyze functions using different representations.
F.IF.7b, c, e (★) F.IF.8b - c F.IF.9

Building Functions (F.BF)

A. Build a function that models a relationship between two quantities.
F.BF.1b – c

B. Build new functions from existing functions.
F.BF.3 F.BF.4a-b F.BF.5

Linear, Quadratic, and Exponential Models (★) (F.LQE)

A. Construct and compare linear, quadratic, and exponential models and solve problems.
F.LQE.1 (★) F.LQE.2 (★)

High School – Statistics & Probability

Interpreting Categorical and Quantitative Data (S.ID)

C. Interpret linear models.
S.ID.7 S.ID.8

High School – Modeling

Domain: Modeling (★)

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See ★ standards on the overview page.

Explanations and Examples:

The goal for this section is to expand on the information in the Modeling section of the standards by adding information from research using an article summarizing our current knowledge base “Quality Teaching of Mathematical Modeling: What Do We Know, What Can we Do?” from Werner Blum.

The word “modeling” is a word that is difficult to define because it is used to describe both a process and a product. The process of *modeling* creates a product called a *model*. The standards expect students can successfully use the process to create a model and that, given a model; they can successfully interpret and understand how the math model is related to the real world situation. But what *exactly* is a model? Niss (2007) defines a model as “a deliberately simplified and formalized image of some part of the real world, formally speaking: a triple (D, M, f) consisting of a domain D of the real world, a subset M of the mathematical world and a mapping from D to M (Niss et al. 2007).”

The standards describe a six step modeling cycle:

1. Identify the variables in the situation and select those that represent essential features.
2. Formulate a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
3. Analyze and perform operations on these relationships to draw conclusions.
4. Interpret the results of the mathematics in terms of the original situation.
5. Validate the conclusions by comparing them with the situation and then either improve the model or, if is acceptable, move to step 6.
6. Report the conclusions and the reasoning behind them.

Throughout the cycle, students will make choices, assumptions, and approximations.

Blum, in his research summary, identifies an important first step that is not explicitly described in the modeling process- to construct a mental model of the situation. This requires understanding the situation, being able to mentally imagine all the parts of the situation. Research has found that many students get stuck here, in this “pre-step.” The reason some students don’t gain entry into the process is because they have been *taught* to ignore the context, find the numbers, and apply a familiar operation. This has been labeled by researchers as the "suspension of sense-making" and occurs whenever students are processing any word problem. Robert Kaplinksy created a video illustrating this phenomenon. He asked 32 8th grade students the following question:

"There are 125 sheep and 5 dogs in a flock. How old is the shepherd?"

Sadly 75% of students performed math operations and provided a numerical answer. This question has been replicated across a variety of settings since 1993 with the same consistent results.

After students have created a mental model of the situation, they are ready to begin the modeling process. The first step is to simplify the mental model down to the critical elements. This requires making assumptions and estimating any missing information. This is another source of difficulties for students- they are afraid to make assumptions. For example, one PISA task that asks students to make assumptions to solve the problem had low success rates across multiple countries.

Fig. 6 PISA task "Rock Concert"

For this PISA task, given to 15 year olds, the success rates were:

- Finland- 37%
- Korea- 21%
- USA- 26%

ROCK CONCERT

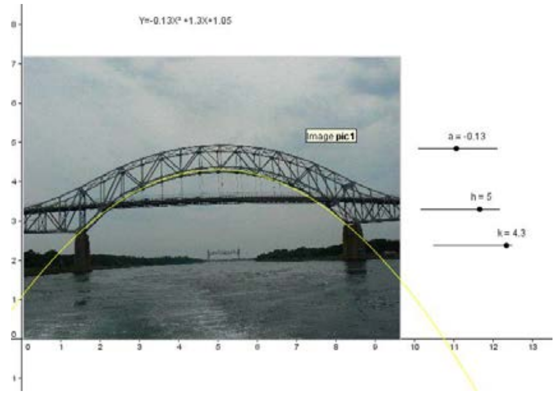
For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

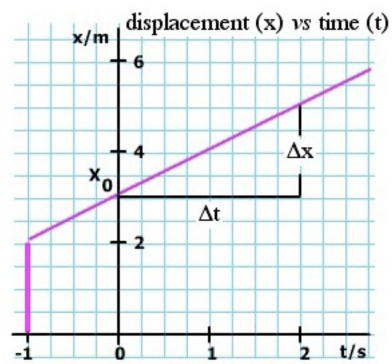
A 2 000
B 5 000
C 20 000
D 50 000
E 100 000

This question required students to estimate about how many people could fit in a square meter and an assumption that each square meter had the same density. The students should have realized that A and B did not match the scenario of a full stadium because it would be one or fewer persons per square meter. Choices D and E were also poor estimates because those choices require 10 or 20 people per square meter. Leaving the only reasonable estimate choice C. Even countries more familiar with metric measurement than the USA struggled with this type of estimation.

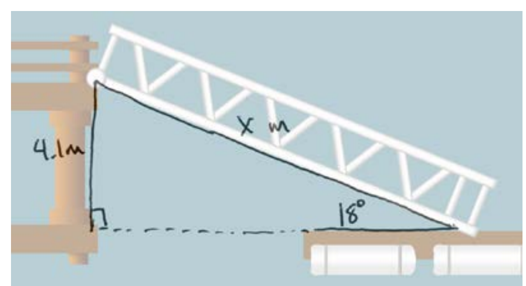
The real world is messy, filled with irrelevant data and partial information. If students are only presented problems that have been simplified and all the assumptions are made, then they do not get practice with this critical step. Developmentally, once given the freedom to find holes or irrelevant information in a problem, adolescents are often excited to explore a problem in this way.



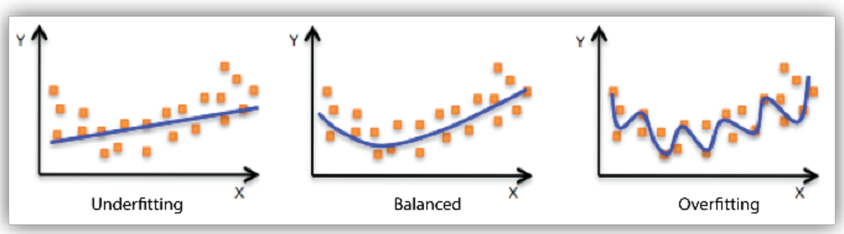
Step 2 is to mathematize the problem by creating geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables. This requires students recognize the general structure of a problem, to understand how the rate of change identifies the function family, the parameters which describe the situation, and possibly the representation that best demonstrates the relationship and can be used to find the needed information.



Step 3 and 4 are the steps we ask students to complete in a typical word problem. Unfortunately these are often the only two steps the students are regularly asked to work. The problem has been sufficiently mathematized and structured so that there are few questions about the correct structure for the problem. Students must "do the math" and interpret the results.



Step 5- validating the conclusions- involves more than interpreting the results and is another step often skipped by students. This step involves determining if the model is suitable in the real world. For example, F.LQE asks students to compare linear, quadratic, and exponential models and use the model to solve problems. If a student selects a model inappropriate to the situation, they will still be able to complete steps 3 and 4. It is once they have reached step 5 to validate the conclusions that they are given the opportunity to re-evaluate the model. Or the validation might decide the function family is correct but that the parameters chosen could be modified to better represent the situation. This step is when, in statistics, the researcher must try to fit the model to the data but also be careful to not over fit the data so that it can't be generalized to other similar situations.



The final step is to have students write up the process and conclusions. Asking students to convince a friend that they are correct can help students structure their persuasive and descriptive argument. Another reason that writing up the process, the assumptions, the simplified structure, etc. is difficult is because the problems we provide are not truly modeling problems- they are word problems. There is one solution path and it isn't messy. Providing problems with multiple viewpoints and different conclusions will help students have something to talk about. For example, when analyzing data do not clean the data for the students. Let them decide how to approach incorrect data and outliers. Using Dan Meyer's Three-Act Math Videos can provide common input data for the students with multiple paths to the solution.

These six steps are cognitively demanding and difficult because they require math knowledge, non-math knowledge, and a specific set of beliefs and attitudes about ones ability to do mathematics. So *why* do we take valuable class time to go through this entire process? Research has identified there are four different justifications and perspectives which drive the modeling process, depending on the type of problem presented to students. Understanding these justifications and perspectives can help the teacher present a wide variety of problem types and to be more intentional about highlighting the purpose for the chosen problem.

1. Applied Math: Applied mathematics justifies the modeling process because the mathematics will help the learner understand the real world. The other three justifications use the situation to support math understanding so applied mathematics is the only justification with the purpose of supporting a deeper theoretical understanding about the world. When working these modeling problems, students are practicing sensemaking through understanding the real-world.
2. Educational Modeling: Another reason to practice the modeling process is to formatively assess the thinking of students. For these problems, the examples are concrete and authentic. They are cognitively rich and include time for students to reflect on their process. When the purpose is educational modeling, students are making sense of the problem through the lens of their own growth.
3. Cultural Modeling: Modeling has the ability to connect the outside world to the math classroom, to allow students to see how math can *help* the world around them. The problems that students work will be authentic and will show how math shapes the world around them or will allow the student to see that mathematics is a science. Students will make sense of these problems by seeing the role of math in the real-world.
4. Pedagogical Modeling: Psychologically modeling problems have the ability to spark interest, motivate students, and increase retention in STEM fields. These problems are interesting, illustrating how math will benefit the student, or are rich enough to deepen students understanding of a mathematics concept (sometimes called conceptual modeling). Students will make sense by finding joy in mathematics or puzzling through a math concept.

It is clear that modeling is an important process in mathematics but also that modeling is demanding. There must be significant efforts to make this process accessible for all learners. There are many resources available by performing an internet search for STEM problems. Below are four examples to start you on your journey.

Resources/Tools:

Quantamagazine:

- https://www.quantamagazine.org/the-math-of-causation-puzzle-20180530/
- https://www.quantamagazine.org/puzzle-solution-bongard-problems-20170628/

- https://www.nytimes.com/spotlight/learning-stem

NRICH Math:

- https://nrich.maths.org/6458

Dan Meyer's Three-Act Math Tasks:

- https://docs.google.com/spreadsheets/d/1jXSt_CoDzyDFeJimZxnhgwOVsWkTQEsfqouLWNNC6Z4/edit#gid=0

High School – Number and Quantity

Domain: The Real Number System (N.RN)

◆ **Cluster:** *Use properties of rational numbers and irrational numbers.*

Standard: **N.RN.2**

(11) Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{\frac{1}{3}}$ *to be the cube root of 5 because we want* $\left(5^{\frac{1}{3}}\right)^3 = 5^{\left(\frac{1}{3}\right)^3}$ *to hold, so* $\left(5^{\frac{1}{3}}\right)^3$ *must equal 5.* **(N.RN.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.

Connections: N.RN.1

Explanations and Examples

Understand that the denominator of the rational exponent is the root index and the numerator is the exponent of the radicand. For example, $5^{\frac{1}{2}} = \sqrt{5}$.

Students extend the properties of exponents to justify that $(5^{\frac{1}{2}})^2 = 5$

Examples:

The goal of this task is to develop a conception understanding for the definition of rational exponents. However, it also raises important issues about distinguishing between linear and exponential behavior (F.LQE.1c) and it requires students to create an equation to model a context (A-CED.2)

- A biology student is studying bacterial growth. She was surprised to find that the population of the bacteria doubled every hour.
  a. Complete the following table and plot the data.

Hours into study	0	1	2	3	4
Population (thousands)	4				

  b. Write an equation for *P*, the population of the bacteria, as a function of time, *t*, and verify that it produces correct populations for *t* = 1, 2, 3, and 4.

  c. The student conducting the study wants to create a table with more entries; specifically, she wants to fill in the population at each half hour. However, she forgot to make these measurements so she wants to estimate the values.

  Instead she notes that the population increases by the same factor each hour, and reasons that this property should hold over each half-hour interval as well. Fill in the part of the below table that you've already computed, and decide what constant factor she should multiply the population by each half hour in order to produce consistent results. Use this multiplier to complete the table:

Hours into study	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Population (thousands)	4						

d. What if the student wanted to estimate the population every 20 minutes instead of every 30 minutes. What multiplier would be necessary to be consistent with the population doubling every hour? Use this multiplier to complete the following table:

Hours into study	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
Population (thousands)	4						

e. Use the population context to explain why it makes sense that we define $2^{\frac{1}{2}}$ to be $\sqrt{2}$ and $2^{\frac{1}{3}}$ as $\sqrt[3]{2}$

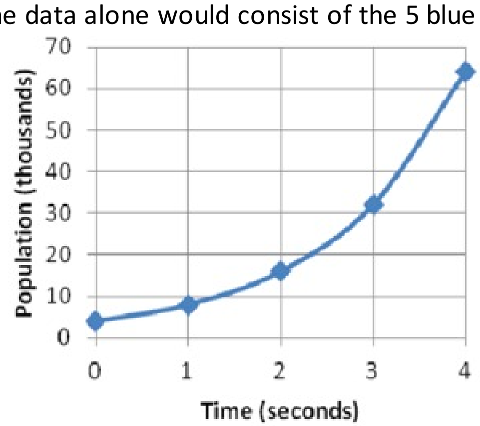
f. Another student working on the bacteria population problem makes the following claim:
If the population doubles in 1 hour, then half that growth occurs in the first half-hour and the other half occurs in the second half-hour. So, for example, we can find the population at $t = \frac{1}{2}$ *by finding the average of the populations at* $\boldsymbol{t} = 0$ *and* $\boldsymbol{t} = 1$.

Comment on this idea. How does it compare to the multipliers generated in the previous problems? For what kind of function would this reasoning work?

Solution:

Hours into study	0	1	2	3	4
Population (thousands)	4	8	16	32	64

a. Students would be expected to find these values by repeatedly multiplying by 2. The plot below consists of the exponential function $P(t) = 4 \cdot 2t$ which students will derive in the next part. The plot of the data alone would consist of the 5 blue points.



b. The equation is $P = 4(2)^t$, since as we tallied above via repeated multiplication, we have $4(2)^1 = 8$, $4(2)^2 = 16$, $4(2)^3 = 32$, $4(2)^4 = 64$, etc.

c.

Hours into study	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Population (thousands)	4	5.657	8	11.314	16	22.627	32

Let x be the multiplier for the half-hour time interval. Then since going forward a full hour in time has the effect of multiplying the population by x^2, we must have $x^2 = 2$, and so the student needs to multiply by $\sqrt{2}$ every half hour.

d.

Hours into study	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
Population (thousands)	4	5.010	6.350	8	10.079	12.699	16

Similarly, since waiting three 20-minute intervals should double the population, the new multiplier has to satisfy $x \cdot x \cdot x = 2$, which gives $x^3 = 2$. So you would need to multiply by $\sqrt[3]{2}$ every 20 minutes to have the effect of doubling every hour.

e. We already know that the equation for population is $P = 4(2)^t$ when t is a natural number. Given this, it's reasonable to use the expression $P\left(\frac{1}{2}\right) = 4(2)^{\frac{1}{2}}$ to *define* $\sqrt{2}$. However, we reasoned above that $P\left(\frac{1}{2}\right) = 4\sqrt{2}$, and equating the two gives $2^{\frac{1}{2}} = \sqrt{2}$. Similarly, equating the expression $P\left(\frac{1}{3}\right) = 2^{\frac{1}{3}}$ with the calculation $P\left(\frac{1}{3}\right) = \sqrt[3]{2}$ gives the reasonable definition $2^{\frac{1}{3}} = \sqrt[3]{2}$.

f. The reasoning mistakenly assumes linear growth within each hour, i.e., that the amount of population growth is the same each half hour. We know instead that the *percentage* growth is constant, not the raw change in population. If we were to apply the faulty reasoning to the first hour, we would get the following values:

Hours into study	0	$\frac{1}{2}$	1
Population (thousands)	4	6	8

However, this does not have constant percentage growth: from $t = 0$ to $t = \frac{1}{2}$ this population grew by 50% (ratio = 1.5), but then from $t = \frac{1}{2}$ to $t = 1$ the ratio is only 1.33. If you graphed this data, instead of seeing a smoothly increasing curve, you would see a series of connected line segments of increasing slopes.

- Complete the table below:

Expression	Numerical Value	Expression	Numerical Value
$4^{\frac{1}{2}} =?$		$\sqrt[2]{4^1} = \sqrt{4} =?$	
$64^{\frac{1}{2}} =?$		$\sqrt[2]{64^1} =?$	
$8^{\frac{2}{3}} =?$		$\sqrt[2]{8^2} =?$	
$16^{\frac{1}{4}} =?$		$\sqrt[4]{16^1} =?$	
$25^{-\frac{1}{2}} =?$		$\left(\sqrt[2]{25}\right)^{-1} = \frac{1}{\sqrt{25}} =?$	
$(2^3)^{\frac{1}{2}} =?$		$\sqrt[2]{(2)^3} =?$	

a. What do you notice about your answers to the problems in the same row?

b. Is there some pattern that relates the two expressions in each row to one another? Describe the pattern.

c. Given the expression $(5^3)^{\frac{1}{4}}$, what expression using a root symbol would yield the same numerical value?

d. Given the expression $\sqrt[3]{54}$ what expression utilizing a fractional exponent would yield the same numerical value?

Instructional Strategies:

In the traditional pathway, for Algebra I in implementing the standards in curriculum, these standards should occur before discussing exponential functions with continuous domains.

The goal is to show that a fractional exponent can be expressed as a radical or a root. For example, an exponent of ⅓ is equivalent to a cube root; an exponent of ¼ is equivalent to a fourth root.

Review the power rule, $(b^n)^m = b^{nm}$ for whole number exponents i.e., $(7^2)^3 = 7^6$.

Compare examples, such as $(7^{\frac{1}{2}})^2 = 7^1 = 7$ and $(\sqrt{7})^2 = 7$ to help students establish a connection between radicals and rational exponents: $7^{\frac{1}{2}} = \sqrt{7}$ and, in general, $b^{\frac{1}{2}} = \sqrt{b}$.

Provide opportunities for students to explore the equality of the values using calculators.

Offer sufficient examples and exercises to prompt the definition of fractional exponents, and give students practice in converting expressions between radical and exponential forms.

When n is a positive integer, generalize the meaning of $b^{\frac{1}{n}} = \sqrt[n]{b^1}$ and then to $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ where *n* and *m* are integers and *n* is greater than or equal to 2. When *m* is a negative integer, the result is the reciprocal of the root $b^{\frac{-m}{n}} = \frac{1}{\sqrt[n]{b^m}}$.

Stress the two rules of rational exponents: 1) the numerator of the exponent is the base's power and 2) the denominator of the exponent is the order of the root. When evaluating expressions involving rational exponents, it is often helpful to break an exponent into its parts – a power and a root – and then decide if it is easier to perform the root operation or the exponential operation first.

Model the use of precise mathematics vocabulary (e.g., base, exponent, radical, root, cube root, square root etc.). The rules for integer exponents are applicable to rational exponents as well; however, the operations can be slightly more complicated because of the fractions. When multiplying exponents, powers are added . $b^n * b^m = b^{n+m}$When dividing exponents, powers are subtracted $\frac{b^n}{b^m} = b^{n-m}$. When raising an exponent to an exponent, powers are multiplied $(b^n)^m = b^{nm}$.

Common Misconceptions:

Students sometimes misunderstand the meaning of exponential operations, the way powers and roots relate to one another, and the order in which they should be performed. Attention to the base is very important.

Consider examples: $(-81^{\frac{3}{4}})$ and $(-81)^{\frac{3}{4}}$. The position of a negative sign of a term with a rational exponent can mean that the rational exponent should be either applied first to the base, 81, and then the opposite of the result is taken, $(-81^{\frac{3}{4}})$ or the rational exponent should be applied to a negative term $(-81)^{\frac{3}{4}}$. The answer to $\sqrt[4]{-81}$ will be not real because the denominator of the exponent is even. If the root is odd, the answer will be a negative number.

Students should be able to make use of estimation when incorrectly using multiplication instead of exponentiation

Resources/Tools:

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-RN.A.1
    - Extending the Definitions of Exponents, Variation 2

Domain: The Real Number System (N.RN)

◆ **Cluster:** *Use properties of rational numbers and irrational numbers.*

Standard: **N.RN.3**

(11) Rewrite expressions involving radicals and rational exponents using the properties of exponents. **(N.RN.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: N.RN.1-3

Explanations and Examples:

Convert from radical representation to using rational exponents and vice versa.

Students can explain and apply mathematical concepts and carry out mathematical procedures with precision and fluency.

Examples:

1. $\sqrt[3]{5^2} = 5^{\frac{2}{3}}$; $5^{\frac{2}{3}} = \sqrt[3]{5^2}$
2. Rewrite using fractional exponents: $\sqrt[5]{16} = \sqrt[5]{2^4} = 2^{\frac{4}{5}}$
3. Rewrite $\frac{\sqrt{x}}{x^2}$ in at least three alternate forms.
4. Rewrite $\sqrt[4]{2^{-4}}$ using only rational exponents.
5. Rewrite $\sqrt[3]{x^3 + 3x^2 + 3x + 1}$ in simplest form.
6. For items 1a – 1e, determine whether each equation is True or False.

1a.	$\sqrt{32} = 2^{\frac{5}{2}}$	T
1b.	$16^{\frac{3}{2}} = 8^2$	T
1c.	$4^{\frac{1}{2}} = \sqrt[4]{64}$	F
1d.	$2^8 = \left(\sqrt[3]{16}\right)^6$	T
1e.	$\left(\sqrt{64}\right)^{\frac{1}{3}} = 8^{\frac{1}{6}}$	F

Instructional Strategies: See N.RN.2

Resources/Tools:

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-RN.A.2
    - Rational or irrational?
    - Checking a calculation of a decimal exponent

Domain: Quantities ★ (N.Q)

Cluster: *Reason quantitatively and use units to solve problems.*

Standard: N.Q.1 ★ (all)

Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. **(N.Q.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: Algebra and Functions

Explanations and Examples:

Across a wide variety of problems and applications, units can and should be used a way to understand a problem and make an effective problem solving tool, guiding the student to the relevant measurements and operations. Interpreting units consistently could be as simple as interpreting the meaning of the y-intercept to as complicated as using the units to support the selection of the appropriate regression model.

Particular attention should be paid to creating graphs that follow standard mathematical and scientific conventions for graphing or discussing decisions made when graphing in cases of no consensus.

- Graphs must be partitioned into equal intervals.
- Intervals should be chosen so that the area interest is easily visible (for example, small enough to see an intersection or large enough to view the vertex of a parabola).
- Intervals should allow global analysis of direction of change, maximum/minimum, end behavior, etc. For example, it is possible to zoom in or out so much that a nonlinear graph appears linear.

Things to consider:

- Is it more important for the graph to take up the majority of the graphing space or should the intervals on the domain and range be the same? Taking up more space might make it easier to see the key features of interest but can distort the appearance of rate of change. Keeping the intervals the same helps create a visual of the rate of change but might not make sense if the domain is $0 \leq x \leq 2$ while the range is $150 \leq y \leq 500$.
- Should the graph include the origin or use a "compressed scale" to begin the scale at a higher number? Compressing the y-axis has the benefit of using more of the graphable space but might create a false y-intercept.

This is an "all" standard because there is no one right answer to most of these questions. Fluency and skill with making these decisions and interpreting the decision of others comes only after consistent and explicit discussion during learning.

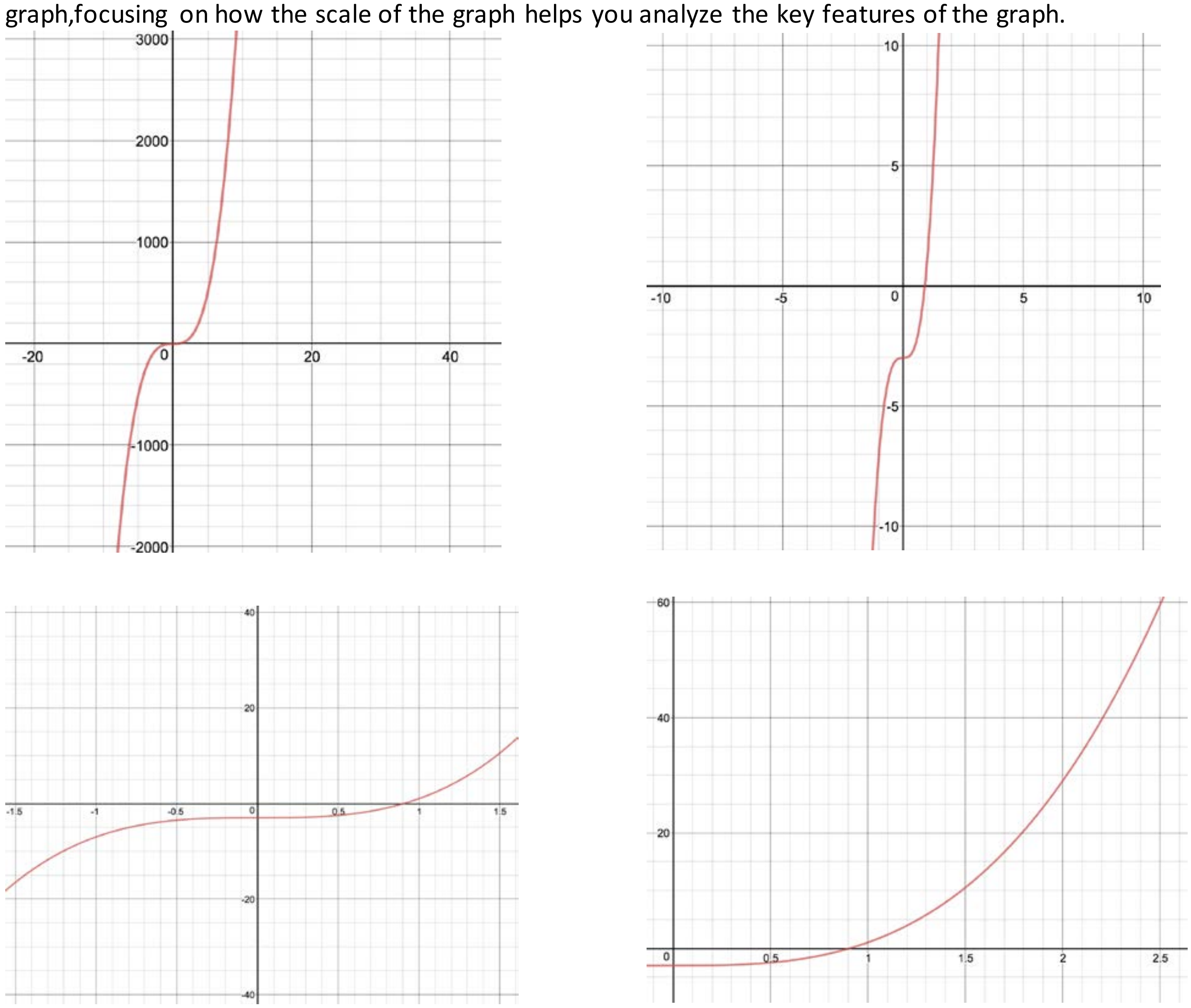
Algebra 2 Examples:

The following questions are adapted from Illustrated Math. Part b) was added to both questions.

Task:

1. The height of a diver over the water is modeled by the equation $h = -5t^2 + 8t + 3$ where *h* denotes the height of the diver over the water (in meters) and *t* is time measured in seconds.
   a) Rewrite this equation, finding *t* in terms of *h*.
   b) Perform a unit analysis using the new equation. According to the unit analysis, what are the units for *t* (when *t* becomes a function a *h*). Discuss why the units are the same or different than the units given in the question.
2. A bacteria population *P* is modeled by the equation $P = P_0 10^{kx}$ where time *t* is measured in hours, *k* is a positive constant, and P_0 is the bacteria population at the beginning of the experiment.
   a) Rewrite this equation to find *t* in terms of *P.*
   b) Perform a unit analysis to determine the unit for the constant *k* which would allow the formula to be

1) Below are versions the same graph, $y = 4x^3 - 3$. Discuss the pro's (if any) and con's (if any) with each graph,focusing on how the scale of the graph helps you analyze the key features of the graph.



Instructional Strategies:

As you think about this the standard, the first few words of the standard should guide you "Use units *as a way to understand problems…"* This standard should be highlighted when it will enhance student understanding and not as part of a procedural checklist or as an addition to question that confuses more than it supports. Remember that this is an ALL standard because mastery is developed *over time*. Initially the conversations will be difficult but students should progress in sophistication throughout their time in high school mathematics. Think instructionally about how you can monitor, assess, and provide feedback to students on growth in this area, as well as all areas in the "ALL" category.
This standard also provides an opportunity to ensure alignment with other departments. The science department likely has some criteria for the graphs that they draw, which might be discussed during class. Sharing the outcome for this standard with other departments might give them some ideas for supporting mathematics within their class.

Some contextual problems may require an understanding of derived measurements and capability in unit analysis. Keeping track of derived units during computations and making reasonable estimates and rational conclusions about accuracy and the precision of the answers help in the problem-solving process.

Common Misconceptions:

Students may not realize the importance of the units' conversions in conjunction with the computation when solving problems involving measurements. Students often have difficulty understanding how ratios expressed in different units can be equal to one**.** For example, $\frac{5280\ ft}{1\ mile}$ is simply one, and it is permissible to multiply by that ratio.

Students need to make sure to put the quantities in the numerator or denominator so that the terms can cancel appropriately. Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than the required.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

N.Q.1-3

"Relationships Between Quantities & Reasoning with Equations & Their Graphs" - EngageNY Algebra 1 Module 1:
In this module students analyze and explain precisely the process of solving an equation. Through repeated reasoning, students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and make conjectures about the form that a linear equation might take in a solution to a problem. They reason abstractly and quantitatively by choosing and interpreting units in the context of creating equations in two variables to represent relationships between quantities. They master the solution of linear equations and apply related solution techniques and the properties of exponents to the creation and solution of simple exponential equations.

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A
  - Traffic Jam
  - Weed Killer

N.Q.1

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.1
  - Ice Cream Van
  - How Much is a Penny Worth?
  - Selling Fuel Oil at a Loss
  - Fuel Efficiency
  - Runners' World

Domain: Number and Quantities ★ (N.Q)

Cluster: *Reason quantitatively and use units to solve problems.*

Standard: N.Q.2 ★ (all)

Define appropriate quantities for the purpose of descriptive modeling. **(N.Q.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

Connections:

The Modeling domain provides a list of standards connected to this standard.

Explanations and Examples:

This standard focuses on critical aspect of modeling and these three words are each essential for students to use consistently: define, appropriate, and quantities.

First, students must clearly define the meaning for the variable. This requires them to attend to precision. For example, $t = time$. Time on the clock, time since the event started, time till the event ends?

It is critical to clearly define the variables to ensure that everyone understands what kind of input is expected and acceptable.

Second, the variable assignment must be appropriate. This means that students should be able to define the independent and dependent variables correctly. It also means they should be able to sift through extra information to identify the required information to answer the question. A research study from Dr. Marilyn Carlson and her team found that students struggle with identifying the appropriate variables. Students were asked to draw a graph showing the height of the fluid given the amount of fluid in the bottle. They found most misidentified the independent variable as the height and the dependent variable as the volume. Even more surprising, the students felt like *time* was an additional variable (i.e. if the water was poured in faster, the rate of change would be greater, if the water was poured and then stopped and then poured again, the graph would reflect those changes). This study illustrates the value in not only clearly defining variables but ensuring that students have made appropriate identifications and are not distracted by incorrect ideas about rate.

Finally, students *must* define the variable as quantities and use the variables as quantities. On Bill McCallum's forum about the standards, a question was asked about function notation which illustrates the importance of this concept:

> …"it can't literally be true that *f(x)* is a function, because it's a number, and a number is not a function. The letter *x* refers to a specific but unspecified number in the domain of *f*, and *f(x)* refers to the corresponding output. That's the way function notation works. I would worry that not being precise in this usage leads to confusion and misconceptions later on. I think your desire to use *f(x)* to refer to the function comes from a sense that *x* in some way represents all the input values at once. But this itself is dangerous, I think: **a lot of the trouble students have with algebra comes from a feeling that *x*** (or

whatever letter you are using) **isn't really a number** but is some vague mystical thing they have to perform mysterious rites on. So the more we can **keep students anchored** in the idea that the letters in algebraic expressions and equations are **just numbers,** and that the things you do to expressions and equations are just the things you can do to numerical expressions, the better.

Algebra 2 Example:

The following is an Illustrated Math Task:

1. On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.
   a) When will the lake be covered half-way?
   b) On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?
   c) On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?
   d) Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

Instructional Strategies: See N.Q.1

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.2
  - Harvesting the Fields

Domain: Quantities ★ (N.Q)

Cluster: *Reason quantitatively and use units to solve problems.*

Standard: **N.Q.3 ★ (all)**

Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. **(N.Q.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See N.Q.1

Explanations and Examples:

Determine the accuracy of values based on their limitations in the context of the situation.

The margin of error and tolerance limit varies according to the measure, tool used, and context.

Examples:

- Determining price of gas by estimating to the nearest cent is appropriate because you will not pay in fractions of a cent but the cost of gas is $\frac{\$3.479}{gallon}$.
- A liquid weed-killer comes in four different bottles, all with the same active ingredient. The accompanying table gives information about the concentration of active ingredient in the bottles, the size of the bottles, and the price of the bottles. Each bottle's contents is made up of active ingredient and water.

	Concentration	Amount in Bottle	Price of Bottle
A	1.04%	64 fl oz.	$12.99
B	18.00%	32 fl oz.	$22.99
C	41.00%	32 fl oz.	$39.99
D	1.04%	24 fl oz.	$5.99

The margin of error and tolerance limit varies according to the measure, tool used, and context.

a) You need to apply a 1% solution of the weed killer to your lawn. Rank the four bottles in order of best to worst buy. How did you decide what made a bottle a better buy than another?

b) The size of your lawn requires a total of 14 fl. oz. of active ingredient. Approximately how much would you need to spend if you bought only the A bottles? Only the B bottles? Only the C bottles? Only the D bottles?

Supposing you can only buy one type of bottle, which type should you buy so that the total cost to you is the least for this particular application of weed killer?

The principal purpose of the task is to explore a real-world application problem with algebra, working with units and maintaining reasonable levels of accuracy throughout. Of particular interest is that the optimal solution for long-term purchasing of the active ingredient is achieved by purchasing bottle C, whereas minimizing total cost for a *particular* application comes from purchasing bottle B. Students might need the instructor's aid to see that this is just the observation that buying in bulk may not be a better deal if the extra bulk will go unused.

Solution:

a) All of the bottles have the same active ingredient, and all can be diluted down to a 1% solution, so all that matters in determining value is the cost per fl. oz. of active ingredient. We estimate this in the following table:

	Amount active in Bottle	Price of bottle	Cost per ounce
A	1.04% × 64 ≈ 0.64 fl oz	\$12.99 ≈ \$13	$\frac{13}{0.64} \approx \$20\ per\ fl\ oz$
B	18.00% × 32 ≈ 6 fl oz	\$22.99 ≈ \$23	$\frac{23}{6} \approx \$4\ per\ fl\ oz$
C	41.00% × 32 ≈ 13 fl oz	\$39.99 ≈ \$40	$\frac{40}{13} \approx \$3\ per\ fl\ oz$
D	1.04% × 24 ≈ 0.24 fl oz	\$5.99 ≈ \$6	$\frac{6}{0.24} \approx \$24\ per\ fl\ oz$

If we assume that receiving more active ingredient per dollar is a better buy than less active ingredient per dollar, the ranking in order of best-to-worst buy is C,B,A,D.

b) The A bottles have about 0.64 fl. oz. of active ingredient per bottle so to get 14 fl. oz. we need

c) $\frac{14\ fl.\ oz.}{0.64\ fl.\ oz./bottle} \approx 22\ bottles.$

Purchasing 22 A bottles at about \$13 each will cost about \$286.

The B bottles have a little less than 6 fl. oz. of active ingredient per bottle so to get 14 fl. oz. we need 3 bottles. Purchasing 3 B bottles at about \$23 each will cost about \$69.

The C bottles have a little more than 13 fl. oz. of active ingredient per bottle, so we need 2 bottles. Purchasing 2 C bottles at about \$40 each will cost about \$80.

The D bottles have only 0.24 fl. oz. of active ingredient per bottle, so to get 14 fl. oz. we need

$$\frac{14\ fl.\ oz.}{0.24\ fl.\ oz./bottle} \approx 58\ bottles.$$

Purchasing 58 D bottles at about \$6 each will cost about \$348.

Thus, although the C bottle is the cheapest when measured in dollars/fl. oz., the B bottles are the best deal for this job because there is too much unused when you buy C bottles.

Instructional Strategies: See N.Q.1

Resources/Tools:

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.1
  - Felicia's Drive
  - Calories in a sports drink
  - Dinosaur Bones
  - Bus and Car

Domain: The Complex Number System N.CN

◆ **Cluster:** *Perform arithmetic operations with complex numbers.*

Standard: **N.CN.1**

(11) Know there is a complex number *i* such that $i^2 = -1$, and every complex number has the form $a + bi$ with *a* and *b* real. **(N.CN.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: N.RN.1-3, N.CN

Explanations and Examples:

Every complex number can be written in the form *a* + *bi* where *a* and *b* are real numbers. The square root of a negative number is a complex number. Complex numbers can be added, subtracted, and multiplied like binomials.

The commutative, associative, and distributive properties hold true when adding, subtracting, and multiplying complex numbers.

Examples:

	Problem	Solution	*bi Form*
1.	$\sqrt{-36}$	$\sqrt{-36} = \sqrt{-1} \bullet \sqrt{36} = 6i$	$6i$
2.	$2\sqrt{-49}$	$2\sqrt{-49} = 2\sqrt{-1} \bullet \sqrt{49} = 2 \bullet 7i = 14i$	$14i$
3.	$-3\sqrt{-10}$	$-3\sqrt{-10} = -3\sqrt{-1} \bullet \sqrt{10} = -3 \bullet i \bullet \sqrt{10} = -3i\sqrt{10}$	$-3i\sqrt{10}$
4.	$5\sqrt{-8}$	$5\sqrt{-8} = 5\sqrt{-1} \bullet \sqrt{8} = 5 \bullet i \bullet 2\sqrt{2} = 10i\sqrt{2}$	$10i\sqrt{2}$

5. $\sqrt{-1} = i$
6. $\sqrt{-4} = 2i$
7. $\sqrt{-7} = i\sqrt{7}$

Instructional Strategies:

Before introducing complex numbers, revisit simpler examples demonstrating how number systems can be seen as "expanding" from other number systems in order to solve more equations. For example, the equation *x* + 5 = 3 has no solution as a whole numbers, but it has a solution *x* = -2 as an integers. Similarly, although 7*x* = 5 has no solution in the integers, it has a solution $x = \frac{5}{7}$ in the rational numbers. The linear equation *ax* + *b* = *c*, where *a*, *b*, and *c* are rational numbers, always has a solution *x i*n the rational numbers: $x = \frac{(c-b)}{a}$.

When moving to quadratic equations, once again some equations do not have solutions, creating a need for larger number systems. For example, $x^2 - 2 = 0$ has no solution in the rational numbers. But it has solutions $\pm\sqrt{2}$ in the real numbers. (The real number line augments the rational numbers, completing the line with the irrational numbers.)

Point out that solving the equation $x^2 - 2 = 0$ in terms of x is equivalent to finding x-intercepts of a graph of $y = x^2 - 2$, which crosses the x-axis at $\left(-\sqrt{2}, 0\right)$ and $(\sqrt{2}, 0)$Thus, the graph illustrates that the solutions are $x = \pm\sqrt{2}$.

Next, use an example of a quadratic equation with real coefficients, such as $x^2 + 1 = 0$, which can be written equivalently as $x^2 = -1$. Because the square of any real number is non-negative, it follows that $x^2 = -1$ has no solution in the real numbers. One can see this graphically by noticing that the graph of $y = x^2 + 1$ does not cross the x-axis.

The "solution" to this "impasse" is to introduce a new number, the imaginary unit i, where $i^2 = -1$, and to consider complex numbers of the form $a + bi$, where a and b are real numbers and i is not a real number. Because i is not a real number, expressions of the form $a + bi$ cannot be simplified.

The existence of i, allows every quadratic equation to have two solutions of the form $a + bi$ – either real when $b = 0$, or complex when $b \neq 0$.

In order to find solutions of quadratic equations or to create quadratic equations from its solutions, introduce students to the condition of equality of complex numbers, with addition, subtraction and multiplication of complex numbers.

Stress the importance of the relationships between different number sets and their properties. The complex number system possesses the same basic properties as the real number system: that addition and multiplication are commutative and associative; the existence of additive identity and multiplicative identity; the existence of an additive inverse for every complex number and the existence of multiplicative inverse or reciprocal for every non-zero complex number; and the distributive property of multiplication over the addition. An awareness of the properties minimizes students' rote memorization and links the rules for manipulations with the complex number system to the rules for manipulations with binomials with real coefficients of the form $a + bx$.

Common Misconceptions:

If irrational numbers are confused with non-real or complex numbers, remind students about the relationships between the sets of numbers.

If an imaginary unit i is misinterpreted as -1 instead of $\sqrt{-1}$, re-establish a definition of i.

Some properties of radicals that are true for real numbers are not true for complex numbers. In particular, for positive real numbers a and b, $\sqrt{a} * \sqrt{b} = \sqrt{(a * b)}$ but $\sqrt{-a} * \sqrt{-b} \neq \sqrt{(-a)(-b)}$ and $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ but $\frac{\sqrt{-a}}{\sqrt{-b}} \neq \sqrt{\frac{-a}{-b}}$.

If those properties are getting *misused*, provide students with an example, such as $10 = \sqrt{100} = \sqrt{(-25)(-4)} = \sqrt{-25} * \sqrt{-4} = 5i * 2i = 10i^2 = -10$ which leads to a contradiction that a positive real number is equal to a negative number.

Resources/Tools:

"Polynomial, Rational, and Radical Relationships" - EngageNY Algebra II Module 1: In this module, students draw on their foundation of the analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property. Students identify and make connections between zeros of polynomials and solutions of polynomial equations. The role of factoring, as both an aid to the algebra and to the graphing of polynomials, is explored. Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations. An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers.

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-CN.A
    - Complex number patterns

Domain: The Complex Number System N.CN

◆ **Cluster:** *Perform arithmetic operations with complex numbers.*

Standard: N.CN.2

(11) Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. **(N.CN.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: N.CN.1-3, 7.NS, 7.EE, N.RN.1-3, A.APR 1-3

Explanations and Examples:

- Apply the fact that the complex number $i^2 = -1$.
- Recognize that $i^4 = i^8 = i^{12} = i^{16} = \cdots = i^{4k} = 1$ *where k is a positve integer*, and use the relation $i^2 = -1$ to justify this fact.
- Recognize that $i^2 = i^6 = i^{10} = i^{14} = \cdots = i^{4k-1} = -1$ *where k is a positve integer*, and use the relation $i^2 = -1$ to justify this fact.
- Recognize that $i^3 = i^7 = i^{11} = i^{15} = \cdots = i^{4k-1} = -i$ *where k is a positve integer*, and use the relation $i^2 = -1$ to justify this fact.
- Recognize that $i = i^5 = i^9 = i^{13} = \cdots = i^{4k-3} = i$ *where k is a positve integer*, and use the relation $i^2 = -1$ to justify this fact.

Use the associative, commutative, and distributive properties, to add, subtract, and multiply complex numbers.

Examples:

- Simplify the following expression. Justify each step using the commutative, associative and distributive properties.
  $(3-2i)(-7+4i)$

Instructional Strategies: See. N.CN.1 and N.RN.1 and A.APR.2

Resources/Tools:

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-CN.A
  - Computations with Complex Numbers

Domain: The Complex Number System N.CN

Cluster: *Perform arithmetic operations with complex numbers.*

Standard: N.CN.3

(11) Find the conjugate of a complex number. **(N.CN.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: N.CN.1-3, N.RN.1-3, A.APR 1-3

Explanations and Examples:

This standard combines N.CN.1 and N.CN.2 when students can multiply a complex number and it's conjugate to eliminate the imaginary number. This skill is useful to rewrite a complex rational number so that there is no imaginary number in the denominator. Quotients of complex numbers (N.CN.4) is a plus standard so it is the teacher's discretion how far toward complex rational numbers to move students. Regardless, practice multiplying the complex number and its conjugate will help students discover this interesting property.

Examples:

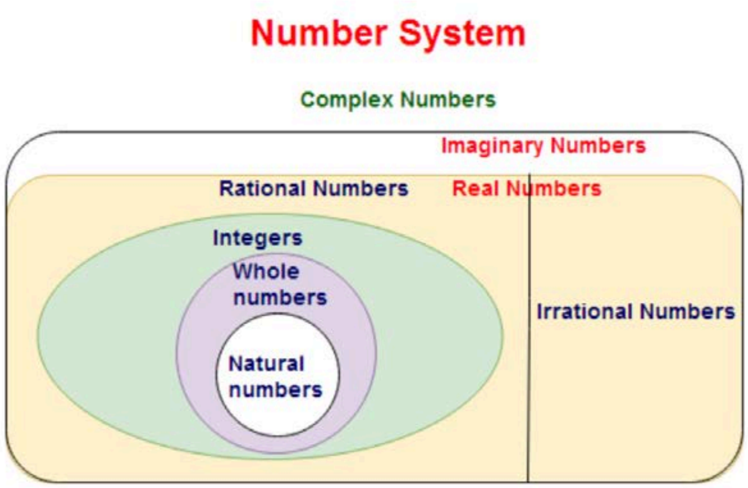
Write the conjugate for each complex number:

a. $3 + 4i$

b. $-4 + i$

c. $-12i$

Use the Venn diagram of the Number System to justify the statement "Every number has a complex conjugate."



Instructional Strategies:

Make sure that students only write the opposite of the imaginary number and not the opposite of the entire complex number. The conjugate of a pure imaginary number should be compared to the conjugate of a real number to help ensure students have a solid understanding of negating the imaginary component only.

Resources/Tools:

Complex Numbers and Transformations: EngageNY Algebra II Module 1: sets the stage for expanding students' understanding of transformations by exploring the notion of linearity. This leads to the study of complex numbers and linear transformations in the complex plane.

Domain: The Complex Number System N.CN

● **Cluster:** *Use complex numbers in polynomial identities and equations.*

Standard: **N.CN.8**

(11) Solve quadratic equations with real coefficients that have complex solutions. **(N.CN.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.7 Look for and make use of structure.

Connections: A.REI.5

Explanations and Examples:

Solve quadratic equations with real coefficients that have solutions of the form $a + bi$ and $a - bi$ (A.REI.5).

Determine when a quadratic equation in standard form, $ax^2 + bx + c = 0$, has complex roots by looking at a graph of $f(x) = ax^2 + bx + c$ or by calculating the discriminant.

Examples:

- Within which number system can $x^2 = -2$ be solved? Explain how you know.
- Solve $x^2 + 2x + 2 = 0$ over the complex numbers.
- Find all solutions of $2x^2 + 5 = 2x$ and express them in the form $a + bi$ (A.REI.5).
- Will a quadratic equation with real coefficients always have real solutions? Why or why not?

Instructional Strategies:

Revisit quadratic equations with real coefficients and a negative discriminant and point out that this type of equation has no real number solution. Emphasize that with the extension of the real number system to complex numbers any quadratic equation has a solution. Since the process of solving a quadratic equation may involve the use of the quadratic formula with a negative discriminant, defining a square root of a negative number becomes $\sqrt{-n} = i\sqrt{n}$, where *N* is a positive real number; *i* is the imaginary unit and $i^2 = -1$. After the square root of a negative number has been defined, emphasize that the quadratic formula can be used without restriction.

While solving quadratic equations using the quadratic formula, students should observe that the quadratic equation always has a pair of solutions regardless of the value of the discriminant.

Common Misconceptions:

In the cases of quadratic equations, when the use of quadratic formula is not critical, students sometime ignore the negative solutions. For example, for the equation $x^2 = 9$, students may mention 3 and forget about (– 3), or mention 3i and forget about (- 3*i*) for the equation $x^2 = -9$. If this misconception persists, advise students to solve this type of quadratic equation either by factoring or by the quadratic formula

Resources/Tools:

"Manipulating Radicals" – Mathematics Assessment Project

This lesson unit is intended to help you assess how well students are able to: •Use the properties of exponents, including rational exponents, and manipulate algebraic statements involving radicals. Discriminate between equations and identities. In this lesson there is also an opportunity to consider the role of the imaginary number, but this is optional.

Domain: Vector and Matrix Quantities N.VM

● **Cluster:** *Perform operations on matrices and use matrices in applications.*

Standard: **N.VM.6**

(11) Use matrices to represent and manipulate data, (e.g. *represent information in a linear programing problem as a matrix or rewriting a system of equations as a matrix.)*

(N.VM.6)

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: N.VM.7-8, S.ID.4

Explanations and Examples:

The goal of this standard is for students to view a table of information abstractly as a matrix of information. This might be helpful when working with a system of equations or when given a situation with multiple unknowns.

Examples:

1. Represent the system of equations using a matrix:

$$2x + 3y = 6$$
$$8x - 3y = -1$$

Solution:

$$\left[\begin{array}{cc:c} 2 & 3 & 6 \\ 8 & -3 & -1 \end{array}\right]$$

2. Represent the system of equations using a matrix:

$$y = -5$$
$$5x + 4y = -20$$

Solution:

$$\left[\begin{array}{cc:c} 0 & 1 & -5 \\ 5 & 4 & -20 \end{array}\right]$$

3. The school that Lisa goes to is selling tickets to the annual talent show. On the first day of ticket sales the school sold 4 senior citizen tickets and 5 student tickets for a total of $102. The school took in $126 on the second day by selling 7 senior citizen tickets and 5 student tickets. Create a matrix that represents the price each of one senior citizen ticket and one student ticket?

   Solution:
   $$\left[\begin{array}{cc|c} 4 & 5 & 102 \\ 7 & 5 & 126 \end{array}\right]$$

Instructional Strategies:

Activate prior knowledge by creating a table with the provided information, labeling the columns and rows appropriately, and comparing the result to the matrix.

Resources/Tools:

System of equations resources can be easily modified to support this standard.

Domain: Vector and Matrix Quantities N.VM

● **Cluster:** *Perform operations on matrices and use matrices in applications.*

Standard: N.VM.7

(11) Multiply matrices by scalars to produce new matrices, *(e.g. as when all of the payoffs in a game are doubled.)* **(N.VM.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: N.VM.6-8

Explanations and Examples:

This standard is limited to multiplying a matrix by a scalar. Multiplication of a matrix by a matrix is N.VM.8.

Examples:

1. $-4\begin{bmatrix} 5 & 1 \\ 6 & 0 \end{bmatrix}$

2. $-4w\begin{bmatrix} -w & -4+u & 0 \\ v & 5v & 3wv \end{bmatrix}$

Instructional Strategies:

Connecting multiplication by a scalar with the distributive property can help students anchor the new learning to their previous knowledge.

Resources/Tools:

CK12:

- https://www.ck12.org/Algebra/Multiplying-Matrices-by-a-Scalar/

Khan Academy

- https://www.khanacademy.org/math/precalculus/precalc-matrices/multiplying-matrices-by-scalars/v/scalar-multiplication

Domain: Vector and Matrix Quantities N.VM

Cluster: *Perform operations on matrices and use matrices in applications.*

Standard: N.VM.8

(11) Add, subtract, and multiply matrices of appropriate dimensions; find determinants of 2×2 matrices. **(N.VM.8)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: N.VM.6-8

Explanations and Examples:

This standard has multiple expectations:

1. Add and subtract matrices
2. Multiply a matrix by a matrix
3. Recognizing the dimensions appropriate for performing the operation
4. Finding determinants of a 2X2 matrix

Connecting this standard back to N.VM.6, representing data using a matrix, and a context when possible might help students connect the abstract process to the quantities in the problem.

Examples:

Add and subtract matrices, if possible.

1. $$\begin{bmatrix} -4 & -5 & 5 \\ 1 & 6 & 3 \\ -2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} -6 & -1 & -6 \\ 6 & -3 & -2 \\ 4 & -1 & -3 \end{bmatrix}$$

2. $$-5\left(\begin{bmatrix} 1 & 0 \\ -2 & -3 \\ 6 & -6 \end{bmatrix} + \begin{bmatrix} -5 & 4 \\ -6 & 0 \\ 4 & 4 \end{bmatrix}\right)$$

3. $$\begin{bmatrix} 5 & -3 & 2 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 2 \\ 12 & -5 \\ 9 & 0 \end{bmatrix}$$

Multiply a matrix by a matrix, if possible.

4. $\begin{bmatrix} 3 & -3 \\ 6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 & 1 \\ 6 & -5 & 4 \end{bmatrix}$

5. $\begin{bmatrix} -1 & 1 \\ 7 & -8 \\ -3 & 10 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 6 & 1 \\ 21 & 8 \end{bmatrix}$

What will be the dimensions of the resulting operation?

6. $2\left(\begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 9 & -3 \end{bmatrix}\right)$

7. $\begin{bmatrix} 2 & 1 & 4 \\ -5 & 0 & 10 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 \\ 6 & 1 \\ 5 & 7 \end{bmatrix}$

Find the determinant of the matrix:

8. $\begin{bmatrix} 3 & 12 \\ 5 & 7 \end{bmatrix}$

Instructional Strategies:

Allocate plenty of instructional time to master the large variety of expectations with this standard

Resources/Tools:

Matrix operations: https://www.khanacademy.org/math/precalculus/precalc-matrices

High School – Algebra

Domain: Seeing Structure in Expressions A.SSE

Cluster: *<u>Interpret the structure of expressions.</u>*

Standard: A.SSE.1 ★ (all)

Interpret expressions that represent a quantity in terms of its context.

A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients. **(A.SSE.1a)**

A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and* $(1+r)^n$. **(A.SSE.1b)**

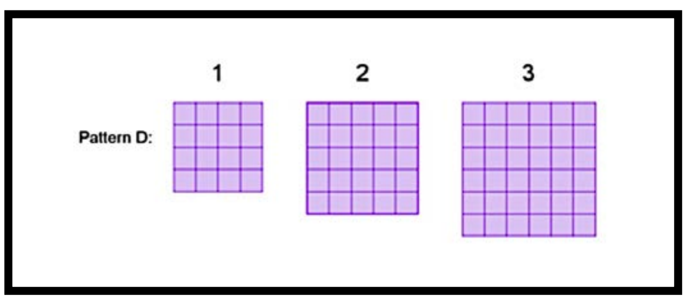
Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections: A.SSE.2

Explanations and Examples:

Viewing this standard as part of the Modeling Domain helps clarify that the purpose of this standard is to interpret in the context of the situation. Students are asked to reflect on the interaction between the situation and the equation. Students should be able to explain the individual parts and, as part of A.SSE.2, explain how the parts of an equivalent expression written in a different form continues to describe the same situation. A question asking students to describe a pattern algebraically provides a great opportunity for students to explain how they see the pattern algebraically.



For example, the following Illustrated Math task could be extended or adapted to demonstrate this standard. The number of tiles in step *n* of Pattern D is defined by $d(n) = (n+3)^2$.

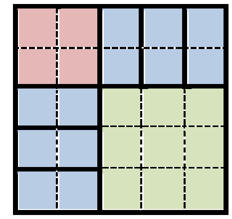
a) Explain "n+3" in the context of the situation?

The length of each side is three more than the number of the step.

b) Explain why the formula has an exponent of 2?

The pattern is describing the area of a square. The formula for the area of the square is $A = s^2$ with $s = n + 3$. This is an example of viewing a one or more of their parts as a single entity. Being able to see the similarity between the area formula and this pattern helps students write their own equations to model a situation.

c) Expanding the pattern into standard form, explain how each component can still be seen the pattern. $d(n) = n^2 + 6n + 9$



It is unlikely that someone would notice the connection between this equation and the pattern and generate this model independently. But asking students to connect an equation they did not generate to a pattern is a good way to assess students' ability to connect an equation to its geometric representation. Students might see different parts "growing" in different ways. One possible interpretation is that the red squares in the corner represent n^2 and that square grows by one row and column each time. The green square is a constant 9 units each time. The blue represents the *6n*, meaning I have 6 groups of size 2. The next iteration will have 6 groups of size 3.

The analysis of this question illustrates how A.SSE 1 and 2 can work together to reveal new information about the problem. Explaining each part in context can create reach conversations with students, as well as reinforce the meaning behind the mathematics.

Algebra 2 Example:

This is an Illustrative Math Task:

Most savings accounts advertise an annual interest rate, but they actually compound that interest at regular intervals during the year. That means that, if you own an account, you'll be paid a portion of the interest before the year is up, and, if you keep that payment in the account, you'll start earning interest on the interest you've already earned.

For example, suppose you put \$500 in a savings account that advertises 5% annual interest. If that interest is paid once per year, then your balance B after t years could be computed using the equation $B = 500(1.05)^t$, since you'll end each year with 100% + 5% of the amount you began the year with.

On the other hand, if that same interest rate is compounded monthly, then you would compute your balance after t years using the equation: $B = 500\left(1 + \frac{.05}{12}\right)^{12t}$.

- Why does it make sense that the equation includes the term $\frac{.05}{12}$? That is, why are we dividing .05 by 12?
- How does this equation reflect the fact that you opened the account with \$500?
- What do the numbers 1 and $\frac{.05}{12}$ represent in the expression $(1+\frac{.05}{12})$?
- What does the "12t" in the equation represent?

Instructional Strategies:

Using visuals to highlight the connection between the situation, the problem, and the equation is a great strategy to help students not get lost in the problem. Highlighting one piece of information, say time, the same color throughout the problem can help show where this piece of information goes throughout the problem. Another strategy is to use post-it notes to physically cover larger pieces of information with "a single entity" to help students see it as one large chunk. Flipping back and forth between the "big" piece written on one side and the "small" piece written on the other can help students view it as a chunk.

Without going into more detail than you might need here, let me briefly name what we asking students to do in part b of the standard so that you have the concept on hand for further research.

> "**The theory of reification** (Sfard, 1987, 1988, 1991, 1992, 1994; Sfard & Linchevski, 1994) describes how concepts come into existence from a cognitive perspective. The theory is based on the fact that many

> **mathematical concepts** are **conceived in two complementary ways, operationally and structurally**. Operational conceptions are "**about processes, algorithms, and actions rather than about objects**" (emphasis in original, Sfard, 1991, p. 4), in contrast to **structural conceptions** where mathematical entities are conceived as **objects, wholes, or as the result of a process instead of the process itself...** Reification is 'an ontological shift- a sudden ability to see something familiar in a totally new light" (Sfard, 1991, p. 19); what was previously only a process can now be seen as an object also."

Reification is difficult to achieve, thus, its placement as an ALL standard. It will require consistent practice.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

"Polynomial and Quadratic Expressions, Equations, and Functions" - EngageNY Algebra I Module 4:
In this module, students continue to interpret expressions, create equations, rewrite equations and functions in different but equivalent forms, and graph and interpret functions, but this time using polynomial functions, and more specifically quadratic functions, as well as square root and cube root functions.

"Interpreting Algebraic Expressions" – Mathematics Assessment Project:
This lesson unit is intended to help you assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It will help you to identify and support students who have difficulty in: Recognizing the order of algebraic operations. / Recognizing equivalent expressions. or Understanding the distributive laws of multiplication and division over addition (expansion of parentheses).

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.A.1
  - Mixing Fertilizer
  - Increasing or Decreasing? Variation 1
  - Throwing Horseshoes
  - Quadrupling Leads to Halving
  - Kitchen Floor Tiles
  - Mixing Candles
  - The Bank Account
  - Delivery Trucks
  - Radius of a Cylinder
  - The Physics Professor

Domain: Seeing Structure in Expressions ★ A.SSE

Cluster: *Interpret the structure of expressions.*

Standard: **A.SSE.2 ★ (all)**

Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.* **(A.SSE.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.

Connections: A.SSE.1

Explanations and Examples:

This standard partners well with A.SSE.1 to support explorations when modeling (see A.SSE.1 for additional information) but the standard can also stand alone as an algebraic standard. There are many standards that focus on the specifics of re-writing an expression (i.e. factoring, completing the square, laws of exponents, trig identities, etc.) but this standard is not focused on typical algebraic manipulation. The goal is for students to take a step back and *see the structure* and connect the structure to the procedures. As with most ALL standards, there is not a specific procedure to teach students; rather this is a skill that develops over time and through intentional questions. For example, in Algebra 1 students learn the formula for slope ($m = \frac{y_2-y_1}{x_2-x_1}$). After developing an understanding of slope, they move into linear functions and look for the general form for a linear function. This standard supports the student development for the forms of a linear function because, through the structure, they can manipulate the equation into a more familiar form.

Point Slope Form: What equations could be written if we know the slope, *m,* and one point (x_1, y_1)?

$m = \frac{y-y_1}{x-x_1}$ ---We can multiply both sides by $x - x_1$

$m(x - x_1) = y - y_1$ **Point slope form**

$y = m(x - x_1) + y_1$ Not a typical form but it is structurally the same as transformations of functions. How can a linear function be

Slope Intercept Form: What equations could be written if we know the slope, *m,* and the y-intercept (0, b)?

$m = \frac{y-b}{x-0}$ ---We can multiply both sides by $x - 0$, simply *x.*

$mx = y - b$

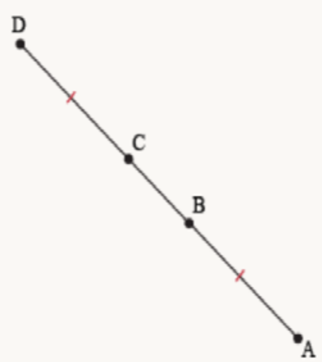
$y = mx + b$

There are many ways that these three examples rely on the structure of the equation.

- First, many students do not realize that understand that (x_1, y_1) is the convention for writing a specific point or value when generalizing an equation but that an equation must still have variables, such as *x* and *y.*
- They also forget that the equation must have an equal sign. This seems like basic understanding but, when working a new type of problem such as finding the general form of an equation, they tend to forget these basic structural requirements.
- Teachers tend to direct students toward the end result that we see (Point-Slope Form or Slope-Intercept Form) rather than letting students play with the equation to see what results they arrive at. As math teachers, if provided with an unknown equation to rewrite into a known form, we would naturally eliminate fractions. Students need to see this same structure and know that eliminating fractions is often valuable. Notice that neither result in the Point-Slope Form example distributed the *m* to $(x - x_1)$ but students would likely think this is a good next step. They need to learn that, structurally, there is often more information gained from the factored form than the distributed form.
- Another common missed opportunity is the ability to perform arithmetic to make an equation less complicated. Students need to recognize the structure of adding or subtracting 0 and multiplying or dividing by 1 and make use of these properties whenever possible.

Student's weakness with the structure of expressions is especially apparent in Geometry class. When students are asked to prove something by performing algebraic operations on geometric statements, they often fail to see that the structure of this equation is the same as the structure learned in Algebra. For example, see the two geometry problems below, which use the structure of the expression to identify ways to rewrite it.

Given the diagram below, show that $\overline{AC} \cong \overline{BD}$.



Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	Given (in figure)
2. $AB = CD$	Definition of congruence (1)
3. $AB + BC = BC + CD$	Addition of BC to (2)
4. $AC = BD$	Segment addition postulate (3)
5. $\overline{AC} \cong \overline{BD}$	Definition of congruence (4)

Many of the steps that rely on the structure of the equation are not written explicitly in the proof. How could additional steps be added to highlight the structure of the equation in relation to the geometric shape?

Show that if two angles are complementary to the same angle, then they're congruent.

Statements	Reasons
1. ∠1 and ∠3 are complementary	Given
2. ∠2 and ∠3 are complementary	Given
3. m∠1 + m∠3 = 90	Definition of complementary angles
4. m∠2 + m∠3 = 90	Definition of complementary angles
5. m∠1 = 90 − m∠3	Subtract m∠3 from (3)
6. m∠2 = 90 − m∠3	Subtract m∠3 from (4)
7. m∠1 = m∠2	Transitive property of equality (5 and 6)
8. ∠1 ≅ ∠2	Definition of congruence (7)

Similarly, students must recognize that that one solution path is to create the same expression in both equations in order to apply the transitive property. Another strategy could have been to solve for $m\angle 3$, set the resulting expressions equal and use the algebraic structure to solve. Both of these approaches rely on students' ability to recognize the structure of the expressions in order to create a strategy that will arrive at the needed proof statement. These types of problems are a struggle for students because there isn't a predictable algorithm and they find it difficult to see how the structure of the expression helps them identify a strategy. The ability to continually reinforce this standard, along with the need to develop this skill slowly over years of instruction is why this standard is an "ALL" standard.

Instructional Strategies:

Strategies from A.SSE.1 will also be useful here. In addition, teachers should use "think alouds" to focus on decision the decision they, as an expert, made as a result of the structure. Comparing and contrasting multiple correct solution strategies is another way to highlight how the structure of an expression can help the student rewrite it.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.A.2
  - Equivalent Expressions
  - Sum of Even and Odd
  - A Cubic Identity
  - Seeing Dots
  - Animal Populations

Domain: Seeing Structure in Expressions A.SSE

▶ **Cluster:** *Write expressions in equivalent forms to solve problems.*

Standard: **A.SSE.3 ★ (all)**

Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★

A.SSE.3b. **(11)** Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. ★ **(A.SSE.3b)**

A.SSE.3c. **(11)** Use the properties of exponents to transform expressions for exponential functions. *For example, the expression* 1.15^t *can be rewritten as* $\left(1.15^{\frac{1}{12}}\right)^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*★ **(A.SSE.3c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.4 Model with mathematics
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF.7, N.RN.1-3

Explanations and Examples:

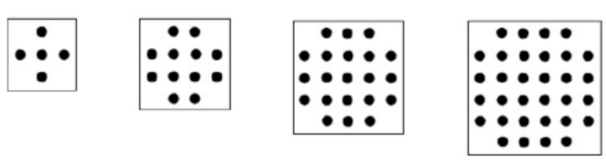
Write expressions in equivalent forms by completing the square to convey the vertex form, to find the maximum or minimum value of a quadratic function, and to explain the meaning of the vertex.

1. What attributes of a graph will factoring and completing the square reveal about a quadratic function?

2. Consider the following algebraic expressions:

$$(n+2)^2 - 4 \text{ and } n^2 + 4n$$

a. Use the figures below to illustrate why the following expressions are equivalent



b. Find some algebraic deductions of the same result.

Solution:

a. A geometric approach to the problem proceeds by identifying, somewhere in the *n* -th figure, the value *n*, and seeing two ways of looking at the dots, giving both $(n+2)^2 - 4 \; and \; n^2 + 4n$. One such approach (among many) is below.

   Let *n* be the number of the figure, with $n = 1$ at the left. We count the dots in each figure in terms of *n* in two different ways. One represents $n^2 + 4n$ and the other represents $(n+2)^2 - 4$.

   Visualizing $n^2 + 4n$:
   - n^2 is the inside full square.
   - $4n$ is the four outside borders with *n* in each.

   Visualizing $(n+2)^2 - 4$:
   - Imagine the larger square with the four additional dots filled in at the corners.

   Then $(n+2)^2$ is the number of dots in the larger square.
   - 4 is the number of dots added.

b. Perhaps most directly, we have $(n+2)^2 - 4 = (n^2 + 4n + 4) - 4 = n^2 + 4n$.

   Alternatively, reversing the steps in this series of equalities is precisely the process of completing the square for the expression $n^2 + 4n$. Similarly, the left-hand side could be viewed as a difference of two squares, in which case we can reason:

$$(n+2)^2 - 4 = \big((n+2)+2\big)\big((n+2)-2\big) = (n+4)(n) = n^2 + 4n.$$

The purpose of above task is to identify the structure in the two algebraic expressions by interpreting them in terms of a geometric context. Students will have likely seen this type of process before, so the principal source of challenge in this task is to encourage a multitude and variety of approaches, both in terms of the geometric argument and in terms of the algebraic manipulation.

Some students might show the equivalence algebraically from the start, either by expanding or by factoring. The algebraic approach should be rewarded, not discouraged. A student who expands could be asked if there is another algebraic method; a student who factors could be asked if there is a way of relating this form to the figure. Observe that the factored form, $(n+4)(n)$, can be related to the figures as follows: If you take the top and bottom borders, turn them vertically, and place them next to the rest of the figure, you get an $(n+4)(n)$ rectangle of dots.

There is also an opportunity here to discuss the process of justifying an algebraic identity. For one, the algebraic solution in part (b) applies to all real numbers *n,* whereas the proof by pictures only directly applies to the case that *n* is a positive integer (though students could be encouraged to replace "numbers of dots" with "areas of regions" to give a version of the geometric proof that works for all positive real numbers.

Use properties of exponents (such as power of a power, product of powers, power of a product, and rational exponents, etc.) to write an equivalent form of an exponential function to reveal and explain specific information about its approximate rate of growth or decay.

3. Write the expression below as a constant times a power of *x* and use your answer to decide whether the expression gets larger or smaller as *x* gets larger.

$$\frac{(2x^3)^2(3x^4)}{(x^2)^3}$$

4. If *x* is positive and $x \neq 0$, simplify: $\frac{\sqrt{x}}{x^3}$

Instructional Strategies:

It is important to balance conceptual understanding and procedural fluency in work with equivalent expressions. For example, development of skill in factoring and completing the square goes hand-in-hand with understanding what different forms of a quadratic expression reveal.

This cluster focuses on linking expressions and functions, i.e., creating connections between multiple representations of functional relations – the dependence between a quadratic expression and a graph of the quadratic function it defines, and the dependence between different symbolic representations of exponential functions. Teachers need to foster the idea that changing the forms of expressions, such as factoring or completing the square, or transforming expressions from one exponential form to another, are not independent algorithms that are learned for the sake of symbol manipulations. They are processes that are guided by goals (e.g., investigating properties of families of functions and solving contextual problems).

Factoring methods that are typically introduced in elementary algebra and the method of completing the square reveals attributes of the graphs of quadratic functions, represented by quadratic equations.

- The solutions of quadratic equations solved by factoring are the *x* – intercepts of the parabola or zeros of quadratic functions.
- A pair of coordinates (*h*, *k*) from the general form $f(x) = a(x-h)^2 + k$ represents the vertex of the parabola, where *h* represents a horizontal shift and *k* represents a vertical shift of the parabola $y = x^2$ from its original position at the origin.
- A vertex (*h, k*) is the minimum point of the graph of the quadratic function if $a > 0$ and is the maximum point of the graph of the quadratic function if $a < 0$. Understanding an algorithm of completing the square provides a solid foundation for deriving a quadratic formula.

Translating among different forms of expressions, equations and graphs helps students to understand some key connections among arithmetic, algebra and geometry. The reverse thinking technique (a process that allows working backwards from the answer to the starting point) can be very effective. Have students derive information about a function's equation, represented in standard, factored or general form, by investigating its graph.

Offer multiple real-world examples of exponential functions. For instance, to illustrate an exponential decay, students need to recognize that in the equation for an automobile cost $C(t) = 20{,}000(0.75)t$, the base is 0.75 and between 0 and 1 and the value of $20,000 represents the initial cost of an automobile that depreciates 25% per year over the course of *t* years.

Similarly, to illustrate exponential growth, in the equation for the value of an investment over time $A(t) - 10{,}000(1.03)^t$, where the base is 1.03 and is greater than 1; and the $10,000 represents the value of an investment when increasing in value by 3% per year for *x* years.

Common Misconceptions:

Some students may believe that factoring and completing the square are isolated techniques within a unit of quadratic equations. Teachers should help students to see the value of these skills in the context of solving higher degree equations and examining different families of functions.

Students may think that the minimum (the vertex) of the graph of $y = (x + 5)^2$ is shifted to the right of the minimum (the vertex) of the graph $y = x^2$ due to the addition sign. Students should explore examples both analytically and graphically to overcome this misconception.

Some students may believe that the minimum of the graph of a quadratic function always occur at the *y*-intercept.

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.B.3
  - Increasing or Decreasing?, Variation 2
  - Ice Cream
  - Profit of a Company
  - Forms of exponential expressions

Domain: Arithmetic with Polynomials and Rational Expressions A.APR

▶ **Cluster:** *Perform arithmetic operations on polynomials.*

Standard: **A.APR.2**

(11) Factor higher degree polynomials; identifying that some polynomials are prime. **(2017)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.

Connections: A.APR, A.SSE

Explanations and Examples:

Understand how this standard relates to A.SSE.3a.

Understand that *a* is a root of a polynomial function if and only if $x - a$ is a factor of the function.

The Remainder theorem says that if a polynomial *p*(*x*) is divided by $x - a$ for some number *a*, then the remainder is the constant $p(a)$. That is, $p(x) = q(x)(x - a) + p(a)$. So if $p(a) = 0\ then\ p(x) = q(x)(x - a)$.

Examples:

1. Let $p(x) = x^3 - 3x^4 + 8x^2 - 9x + 30$. Evaluate $p(-2)$
   What does your answer tell you about the factors of *p*(*x*)?

   Solution: $p(-2) = 0$, so $x + 2$ is a factor of $p(x)$.

2. Consider the polynomial function: $P(x) = x^4 - 3x^3 + ax^2 - 6x + 14$, where *a* is an unknown real number. If $(x - 2)$ is a factor of this polynomial, what is the value of *a*?

   Solution: By the Remainder Theorem, if $(x - 2)$ is a factor of $P(x)$, then $P(2)$ must equal zero. Therefore, $P(2) = 16 - 3 \cdot 8 + a \cdot 4 - 6 \cdot 2 + 14 = 0$. Simplifying, we find that $4a - 6 = 0, and\ a = \frac{3}{2}$.

The purpose of this task is to emphasize the use of the Remainder Theorem as a method for determining structure in polynomial in equations, and in this particular instance, as a replacement for division of polynomials.

One possible solution path is to use polynomial division to divide $P(x)$ by $(x - 2)$ and determine the remainder in terms of *a* and then solve for *a* by setting the remainder equal to zero. However, the division operation becomes unwieldy with the unknown parameter *a* in play. A more straightforward approach is to use the Remainder Theorem.

Instructional Strategies:

As discussed for the previous cluster (**Perform arithmetic operations on polynomials**), polynomials can often be factored. Even though polynomials (i.e., polynomial expressions) can be explored as mathematical objects without consideration of functions, in school mathematics, polynomials are usually taken to define functions. Some equations may include polynomials on one or both sides. The importance here is in distinction between equations that have solutions, and functions that have zeros. Thus, polynomial functions have zeros.

This cluster is about the relationship between the factors of a polynomial, the zeros of the function defined by the polynomial, and the graph of that function. The zeros of a polynomial function are the *x*-intercepts of the graph of the function.

Through some experience with long division of polynomials by $(x - a)$, students get a sense that the quotient is always a polynomial of a polynomial that is one degree less than the degree of the original polynomial, and that the remainder is always a constant. In other words, $p(x) = q(x)(x - a) + r$. Using this equation, students reason that $p(a) = r$. Thus, if $p(a) = 0$, then the remainder $r = 0$, the polynomial $p(x)$ is divisible by $(x - a)$ and $(x - a)$ is a factor of $p(x)$. Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

Whereas, the first standard specifically targets the relationship between factors and zeros of polynomials, the second standard requires more general exploration of polynomial functions: graphically, numerically, verbally and symbolically.

Through experience graphing polynomial functions in factored form, students can interpret the Remainder Theorem in the graph of the polynomial function. Specifically, when $(x - a)$ is a factor of a polynomial $p(x)$, then $p(a) = 0$ and therefore $x = a$ is an *x*-intercept of the graph$y = p(x)$. Conversely, when students notice an *x*-intercept near $x = b$ in the graph of a polynomial function $p(x)$, then the function has a zero near $x = b$, and $p(b)$ is near zero. Zeros are located approximately when reasoning from a graph. Therefore, if $p(b)$ is not exactly zero, then $(x - b)$ is not a factor of $p(x)$.

Students can benefit from exploring the rational root theorem, which can be used to find all of the possible rational roots (i.e., zeros) of a polynomial with integer coefficients. When the goal is to identify all roots of a polynomial, including irrational or complex roots, it is useful to graph the polynomial function to determine the most likely candidates for the roots of the polynomial that are the *x*-intercepts of the graph.

When at least one rational root $x = r$ is identified, the original polynomial can be divided by $x = r$, so that additional roots can be sought in the quotient. Long division will suffice in simple cases. Synthetic division is an abbreviated method that is less prone to error in complicated cases, but Computer Algebra Systems may be helpful in such cases.

Graphs are used to understand the end-behavior of n^{th} degree polynomial functions, to locate roots and to infer the existence of complex roots. By using technology to explore the graphs of many polynomial functions, and describing the shape, end behavior and number of zeros, students can begin to make the following informal observations:

- The graphs of polynomial functions are continuous.
- An n^{th} degree polynomial has at most *n* roots and at most $n - 1$ "changes of direction" (i.e., from increasing to decreasing or vice versa).
- An even-degree polynomial has the same end-behavior in both the positive and negative directions: both heading to positive infinity, or both heading to negative infinity, depending upon the sign of the leading

coefficient.

- An odd-degree polynomial has opposite end-behavior in the positive versus the negative directions, depending upon the sign of the leading coefficient.
- An odd-degree polynomial function must have at least one real root.

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-APR.B.2
  - The Missing Coefficient
  - Zeroes and factorization of a quadratic polynomial I
  - Zeroes and factorization of a quadratic polynomial II
  - Zeroes and factorization of a general polynomial
  - Zeroes and factorization of a non polynomial function

Domain: Arithmetic with Polynomials and Rational Expressions A.APR

▶ **Cluster:** *Perform arithmetic operations on polynomials.*

Standard: A.APR.3

(11) Know and apply the **Remainder Theorem**: For a polynomial *p*(*x*) and a number *c*, the remainder on division by $(x - c)$ is $p(c)$, so $p(c) = 0$ if and only if $(x - c)$ is a factor of $p(x)$. **(A.APR.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically. .
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: A.APR.1-3, A.REI.5, F.IF.7

Explanations and Examples:

Identify the multiplicity of the zeroes of a factored polynomial and explain how the multiplicity of the zeroes provides a clue as to how the graph will behave when it approaches and leaves the *x*-intercept.

Sketch a rough graph using the zeroes of a polynomial and other easily identifiable points such as the *y*-intercept.

Graphing calculators or programs can be used to generate graphs of polynomial functions.

Example:

- Factor the expression $x^3 + 4x^2 - 59x - 126$ and explain how your answer can be used to solve the equation $x^3 + 4x^2 - 59x - 126 = 0$.

  Explain why the solutions to this equation are the same as the *x*-intercepts of the graph of the function
  $$f(x) = x^3 + 4x^2 - 59x - 126$$

Instructional Strategies:

Graphs are used to understand the end-behavior of n^{th} degree polynomial functions, to locate roots and to infer the existence of complex roots. By using technology to explore the graphs of many polynomial functions, and describing the shape, end behavior and number of zeros, students can begin to make the following informal observations:

- The graphs of polynomial functions are continuous.
- An n^{th} degree polynomial has at most n roots and at most $n-1$ "changes of direction" (i.e., from increasing to decreasing or vice versa).
- An even-degree polynomial has the same end-behavior in both the positive and negative directions: both heading to positive infinity, or both heading to negative infinity, depending upon the sign of the leading coefficient.
- An odd-degree polynomial has opposite end-behavior in the positive versus the negative directions, depending upon the sign of the leading coefficient.
- An odd-degree polynomial function must have at least one real root.

Resources/Tools:

Khan Academy Remainder Theorem

Domain: Arithmetic with Polynomials with Rational Expressions (A.APR)

● **Cluster:** *Use polynomial identities to solve problems.*

Standard: **A.APR.4**

(9/10/11) Generate polynomial identities from a pattern. *For example, difference of squares, perfect square trinomials, (emphasize sum and difference of cubes in grade* ***11****).* **(A.APR.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: A.APR.1, A.SSE.1-2

Explanations and Examples:

In Grade 6, students began using the properties of operations to rewrite expressions in equivalent forms. When two expressions are equivalent, an equation relating the two is called an *identity* because it is true for all values of the variables. This cluster is an opportunity to highlight polynomial identities that are commonly used in solving problems. To learn these identities, students need experience using them to solve problems.

This standard is both a 9/10 standard and an 11 standard because it is done when factoring is introduced with polynomials (which might be 9th for a traditionally sequenced Algebra 1 or 10th, course 2 for an Integrated Math sequence).In 9/10, students could explore difference of two squares, perfect square binomials, and others (list might not be complete):

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $(x-y)^2 = x^2 - 2xy + y^2$
3. $(x+y)(x-y) = x^2 - y^2$
4. $(x+a)(x+b) = x^2 + x(a+b) + ab$

Eleventh grade will address polynomial identities from cubes.

Examples:

Use the distributive law to explain why $x^2 - y^2 = (x-y)(x+y)$ for any two numbers *x* and *y*.

Derive the identity $(x-y)^2 = x^2 - 2xy + y^2$ from $(x+y)^2 = x^2 + 2xy + y^2$ by replacing y by $-y$.

Use an identity to explain the pattern

$2^2 - 1^2 = 3$
$3^2 - 2^2 = 5$
$4^2 - 3^2 = 7$
$5^2 - 4^2 = 9$

Solution: $(n+1)^2 - n^2 = 2n + 1$ for any whole number n.

Instructional Strategies:

Students have been taught the rules for these identities and sometimes forget that the rules simply provide a shortcut. They forget (or don't realize) that they can factor the problems using the same procedures they use with other quadratics.

Students should be able to explain any of these identities. Furthermore, they should develop sufficient fluency that they can recognize expressions of the form on either side of these identities in order to replace that expression with an equivalent expression in the form of the other side of the identity.

With identities such as these, students can discover and explain facts about the number system. For example, in the multiplication table, the perfect squares appear on the diagonal. Diagonally, next to the perfect squares are "near squares," which are one less than the perfect square. Why?

Why is the sum of consecutive odd numbers beginning with 1 always a perfect square?

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-APR.C.4
  - Trina'a Triangles

Domain: Creating Equations A.CED

Cluster: *Create equations that describe numbers or relationships.*

Standard: **A.CED.1 ★ (all)**

Apply and extend previous understanding to create equations and inequalities in one variable and use them to solve problems. **(A.CED.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

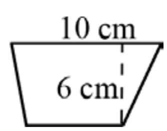
Connections: Modeling and Functions

Explanations and Examples:

Every year, in every course, students should be creating equations and inequalities. There isn't a time when we say, "The students have mastered it! They no longer need to develop this skill." For that reason, this standard is selected as an ALL standard. Algebra 1 focuses on creating equation and inequalities that are linear, quadratic, or exponential. Algebra 2 continues to increase in sophistication with linear, quadratic, and exponential but adds new function families such as rational, square root, logarithmic, and polynomial. Geometry reinforces algebraic skills while learning geometric properties by asking students to solve geometry problems using algebra skills.

Examples:

- Given that the following trapezoid has area $54\ cm^2$, set up an equation to find the length of the unknown base, and solve the equation.



- Lava coming from the eruption of a volcano follows a parabolic path. The height *h* in feet of a piece of lava *t* seconds after it is ejected from the volcano is given by $h(t) = -16t^2 + 64t + 936$. $h(t) = -t^2 + 16t + 936$. After how many seconds does the lava reach its maximum height of 1000 feet?

- The value of an investment over time is given by the equation $A(t) = 10{,}000(1.03)^t$. What does each part of the equation represent?

  Solution: The $10,000 represents the initial value of the investment. The 1.03 means that the investment will grow exponentially at a rate of 3% per year for *t* years.

- You bought a car at a cost of \$20,000. Each year that you own the car the value of the car will decrease at a rate of 25%. Write an equation that can be used to find the value of the car after *t* years.

  Solution: $C(t) = \$20{,}000(0.75)^t$. The base is $1 - 0.25 = 0.75$ and is between 0 and 1, representing exponential decay. The value of \$20,000 represents the initial cost of the car.

- Suppose a friend tells you she paid a total of \$16,368 for a car, and you'd like to know the car's list price (the price before taxes) so that you can compare prices at various dealers. Find the list price of the car if your friend bought the car in:
  a) Arizona, where the sales tax is 5.6%.
  b) New York, where the sales tax is 8.25%.
  c) A state where the sales tax is *r*.

  Solution:
  a) If *p* is the list price in dollars then the tax on the purchase is 0.056*p*. The total amount paid is $p + 0.056p$, so

$$p + 0.056p = 16{,}368$$

$$(1 + 0.056)p = 16{,}368$$

$$p = \frac{16{,}368}{1 + 0.056} = \$15{,}500$$

$$p = \$15{,}500$$

  b) The total amount paid is $p + 0.0825p$ so

$$p + 0.0825p = 16{,}368$$

$$(1 + 0.0825)p = 16{,}368$$

$$p = \frac{16{,}368}{1 + 0.0825} =$$

$$p = \$15{,}120.55$$

  c) The total amount paid is $p + rp$ so

$$p + rp = 16{,}368$$

$$(1 + r)p = 16{,}368$$

$$\frac{16{,}368}{1 + r}$$

$p = \frac{16{,}368}{1+r}$ dollars.

Instructional Strategies:

Reading and comprehension strategies such as highlighting and annotating will help students make meaning from the problem. It is also important that the students understand the context and can visualize what is happening in the problem. These general strategies are well-known and can be effective literacy strategies that support writing equations.

A math specific strategy is to try and identify a general structure to math problems what can be applied to a variety of different situations. For example, there are many types of situations that fit into a part/part/whole or start/change/unknown situation in Algebra 1. Mixture problems is another general problem type with a fairly consistent problem structure. Identifying additional general problem types will help students see the larger structure present in Algebra problems. The appendix at the back of the Standards and at the back of each flip book provides a general structure that is useful scaffolding for students. Many situations will fit within these computation situations and can help students see the pattern across a wide array of problems.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.1
  - Planes and wheat
  - Paying the rent
  - Buying a car
  - Sum of angles in a polygon
- A-CED.A.2
  - Throwing a Ball

Common Misconceptions:

Students may believe that equations of linear, quadratic and other functions are abstract and exist only "in a math book," without seeing the usefulness of these functions as modeling real-world phenomena.

Additionally, they believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.

Students may interchange slope and y-intercept when creating equations. For example, a taxi cab costs $4 for a dropped flag and charges $2 per mile. Students may fail to see that $2 is a rate of change and is slope while the $4 is the starting cost and incorrectly write the equation as $y = 4x + 2$ instead of $y = 2x + 4$.

Given a graph of a line, students use the x-intercept for b instead of the y-intercept.

Given a graph, students incorrectly compute slope as run over rise rather than rise over run. For example, they will compute slope with the change in x over the change in y.

Students do not know when to include the "or equal to" bar when translating the graph of an inequality.

Students do not correctly identify whether a situation should be represented by a linear, quadratic, or exponential function.

Students often do not understand what the variables represent. For example, if the height h in feet of a piece of lava t seconds after it is ejected from a volcano is given by $h(t) = -16t^2 + 64t + 936$ and the student is asked to find the time it takes for the piece of lava to hit the ground, the student will have difficulties understanding that $h = 0$ at the ground and that they need to solve for t.

Domain: Creating Equations A.CED

Cluster: *Create equations that describe numbers or relationships.*

Standard: A.CED.2 ★ (all)

Apply and extend previous understanding to create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **(A.CED.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

Connections: See A.CED.1

Explanations and Examples:

See A.CED.1

Examples:

- The formula for the surface area of a cylinder is given by $A = 2\pi rh + 2\pi r^2$, where *r* represents the radius of the circular cross-section of the cylinder and *h* represents the height. Choose a fixed value for *h* and graph *V* vs. *r*. Then pick a fixed value for *r* and graph *V* vs. *h*. Compare the graphs.

  What is the appropriate domain for *r* and *h*? Be sure to label your graphs and use an appropriate scale.
- Gold is alloyed with different metals to make it hard enough to be used in jewelry. The amount of gold present in a gold ally is measured in 24ths called karats. 24-karat gold is 100% gold. Similarly, 18-karat gold is 75% gold.

  How many ounces of 18-karat gold should be added to an amount of 12-karat gold to make 4 ounces of 14-karat gold? Graph equations on coordinate axes with labels and scales.

- A metal alloy is 25% copper. How much of each alloy should be used to make 1000 grams of a metal alloy that is 45% copper?

- Find a formula for the volume of a single-scoop ice cream cone in terms of the radius and height of the cone. Rewrite your formula to express the height in terms of the radius and volume. Graph the height as a function of radius when the volume is held constant.

- David compares the sizes and costs of photo books offered at an online store. The table below shows the cost for each size photo book.

Book Size	Base Price	Cost for Each Additional Page
7-in. by 9-in.	\$20	\$1.00
8-in. by 11-in.	\$25	\$1.00
12-in. by 12-in.	\$45	\$1.50

1. Write an equation to represent the relationship between the cost, *y*, in dollars, and the number of pages, *x*, for each book size. Be sure to place each equation next to the appropriate book size. Assume that *x* is at least 20 pages.

Book Size	Equation
7-in. by 9-in.	
8-in. by 11-in.	
12-in. by 12-in.	

2. What is the cost of a 12-in. by 12-in. book with 28 pages?

3. How many pages are in an 8-in. by 11-in. book that costs \$49?

Solution:

1. 7-in. by 9-in. $y = x$
   8-in. by 11-in. $y = x + 5$
   12-in. by 12-in. $y = 1.50x + 15$
2. \$57
3. 44 pages

Instructional Strategies: See A.CED.1

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.2
  - Clea on an escalator
  - Silver rectangle

Domain: Creating Equations A.CED

Cluster: *Create equations that describe numbers or relationships.*

Standard: A.CED.3 ★ (all)

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★ **(A.CED.3)**

Suggested Standards for Mathematical Practice (MP):

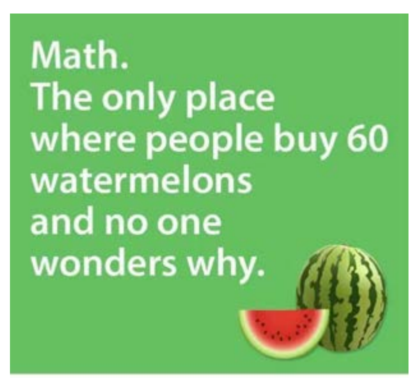
- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

Connections: See A.CED.1

Explanations and Examples:

Part 1: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities.
A constraint is often thought about (by math teachers) as a linear programing problem. While that type of problem certainly fits within this standard, it would not be categorized as an ALL standard if that were the entire scope. The word "constrained" can apply to a wide variety of problem types. For example, if child tickets cost $3 and an adult's ticket cost $5, we can purchase tickets in any quantity- unconstrained by additional information. However if additional information were provided about how much money could be spent, then the solution set would be *constrained* to the values that add to no more than that amount. This situation also has a constrained domain; which is constrained to discrete values and that must be between zero and the maximum number of a single variable able to purchase tickets.

Part 2: and interpret solutions as viable or non-viable options in a modeling context.
This part of the standard has frequently been interpreted to address extraneous solutions but the phrase "in a modeling context" should redirect instruction back to thinking about the constraints on a problem, which could include constraints on the solution. For example, in the situation described above, it isn't possible to purchase half a ticket. Therefore, viable solutions would be whole numbers. Negative numbers are also not possible. The viable has broader meaning than simply possible or not possible. Is the solution "viable" or feasible? If we were working a rate problem, it might be possible for the speed of a motorcycle to be 350 mph according to all the constraints given in the situation but is it a viable solution? Thinking about the asymptote on an exponential function, as the value gets infinitesimally close to zero, does it remain viable?



desmos.com

As seen in many of the ALL standards, there is not a simple procedure or algorithm that will always produce the correct answer. A classroom discussion about the variety of factors that might constrain the equation or solution provides a rich opportunity for students to learn how to think about the world mathematically, in a truly real life situation. The statement above, posted frequently on social media, reflects the memes that result from a lack of attention to viability in a modeling context.

Examples:

A club is selling hats and jackets as a fundraiser. Their budget is $1500 and they want to order at least 250 items. They must buy at least as many hats as they buy jackets. Each hat costs $5 and each jacket costs $8.

- Write a system of inequalities to represent the situation.
- Graph the inequalities.
- If the club buys 150 hats and 100 jackets, will the conditions be satisfied?
- What is the maximum number of jackets they can buy and still meet the conditions?

Represent inequalities describing nutritional and cost constraints o combinations of different foods.

The coffee variety *Arabica* yields about 750 kg of coffee beans per hectare, while *Robusta* yields about 1200 kg per hectare. Suppose that a plantation has *a* hectares of *Arabica* and *r* hectares of *Robusta*.

a) Write an equation relating *a* and *r* if the plantation yields 1,000,000 kg of coffee.
b) On August 14, 2003, the world market price of coffee was$1.42 per kg of *Arabica* and $0.73 per kg of *Robusta*. Write an equation relating *a* and *r* if the plantation produces coffee worth $1,000,000.

This task is designed to make students think about the meaning of the quantities presented in the context and choose which ones are appropriate for the two different constraints presented. The purpose of the task is to have students *generate* the constraint equations for each part (though the problem and not to have students *solve* said equations. If desired, instructors could also use this task to touch on such solutions by finding and interpreting solutions to the system of equations created in parts (a) and (b).

Solution:

a) We see that *a* hectares of *Arabica* will yield 750*a* kg of beans, and that *r* hectares of *Robusta* will yield 1200r kg of beans. So the constraint equation is

$$750a + 1{,}200r = 1{,}000{,}000.$$

b) We know that a hectares of *Arabica* yield 750*a* kg of beans worth $1.42/kg for a total dollar value of $1.42(750a) = 1065a$. Likewise, *r* hectares of *Robusta* yield 1200*r* kg of beans worth $0.73/kg for a total dollar value of $0.73(1200r) = 876r$. So the equation governing the possible values of *a* and *r* coming from the total market value of the coffee is

$$1065a + 876r = 1{,}000{,}000$$

Instructional Strategies:

While this standard represents an exciting opportunity to open the math world up to big, wide, real world; that ambiguity can be intimidating and/or distracting. Students will enjoy thinking about ways the problem will be constrained by the context of the problem. Teaching students Talk Moves such as revoicing or adding on can provide some structure to guide the conversation. Another strategy is to restrict the types of constraints the students can discuss either by number or type. Finally, as with each of other ALL standards, the goal is to *enhance* student understanding of functions and how quantities vary together.. Not every solution is possible. Infinite possibilities doesn't mean infinite viable solutions. Do not let the discussion drag you down rabbit hole or cause confusion.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.3
  - Dimes and Quarters
  - Writing Constraints
  - Growing Coffee
  - How Much Folate
  - Bernardo and Sylvia Play a Game

Domain: Creating Equations A.CED

Cluster: *Create equations that describe numbers or relationships.*

Standard: **A.CED.4 ★ (all)**

Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance R.*★ **(A.CED.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: A.SSE.2, A.REI.2

Explanations and Examples:

Rearranging formulas is a critical skill in many applications such as computer programming or spreadsheet formulas. But, often, in the process of solving for the quantity of interest, student misunderstandings about the steps for solving equations will come to the forefront. Mistakes that they wouldn't make when solving other equations will begin to appear. The algebraic reasoning required to correct the flaw causes significant struggle, even when the same problem with numbers would have an immediate and correct answer. Usiskin provided some insight into this area of student difficulty- quoted below.

Quoting from Teaching Mathematics in Grades 6-12:Developing Research Based Instructional Practices:

Figure 8.2 Examples to illustrate the complexity of literal symbols (Usiskin, 1988, p. 40).

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot (\frac{1}{n})$
5. $y = kx$

"Usiskin (1988) provided a poignant example of the complexity involved in understanding the meanings of literal symbols by asking readers to consider the equations shown in Figure 8.2. Although all five symbol strings in Figure 8.2 are equations, each uses literal symbols in different ways. In equation 2, for example, x is often referred to as an unknown whose value can be determined by dividing each side by five. In equation 5, however, we usually think of x as being free to vary and take on numerous different values, making x feel more like a variable than a specific unknown. In equation 5, k is often thought of as a constant that specifies the slope of a line. Therefore, this set of five equations illustrates at least three different ways literal symbols can be used in algebra— to represent unknowns, variables, and constants. In addition, the equations themselves can be used for different purposes. To illustrate this, observe that equation 1 is commonly referred to as a formula, equation 3 as an identity, and equation 4 as a property."

Recognizing that students struggle with the meaning of the variable in a literal equation does not change the reasoning required. It does indicate that shifting back and forth between different meanings for a variable can be confusing for students and addressing this confusion directly might help some students.

So one goal for this standard is for students to become comfortable with different uses of the variable in the equation and to surface any flaws with their algebraic reasoning. Another goal is for students to develop the ability to think ahead to their goal and plan a path to get there. For example, if the variable of interest appears in several different terms, with different exponents, then factoring or completing the square might be required. If the variable of interest is in the exponent, then logarithms might be the strategy. As has been stated before, ALL standards require practice and the development of sophistication over time. With each new function family, an effort to should be made to solve a formula for a given variable.

Examples:

1. Solve for *h*: $A = \frac{1}{2}bh$

   An example like the problem above can highlight how students will move the $\frac{1}{2}$. Dividing by $\frac{1}{2}$ would work if the student remembers how to divide fractions. A more robust solution path would be to multiply by the multiplicative inverse of $\frac{1}{2}$ or 2. A student who writes the answer $h = \frac{A}{\frac{1}{2}b}$ is correctly reasoning about equations but does not see that the fraction divided by a fraction is unnecessary. Even more concerning is the student whose answer is $h = \frac{A}{2b}$ because they incorrectly divided by $\frac{1}{2}$.

2. Solve $A = P + Prt$ for *r.*

   When working this problem, some students arrive at the correct solution $r = \frac{A-P}{Pt}$ but will go one step too far and "cancel" the *P*. These students struggle with the reasoning necessary in A.SSE.2, seeing part of an expression as a single entity. The numerator cannot be separated in this way and should be viewed as a whole piece.

3. Solve $A = P + Prt$ for *P.*

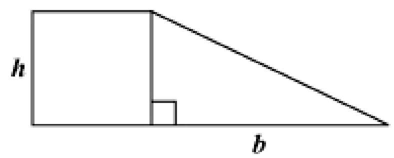
   This kind of problem is particularly challenging for students because they do not see how connection between factoring and distributing. With numbers, they understand that a problem like $2(x + y)$ will "give" the *x* and *y* a two but will not necessarily observe that *2x+2y* can be "undone" by factoring the two. A problem like the one above can highlight that students have missed this connection.

An Illustrated Math Task:

4. A bacteria population *P* is modeled by the equation $P = P_0 10^{kt}$ where time *t* is measured in hours, *k* is a positive constant, and P_0 is the bacteria population at the beginning of the experiment. Rewrite this equation to find *t* in terms of *P.*

   This equation will assess if students understand *when* to use logarithms.

   The figure below is made up of a square with height, *h* units, and a right triangle with height, *h* units, and base length, *b* units.



   The area of this figure is 80 square units.

   Write an equation that solves for the height, *h*, in terms of *b*.
   Show all work necessary to justify your answer.

   Sample Response:

$$h^2 + \frac{1}{2}bh = 80$$

$$h^2 + \frac{1}{2}bh + \frac{1}{16}b^2 = 80 + \frac{1}{16}b^2$$

$$\left(h + \frac{1}{4}b\right)^2 = 80 + \frac{1}{16}b^2$$

$$h + \frac{1}{4}b = \sqrt{80 + \frac{1}{16}b^2}$$

$$h = \sqrt{80 + \frac{1}{16}b^2} - \frac{1}{4}b$$

Instructional Strategies:

Substituting numbers in for the variables and solving it side by side with the literal equation can help scaffold the abstract thinking required.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.4
  - ~ Equations and Formulas

Domain: Reasoning with Equations and Inequalities A.REI

Cluster: *Understand solving equations as a process of reasoning and explain the reasoning.*

Standard: **A.REI.1 (all)**

Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. **(A.REI.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.

Connections: Algebra standards

Explanations and Examples:

In Algebra 1 students should focus on and master A.REI.1 for linear equations and be able to extend and apply their reasoning to other types of equations in future courses. In Algebra 2, extend to simple rational and radical equations.

Assuming an equation has a solution, construct a convincing argument that justifies each step in the solution process. Justifications may include the associative, commutative, and division properties, combining like terms, multiplication by 1, etc.

Properties of operations can be used to change expressions on either side of the equation to equivalent expressions. In addition, adding the same term to both sides of an equation or multiplying both sides by a non-zero constant produces an equation with the same solutions. Other operations, such as squaring both sides, may produce equations that have extraneous solutions.

Examples:

1. Explain why the equation $\frac{x}{2}+\frac{7}{3}=5$ has the same solutions as the equation $3x+14=30$.
   Does this mean that $\frac{x}{2}+\frac{7}{3}$ is equal to $3x+14$?

2. Show that $x=2$ and $x=-3$ are solutions to the equation $x^2+x=6$.
   Write the equation in a form that shows these are the only solutions, explaining each step in your reasoning.

3. Transform $2x-5=7$ to $2x=12$ and tell what property of equality was used.
   Solution:

$$2x-5=7$$
$$2x-5+5=7+5 \quad \text{Addition property of equality .}$$
$$2x=12$$

Instructional Strategies:

Challenge students to justify each step of solving an equation. Transforming $2x - 5 = 7$ to $2x = 12$ is possible because $5 = 5$, so adding the same quantity to both sides of an equation makes the resulting equation true as well. Each step of solving an equation can be defended, much like providing evidence for steps of a geometric proof.

Provide examples for how the same equation might be solved in a variety of ways as long as equivalent quantities are added or subtracted to both sides of the equation, the order of steps taken will not matter.

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ -\quad 2 &=& -2 \\ \hline 3n &=& n - 12 \\ -n &=& -n \\ \hline 2n &=& -12 \\ n &=& -6 \end{array} \qquad \textbf{OR} \qquad \begin{array}{rcl} 3n + 2 &=& n - 10 \\ +\quad 10 &=& +10 \\ \hline 3n + 12 &=& n \\ -3n &=& -3n \\ \hline 12 &=& -2n \\ n &=& -6 \end{array} \qquad \textbf{OR} \qquad \begin{array}{rcl} 3n + 2 &=& n - 10 \\ -n &=& -n \\ \hline 2n + 2 &=& -10 \\ -2 &=& -2 \\ \hline 2n &=& -12 \\ n &=& -6 \end{array}$$

Connect the idea of adding two equations together as a means of justifying steps of solving a simple equation to the process of solving a system of equations. A system consisting of two linear functions such as $2x + 3y = 8$ and $x - 3y = 1$ can be solved by adding the equations together, and can be justified by exactly the same reason that solving the equation $2x - 4 = 5$ can begin by adding the equation $4 = 4$.

Investigate the solutions to equations such as $3 = x + \sqrt{2x - 3}$. By graphing the two functions, $y = 3$ and $y = x + \sqrt{2x - 3}$ students can visualize that graphs of the functions only intersect at one point. However, subtracting $x = x$ from the original equation yields $3 - x = \sqrt{2x - 3}$ which when both sides are squared produces a quadratic equation that has two roots $x = 2$ and $x = 6$. Students should recognize that there is only one solution $(x = 2)$ and that $x = 6$ is generated when a quadratic equation results from squaring both sides; $x = 6$ is extraneous to the original equation. Some rational equations, such as $\frac{x}{(x-2)} = \frac{2}{(x-2)} + \frac{5}{x}$ result in extraneous solutions as well.

Begin with simple, one-step equations and require students to write out a justification for each step used to solve the equation.

Ensure that students are proficient with solving simple rational and radical equations that have no extraneous solutions before moving on to equations that result in quadratics and possible solutions that need to be eliminated.

Provide visual examples of radical and rational equations with technology so that students can see the solution as the intersection of two functions and further understand how extraneous solutions do not fit the model.

It is very important that students are able to reason how and why extraneous solutions arise.
Computer software that generates graphs for visually examining solutions to equations, particularly rational and radical. Examples of radical equations that do and do not result in the generation of extraneous solutions should be prepared for exploration.

Common Misconception:

Students may believe that solving an equation such as $3x + 1 = 7$ involves "only removing the 1," failing to realize that the equation $1 = 1$ is being subtracted to produce the next step.

Additionally, students may believe that all solutions to radical and rational equations are viable, without recognizing that there are times when extraneous solutions are generated and have to be eliminated.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Mathematics Assessment Project:

- "Building and Solving Equations 2"

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.A
  - ~ Same Solutions?
  - ~ How does the solution change?

Domain: Reasoning with Equations and Inequalities A.REI

Cluster: *Solve equations and Inequalities in one variable.*

Standard: A.REI.2 (all)

Apply and extend previous understanding to solve equations, inequalities, and compound inequalities in one variable, including **literal equations** and inequalities. **(A.REI.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: A.CED.4

Explanations and Examples:

See A.CED.4

Examples:

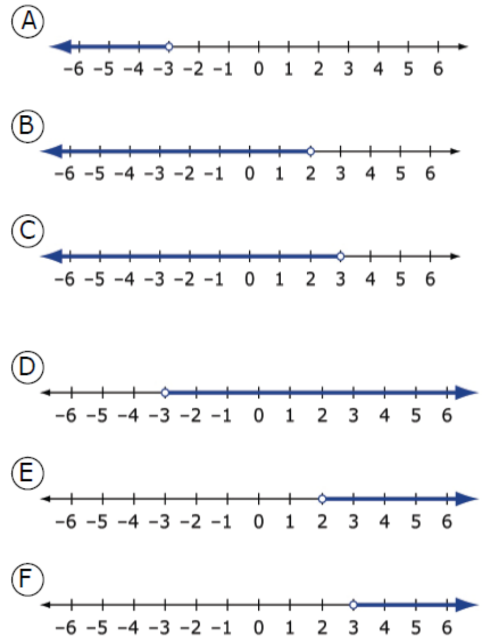
1. Solve for the variable:
   - $\frac{7}{3}y - 8 = 111$
   - $3x > 9$
   - $ax + 7 = 12$
   - $\frac{3+x}{7} = \frac{x-9}{4}$
   - Solve for x: $\frac{2}{3}x + 9 < 18$

2. Match each inequality in items 1 – 3 with the number line in items A – F that represent the solution to the inequality.

1 $-4x < -12$

2 $2(x + 2) < 8$

3 $5 - 2x < 2 - x$



Solutions: **1.** F **2.** B **3.** F

Instructional Strategies:

There are two major reasons for discussing the topic of inequalities along with equations: First, there are analogies between solving equations and inequalities that help students understand them both**.** Second, the applications that lead to equations almost always lead in the same way to inequalities.

In grades 6-8, students solve and graph linear equations and inequalities. Graphing experience with inequalities is limited to graphing on a number line diagram. Despite this work, some students will still need more practice to be proficient. It may be beneficial to remind students of the most common solving techniques, such as converting fractions from one form to another, removing parentheses in the sentences, or multiplying both sides of an equation or inequality by the common denominator of the fractions. Students must be aware of what it means to check an inequality's solution. The substitution of the end points of the solution set in the original inequality should give equality regardless of the presence or the absence of an equal sign in the original sentence**.** The substitution of any value from the rest of the solution set should give a correct inequality.

Careful selection of examples and exercises is needed to provide students with meaningful review and to introduce other important concepts, such as the use of properties and applications of solving linear equations and inequalities. Stress the idea that the application of properties is also appropriate when working with equations or inequalities that include more than one variable, fractions and decimals. Regardless of the type of numbers or variables in the equation or inequality, students have to examine the validity of each step in the solution process.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(B + b)$ when solving for *b* is similar to solving an equation with one variable. Provide students with an opportunity to abstract from particular numbers and apply the same kind of manipulations to formulas as they did to equations. One of the purposes of doing abstraction is to learn how to evaluate the formulas in easier ways and use the techniques across mathematics and science.

Draw students' attention to equations containing variables with subscripts. The same variables with different subscripts (e.g., $x_1\, and\ x_2$) should be viewed as different variables that cannot be combined as like terms. A variable with a variable subscript, such as a_n, must be treated as a single variable – the *n*th term, where variables *a* and *n* have different meaning.

Common Misconceptions:

Some students may believe that for equations containing fractions only on one side, it requires "clearing fractions" (the use of multiplication) only on that side of the equation. To address this misconception, start by demonstrating the solution methods for equations similar to $\frac{1}{4}x + \frac{1}{5}x + \frac{1}{6}x + 46 = x$ and stress that the Multiplication Property of Equality is applied to both sides, which are multiplied by 60.

Students may confuse the rule of changing a sign of an inequality when multiplying or dividing by a negative number with changing the sign of an inequality when one or two sides of the inequality become negative (for ex., $3x > -15\ or\ x < -5$).

Some students may believe that subscripts can be combined as $b_1 + b_2 = b_3$ and the sum of different variables *d* and D is $2D(d + D = 2D)$.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.A.1
  - Reasoning with linear inequalities
- A.REI.B
  - Integer Solutions to Inequality

Domain: Reasoning with Equations and Inequalities (A.REI)

▶ **Cluster:** *Solve equations and inequalities in one variable.*

Standard: **A.REI.3**

Solve equations in one variable and give examples showing how extraneous solutions may arise.

A.REI.3a. **(9/10/11)** Solve rational, absolute value and square root equations. **(A.REI.2)**

(9/10) Limited to simple equations such as, $2\sqrt{x-3}+8=16$, $\frac{x+3}{2x-1}=5, x \neq \frac{1}{2}$.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.

Connections: A.SSE.2, A.CED, A.REI.1, A.REI.2

Explanations and Examples:

Extending multiple standards about solving equations (A.CED, A.REI.1 & 2), this standard specifies that students in 9^{th} or 10^{th} grade will solve equations that extend beyond linear and quadratic. Students will solve simple rational, absolute value, and square root function during the first two years of high school and during 11^{th} grade, these types of equations will become more complicated.

Extraneous solutions arise when students reverse a step that isn't reversible, such as taking the square root, or if the solution is not viable in the context. To avoid extraneous solutions, students must check the solutions when the problem is a type where this is a risk.

Examples:

1. Solve for x:
   - $\sqrt{x+2}=5$
   - $\frac{7}{8}\sqrt{2x-5}=21$
   - $\frac{x+2}{x+3}=2$
   - $\sqrt{3x-7}=-4$
2. Solve the following two equations by isolating the radical on one side and squaring both sides:

   i. $\sqrt{2x+1}-5=-2$

   ii. $\sqrt{2x+1}+5=2$
3. If we raise both sides of an equation a power, we sometimes obtain an equation which has more solutions than the original one. (Sometimes the extra solutions are called extraneous solutions.)

4. Which of the following equations result in extraneous solutions when you raise both sides to the indicated power? Explain.

   i. $\sqrt{x} = 5$, square both sides

   ii. $\sqrt{x} = -5$, square both sides

   iii. $\sqrt[3]{x} = 5$, cube both sides

   iv. $\sqrt[3]{x} = -5$, cube both sides

5. Create a square root equation similar to the one in problem 1 that has an extraneous solution.

   Show the algebraic steps you would follow to look for a solution, and indicate where the extraneous solution arises.

Selected Solutions:

2. i.

$$\begin{aligned}\sqrt{2x+1}-5&=-2\\ \left(\sqrt{2x+1}\right)^2&=(3)^2\\ 2x+1&=9\\ 2x&=8\\ x&=4\end{aligned}$$

Checking:

$$\begin{aligned}\sqrt{2\cdot 4+1}-5&=-2\\ \sqrt{9}-5&=-2\\ 3-5&=-2\\ -2&=-2\end{aligned}$$

So, *x* = 4 is the solution to the equation.

ii.

$$\begin{aligned}\sqrt{2x+1}+5&=2\\ \left(\sqrt{2x+1}\right)^2&=(-3)^2\\ 2x+1&=9\\ 2x&=8\\ x&=4\end{aligned}$$

Checking:

$$\begin{aligned}\sqrt{2\cdot 4+1}+5&=2\\ \sqrt{9}+5&=2\\ 3+5&=2\\ 8&\neq 2\end{aligned}$$

So this equation has no solution.

4. The only one of the equations that produces an extraneous solution is $\sqrt{x} = -5$

The square root symbol (like all even roots) is defined to be the positive square root, so a positive root can never be equal to a negative number. Squaring both sides of the equation will make that discrepancy disappear; the square of a positive number is positive but so is the square of a negative number, so we'll end up with a solution to the new equation even though there was no solution to the original equation.

This is not the case with odd roots - a cube root of a positive number is positive, and a cube root of a negative number is negative. When we cub

Instructional Strategies: See A.REI.1

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.A.2
  - Radical Equations
  - Basketball

Domain: Reasoning with Equations and Inequalities A.REI

▶ **Cluster:** *Solve equations and inequalities in one variable.*

Standard: A.REI.4

(11) Solve radical and rational exponent equations and inequalities in one variable, and give examples showing how extraneous solutions may arise. **(A.REI.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.

Connections: A.REI.9-10

Explanations and Examples:

Extending multiple standards about solving equations (A.CED, A.REI.1 & 2), this standard specifies that students in 9th or 10th grade will solve equations that extend beyond linear and quadratic. Students will solve simple rational, absolute value, and square root function during the first two years of high school and during 11th grade, these types of equations will become more complicated.

Extraneous solutions arise when students reverse a step that isn't reversible, such as taking the square root, or if the solution is not viable in the context. To avoid extraneous solutions, students must check the solutions when the problem is a type where this is a risk.

Examples:

1. Solve for x:
   - $\sqrt{x+2} = 5$
   - $\frac{7}{8}\sqrt{2x-5} = 21$
   - $\frac{x+2}{x+3} = 2$
   - $\sqrt{3x-7} = -4$

2. Solve the following two equations by isolating the radical on one side and squaring both sides:

   i. $\sqrt{2x+1} - 5 = -2$

   ii. $\sqrt{2x+1} + 5 = 2$

3. If we raise both sides of an equation a power, we sometimes obtain an equation which has more solutions than the original one. (Sometimes the extra solutions are called extraneous solutions.)

4. Which of the following equations result in extraneous solutions when you raise both sides to the indicated power? Explain.

   i. $\sqrt{x} = 5$, square both sides

   ii. $\sqrt{x} = -5$, square both sides

   iii. $\sqrt[3]{x} = 5$, cube both sides

   iv. $\sqrt[3]{x} = -5$, cube both sides

5. Create a square root equation similar to the one in part (a) that has an extraneous solution.

   Show the algebraic steps you would follow to look for a solution, and indicate where the extraneous solution arises.

Selected Solutions:

3. i.

$$\begin{aligned} \sqrt{2x+1} - 5 &= -2 \\ \left(\sqrt{2x+1}\right)^2 &= (3)^2 \\ 2x+1 &= 9 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

Checking:

$$\begin{aligned} \sqrt{2 \cdot 4 + 1} - 5 &= -2 \\ \sqrt{9} - 5 &= -2 \\ 3 - 5 &= -2 \\ -2 &= -2 \end{aligned}$$

So, $x = 4$ is the solution to the equation.

ii.

$$\begin{aligned} \sqrt{2x+1} + 5 &= 2 \\ \left(\sqrt{2x+1}\right)^2 &= (-3)^2 \\ 2x+1 &= 9 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

Checking:

$$\begin{aligned} \sqrt{2 \cdot 4 + 1} + 5 &= 2 \\ \sqrt{9} + 5 &= 2 \\ 3 + 5 &= 2 \\ 8 &\neq 2 \end{aligned}$$

So this equation has no solution.

5. The only one of the equations that produces an extraneous solution is $\sqrt{x} = -5$

The square root symbol (like all even roots) is defined to be the positive square root, so a positive root can never be equal to a negative number. Squaring both sides of the equation will make that discrepancy disappear; the square of a positive number is positive but so is the square of a negative number, so we'll end up with a solution to the new equation even though there was no solution to the original equation.

This is not the case with odd roots - a cube root of a positive number is positive, and a cube root of a negative number is negative. When we cube both sides of the last equation, the negative remains, and we end up with a true solution to the equation.

Instructional Strategies: See A.REI.1

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.A.2
  - Radical Equations
  - Basketball

Domain: Reasoning with Equations and Inequalities A.REI

- **Cluster:** *Solve equations and inequalities in one variable.*

Standard: A.REI.5

A.REI.5. Solve quadratic equations and inequalities

A.REI.5b. **(11)** Solve quadratic equations with complex solutions written in the form $a \pm bi$ for real numbers *a* and *b.* **(A.REI.4b)**

A.REI.5c. **(11)** Use the method of completing the square to transform and solve any quadratic equation in *x* into an equation of the form $(x-p)^2 = q$ that has the same solutions. **(A.REI.4a)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: A.REI.2

Explanations and Examples:

Transform a quadratic equation written in standard form to an equation in vertex form $(x-p)^2 = q$ by completing the square.

Derive the quadratic formula by completing the square on the standard form of a quadratic equation.

Solve quadratic equations in one variable by simple inspection, taking the square root, factoring, and completing the square.

Understand why taking the square root of both sides of an equation yields two solutions.

Use the quadratic formula to solve any quadratic equation, recognizing the formula produces all complex solutions. Write the solutions in the form a$\pm bi$, where *a* and *b* are real numbers.

Explain how complex solutions affect the graph of a quadratic equation.

Students should solve by factoring, completing the square, and using the quadratic formula. The zero product property is used to explain why the factors are set equal to zero. Students should relate the value of the discriminant to the type of root to expect. A natural extension would be to relate the type of solutions to:

Value of Discriminant	Nature of Roots	Nature of Graph
$b^2 - 4ac = 0$	1 real roots	intersects *x*-axis once
$b^2 - 4ac > 0$	2 real roots	intersects *x*-axis twice
$b^2 - 4ac < 0$	2 complex roots	does not intersect *x*-axis

Examples:

1. Are the roots of $2x^2 + 5 = 2x$ real or complex? How many roots does it have? Find all solutions of the equation.

2. What is the nature of the roots of $x^2 + 6x + 10 = 0$? Solve the equation using the quadratic formula and completing the square. How are the two methods related?

3. Projectile motion problems, in which the initial conditions establish one of the solutions as extraneous within the context of the problem.
   a. An object is launched at 14.7 meters per second (m/s) from a 49-meter tall platform. The equation for the object's height *s* at time *t* seconds after launch is $s(t) = -4.9t^2 + 14t + 49$, where *s* is in meters. When does the object strike the ground?

      Solution: $0 = -4.9t^2 + 14.7t + 49$
      $0 = -t^2 - 3t - 10$
      $0 = (t + 2)(t - 5)$

      So the solutions for *t* are *t = 5 or t* = –2, but *t* = –2 does not make sense in the context of this problem and therefore is an extraneous solution.

Instructional Strategies:

Completing the square is usually introduced for several reasons to find the vertex of a parabola whose equation has been expanded; to look at the parabola through the lenses of translations of a "parent" parabola $y = x^2$; and to derive a quadratic formula. Completing the square is a very useful tool that will be used repeatedly by students in many areas of mathematics. Teachers should carefully balance traditional paper-pencil skills of manipulating quadratic expressions and solving quadratic equations along with an analysis of the relationship between parameters of quadratic equations and properties of their graphs.

Start by inspecting equations such as $x^2 = 9$ that has two solutions, 3 and -3. Next, progress to equations such as $x^2 = 9$ by substituting $x - 7$ for x and solving them either by "inspection" or by taking the square root on each side:

$$x - 7 = 3 \quad and \quad x - 7 = -3$$
$$x = 10 \qquad\qquad x = 4$$

Graph both pairs of solutions (-3 and 3, 4 and 10) on the number line and notice that 4 and 10 are 7 units to the right of – 3 and 3. So, the substitution of *x* – 7 for *x* moved the solutions 7 units to the right. Next, graph the function $y = (x - 7)^2 - 9$, pointing out that the *x*-intercepts are 4 and 10, and emphasizing that the graph is the translation of 7 units to the right and 9 units down from the position of the graph of the parent function $y = x^2$ that passes through the origin (0, 0). Generate more equations of the form $y = a(x - h)^2 + k$ and compare their graphs using a graphing technology.

Highlight and compare different approaches to solving the same problem. Use technology to recognize that two different expressions or equations may represent the same relationship. For example, since $x^2 - 10a + 25 = 0$ can be rewritten as $(x - 5)(x - 5) = 0 \; or \; (x - 5)^2 = 0 \; or \; x^2 = 25$, these are all representations of the same equation that has a double solution $x = 5$. Support it by putting all expressions into graphing calculator. Compare their graphs and generate their tables displaying the same output values for each expression.

Guide students in transforming a quadratic equation in standard form, $0 = a^2 - bx + c$, to the vertex form $0 = a(x - h)^2 + k$ by separating your examples into groups with *a* = 1 and *a* ≠ 1 and have students guess the number that needs to be added to the binomials of the type $x^2 + 6x, x^2 - 2x, x^2 + 9x, x^2 - \frac{2}{3}x$ to form complete square of the binomial $(x - m)^2$.

Then generalize the process by showing the expression $(b/2)^2$ that has to be added to the binomial $x^2 - bx$. Completing the square for an expression whose x^2 coefficient is not 1 can be complicated for some students. Present multiple examples of the type $0 = 2x^2 - 5x - 9$ to emphasize the logic behind every step, keeping in mind that the same process will be used to complete the square in the proof of the quadratic formula.

Discourage students from giving a preference to a particular method of solving quadratic equations. Students need experience in analyzing a given problem to choose an appropriate solution method before their computations become burdensome. Point out that the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is a universal tool that can solve any quadratic equation; however, it is not reasonable to use the Quadratic Formula when the quadratic equation is missing either a middle term, *bx*, or a constant term, *c*. When it is missing a constant term, (e.g., $3x^2 - 10x = 0$) a factoring method becomes more efficient. If a middle term is missing (e.g., $2x^2 - 15 = 0$), a square root method is the most appropriate. Introduce the concept of discriminants and their relationship to the number and nature of the roots of quadratic equation.

Offer students examples of a quadratic equation, such as $x^2 + 9 = 0$. Since the graph of the quadratic function $y = x^2 + 9$ is situated above the *x* –axis and opens up, the graph does not have *x*–intercepts and therefore, the quadratic equation does not have real solutions. At this stage introduce students to non-real solutions, such as $x = \pm\sqrt{-9}$ or $x = \pm 3\sqrt{-1}$ and a new number type-imaginary unit *i* that equals $\sqrt{-1}$ Using *i* in place of $\sqrt{-1}$ is a way to present the two solutions of a quadratic equation in the complex numbers form *a* ± *bi*, if *a* and *b* are real numbers and *b* ≠ 0. Have students observe that if a quadratic equation has complex solutions, the solutions always appear in conjugate pairs, in the form *a* + *bi* and *a* – *bi*. Particularly, for the equation $x^2 = -9$, a conjugate pair of solutions are 0 +3*i* and 0 – 3*i*.

Common Misconception:

Some students may think that rewriting equations into various forms (taking square roots, completing the square, using quadratic formula and factoring) are isolated techniques within a unit of quadratic equations. Teachers should help students see the value of these skills in the context of solving higher degree equations and examining different families of functions.

Resources/Tools:

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.B.4.b
    - Braking Distance

Domain: Reasoning with Equations and Inequalities (A.REI)

Cluster: *Understand solving equations as a process of reasoning and explaining the reasoning.*

Standard: A.REI.8 (all)

Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). **(A.REI.10)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: Functions Domain

Explanations and Examples:

Quoting from Teaching Mathematics in Grades 6-12: Developing Research Based Instructional Practices:

"Research suggests that multiple representations of functions are not used to their fullest extent in traditional mathematics instruction. Knuth (2000) noted that traditional algebra instruction emphasizes **producing graphs from symbolic representations** of functions (e.g., the task, "Produce a graph of $y = x^2$"), but generally does not ask students to **reason from graphs back to symbolic representations.** To illustrate the detrimental effects of this practice, Knuth gave high school students tasks in which they were to determine the equation of a given linear graph. In one task, students were asked to determine the value of "?" in $?x + 3y = -6$. They were given a graph of the equation to use. Many of the students who gave a **correct solution** to the task used the **inefficient process** of calculating the slope of the graph provided, determining the y-intercept from the graph, writing the equation in slope-intercept form, and then converting it back to standard form. **Students did not seem to recognize that every point on the graph represented a solution** to $?x + 3y = -6$. If they understood this idea, called the **Cartesian connection,** they likely would have chosen any point (x, y) from the graph to substitute into the equation to determine the value of the "?" symbol. Although students in traditional algebra classes produce tables, graphs, and equations for functions, **the act of producing these representations becomes a rote process devoid of meaning** if problems that prompt them to recognize ideas such as the Cartesian connection are not included.

Students who do not fully grasp the Cartesian connection may also **lack skill in choosing the most efficient representations for solving problems.** Slavit (1998) examined the algebraic problem solving strategies of students in a precalculus course where the instructor emphasized graphical representations. Despite this emphasis, some **students persisted in using equations and symbol manipulation even when it was inefficient to do so.** For example, when given a task requiring a solution to $-0.1x^2 + 3x + 80 = x$, one of the students interviewed first tried to factor. When factoring became difficult, she used the quadratic formula. Although she was prompted by the interviewer to discuss other solution strategies, **approaching the problem graphically never occurred to her.** A graphical approach might involve locating the roots of the parabola $y = -0.1x^2 + 2x + 80$ or determining the intersection point of $y = -0.1x^2 + 3x + 80$ and *y* = *x*. Slavit partially attributed the lack of use of graphical representations to past instruction focusing heavily on symbol manipulation. **When instruction emphasizes only one representational system, some students come to see graphical and symbolic representations as two separate systems of procedures to follow rather than as representations that complement one another.**

When students view **function representations as complementary** to one another, they develop **better understanding of the attributes of functions**. Schwarz and Hershkowitz (1999) found that students who explored multiple representations with technology understood functions more deeply than those who did not. Students using the technology were able to generate a broader range of examples of functions and also understood multiple representations as different descriptions of the same function. The zooming and scaling features of the technology helped students develop better part-whole reasoning about function representations. **The results of the study strongly suggest that technology capable of generating multiple representations of functions should be a foundational part of algebra instruction."**

Groth, Randall E.. Teaching Mathematics in Grades 6 - 12: Developing Research-Based Instructional Practices (p. 225). SAGE Publications. Kindle Edition.

Examples:

1. Which of the following points is on the circle with equation $(x-1)^2+(y+2)^2=5$?

   (a) (1, -2) (b) (2, 2) (c) (3, -1) (d) (3, 4)

2. Graph the equation and determine which of the following points are on the graph of $y=3^x+1$.

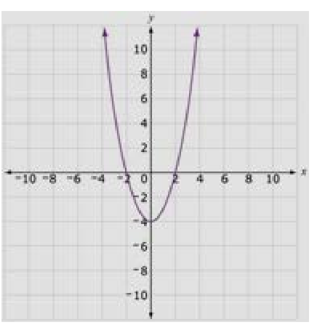
   (a) (2, 7) (b) $\left(-1,\frac{4}{3}\right)$ (c) (2, 10) (d) (0, 1)

$\sqrt{x-4}$

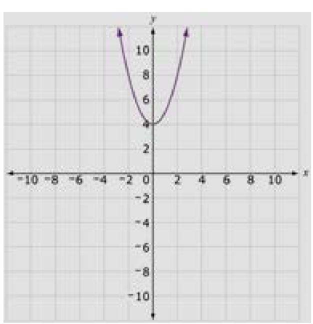
3. Which graph could represent the solution set of $y=\sqrt{x-4}$?

   Solution: B

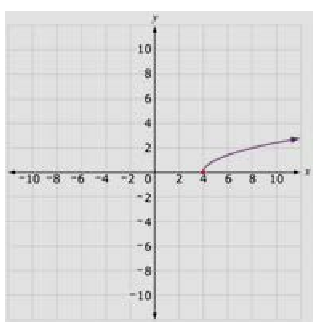
A.



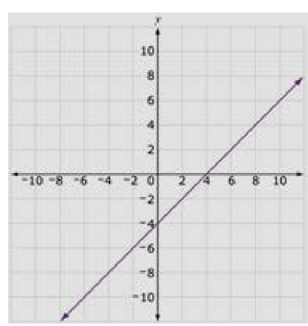
B.



C.



D.



Instructional Strategies:

Beginning with simple, real-world examples, help students to recognize a graph as a set of solutions to an equation. For example, if the equation $y = 6x + 5$ represents the amount of money paid to a babysitter (i.e., $5 for gas to drive to the job and $6/hour to do the work), then every point on the line represents an amount of money paid, given the amount of time worked.

Explore visual ways to solve an equation such as $2x + 3 = x - 7$ by graphing the functions $y = 2x + 3$ and $y = x - 7$. Students should recognize that the intersection point of the lines is at (-10, -17). They should be able to verbalize that the intersection point means that when *x* = -10 is substituted into both sides of the equation, each side simplifies to a value of -17. Therefore, -10 is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear or both.

Using technology, have students graph a function and use the trace function to move the cursor along the curve. Discuss the meaning of the ordered pairs that appear at the bottom of the calculator, emphasizing that every point on the curve represents a solution to the equation.

Begin with simple linear equations and how to solve them using the graphs and tables on a graphing calculator. Then, advance students to nonlinear situations so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions.

Use the table function on a graphing calculator to solve equations. For example, to solve the equation $x^2 = x + 12$, students can examine the equations $y = x^2$ *and* $y = x + 12$ and determine that they intersect when $x = 4$ and when $x = -3$ by examining the table to find where the *y*-values are the same.

Investigate real-world examples of two-dimensional inequalities. For example, students might explore what the graph would look like for money earned when a person earns *at least* $6 per hour. (The graph for a person earning *exactly* $6/hour would be a linear function, while the graph for a person earning at least $6/hour would be a half-plane including the line and all points above it.)

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

"Optimization Problems: Boomerangs" – Mathematics Assessment Project

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.D.10
    - Collinear Points

Domain: Reasoning with Equations and Inequalities (A.REI)

▶ **Cluster:** *Represent and solve equations and inequalities graphically.*

Standard: A.REI.9 ★

(9/10/11) Solve an equation $f(x) = g(x)$ by graphing $y = f(x)$ and $y = g(x)$ and finding the x-value of the intersection point. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. For **(9/10)** focus on linear, quadratic, and absolute value. **(A.REI.11)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

Connections: A.REI, F.IF. Modeling, A.REI.8

Explanations and Examples:

The ALL Flipbook shares research on the Cartesian Connection in A.REI.8. In summary, research says that students do not fully connect all the different representations of a function. Students who can connect across representations are able to flexibly solve problems. One solution strategy that capitalizes on these connections is described in this standard. Let's work backwards, starting with two function rules.

$f(x) = x - 3$

Input	Rule	Output
-2	-2-3=	-5
-1	-1-3=	-1
0	0-3	-3
1	1-3	-2
2	2-3	-1
3	3-3	0
4	4-3	1

Function Rule

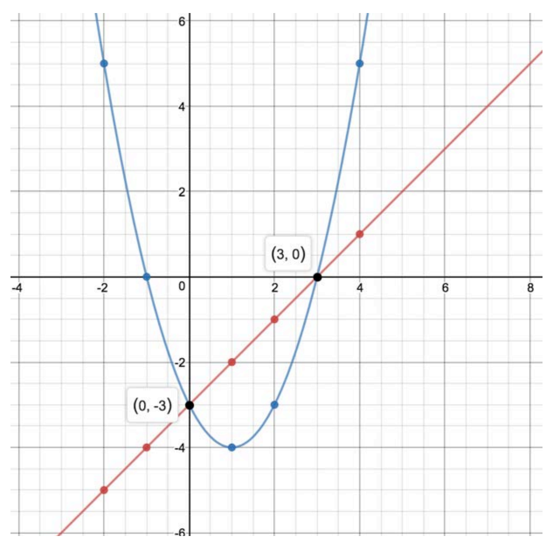
A function takes an input and assigns it to exactly one output.

Solving a system is the process of finding an input for both functions and results in the same output.

$g(x) = x^2 - 2x - 3$

Input	Rule	Output
-2	$(-2)^2 - 2(-2) - 3 =$	5
-1	$(-1)^2 - 2(-1) - 3 =$	0
0	$(0)^2 - 2(0) - 3 =$	-3
1	$(1)^2 - 2(1) - 3 =$	-4
2	$(2)^2 - 2(2) - 3 =$	-3
3	$(3)^2 - 2(3) - 3 =$	0
4	$(3)^2 - 2(3) - 3 =$	5

The ALL standard A.REI.10 expects that students realize the graph represents all possible solutions to an equation. So another view of the system is graphing the equations and identifying the points of intersection.



A.REI.9 uses this type of thinking to solve equations. For example, consider the equation

$$x - 3 = x^2 - 2x - 3$$

This equation can be viewed as a question asking for what input are the two functions equal.

$$f(x) = g(x)$$

When provided an equation to solve, the equation can be separated into two functions to identify when the two functions have equal outputs. See the examples below along with some notes about how this thinking can be used to support student learning.

$$\mathbf{4x - 2 = 10}$$

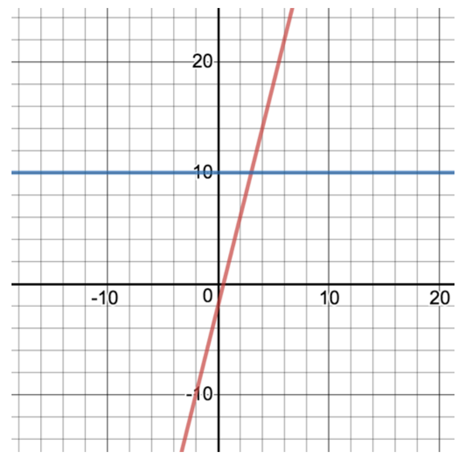
$f(x) = 4x - 2$

This equation can be solved by finding the point on the graph where the function output is 10 *or* by finding the point of intersection.

$g(x) = 10$

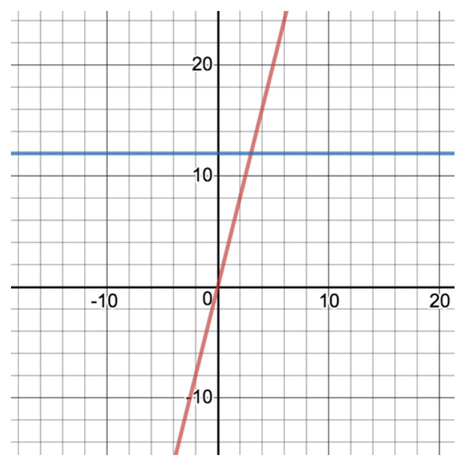
Not only does solving by graphing provide an entry point for all students, it also provides them independence when checking their solutions found through algebraic manipulation. They will begin the problem knowing the answer is 2.

With each step taken toward the solution, the student can graph the equation to ensure the result is still equivalent to the original problem.



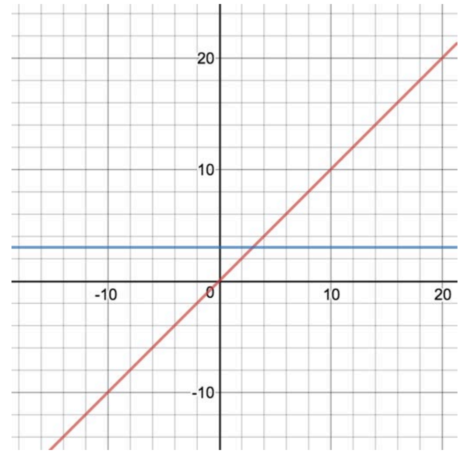
$$f(x) = 4x - 2$$
$$g(x) = 10$$
$$Intersection\ (3,10)$$

The solution is x=3



$$f(x) = 4x$$
$$g(x) = 12$$
$$Intersection\ (3,12)$$

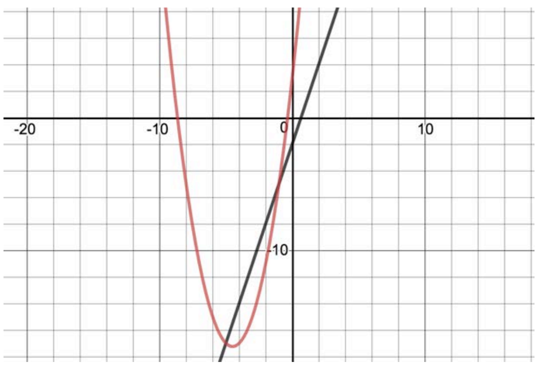
$f(x) = x$
$g(x) = 3$
$Intersection\ (3,3)$



Let's look at another concept that students struggle with: setting a quadratic equal to zero before solving and then referring to the corresponding answer as both solutions, zeros, and x-intercept.

$$3x - 2 = x^2 + 9x + 3$$

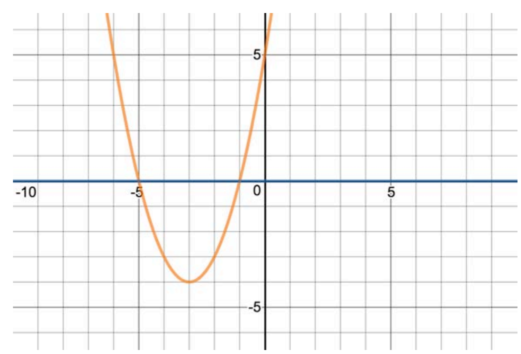
First, graph the pair of functions to identify the solution goal.



The points of intersection are: (-1, -5) and (-5, -17) so the solutions are x=-1 and -5

Setting the equation equal to zero, the new equation is
$x^2 + 6x + 5 = 0$

- The intersection is (-5, 0) and (-1, 0).
- The solutions are x= -1 and -5.
- They are the zeros of the function when the equation is equal to zero.
- They are the x-intercepts when the equation is equal to zero because graphing f(x)=0 creates a line that coincides with the x-axis.



These examples *do not imply* that the only purpose for this standard is to support the teaching and learning of algebraic manipulation. There is value in learning to solve equations by graphing, in addition to solving with tables, reasoning about numbers, *and* algebraic manipulation. Solving equations by graphing is another tool and is not intended to *replace* algebraic manipulation.

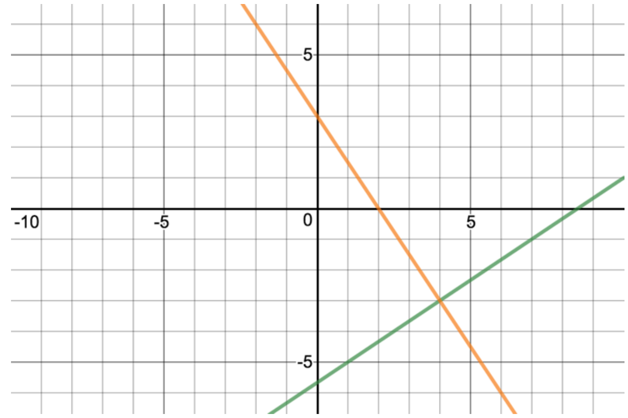
BUT, you might be thinking, if a student learns to solve equations by graphing *why* would they want to learn algebraic manipulation? Excellent point! Excellent question! Teachers need to ask better questions than "find the solution" to encourage students to use all their resources (such as asking students to rewrite the function into a different form, combining functions using arithmetic, finding the exact solution when the solution is irrational, etc.). We want to ask questions with multiple entry points and then have a rich discussion about why the student chose a particular solution strategy. Not only is solving an equation by graphing a valuable tool but it also helps visualize the relationship between the functions.

Examples:

1. Solve and justify, using substitution, that the solution is correct.

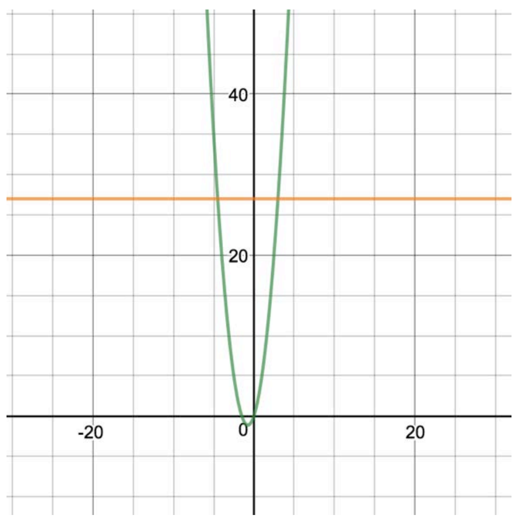
$$\frac{2x-17}{3} = -3(\frac{1}{2}x-1)$$

The intersection is (4, -3). The solution is *x=4*.



2. The length of a rectangle is three more than twice the width. Determine the dimensions that will give a total area of 27 m^2.

<u>Solution:</u>
$w(2w+3) = 27$



Intersections are (-4.5, 27) and (3, 27). When constrained by the situation, the width is 2 and the length is 9.

3. What do the points on the graph of the function $A(w) = w(2w+3)$ represent?

<u>Solution:</u>
Along the x-axis, the *w* variable represents the width of the rectangle. Along the *y*-axis is the output of the function, Area. Each point along the graph represents the width of the rectangle and its associated Area.

4. In the situation above, one student used the function $g(w) = w(2w + 3)$ to find the answer and another used the function $f(w) = 2w^2 + 3w - 27 = 0$. The points for each graph mean different things. Describe the meaning of the points on each graph and how to use the graph to find the width with an Area of 27 m^2.

   Solution:

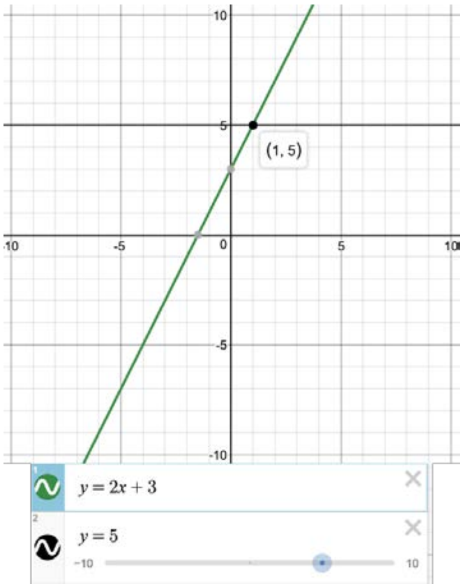
   For the *g* function, each point on the graph represents the width of the rectangle and its associated Area. To find the width, find the point where g(w)=27. The *f* function represents the same rectangle translated down 27 units so that the x-intercept represents an Area of *exactly* 27 m^2. Each point on the graph of *f* represents the width of the rectangle and the increase or decrease of the area above or below 27 m^2.

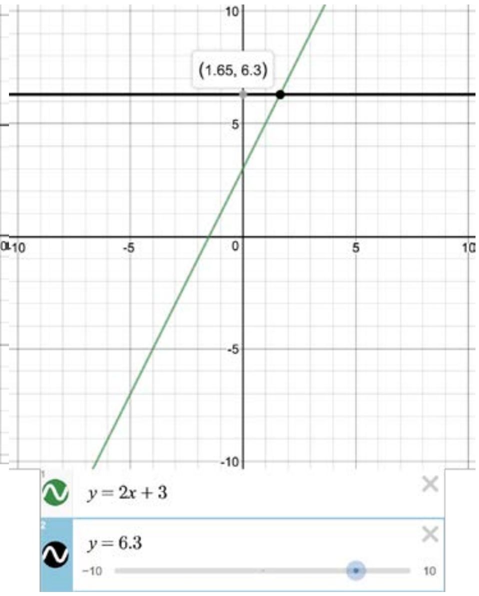
Instructional Strategies:

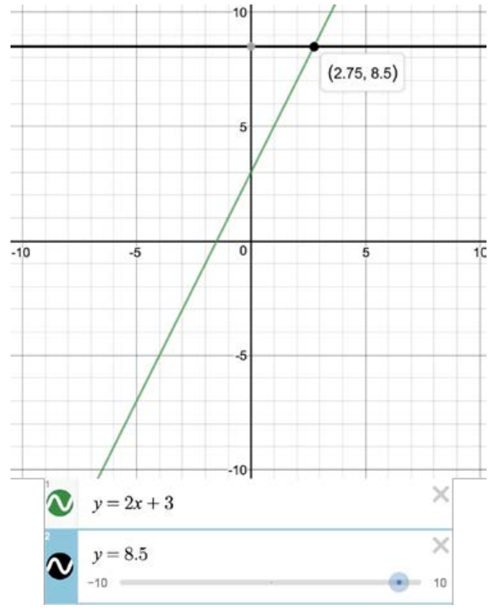
Using technology, students can easily find the points of intersection and explore how the solutions are related to the functions. In particular, desmos easily replaces numbers in the equation with parameters that can become sliders to explore how the solutions changes as the numbers in the equation change. This again expands the classroom discussion beyond "find the answer" into developing number sense and estimation skills. For example, when reviewing solving two-step equations such as $2x + 3 = 4$, you could explore some ideas such as:

1. **As the value that the expression $2x + 4$ is equal increases, how does the solution to the equation change?**

   <u>Solution</u>



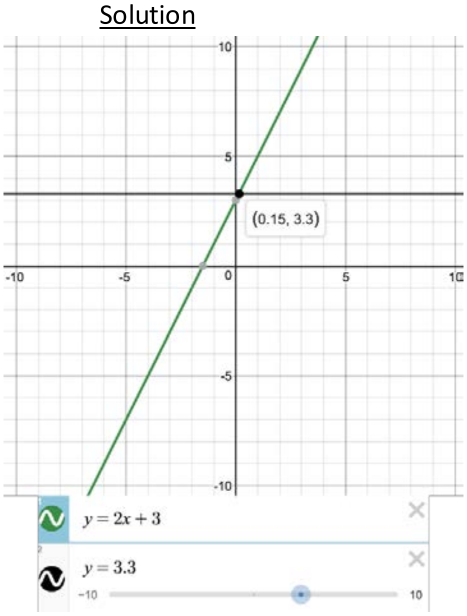


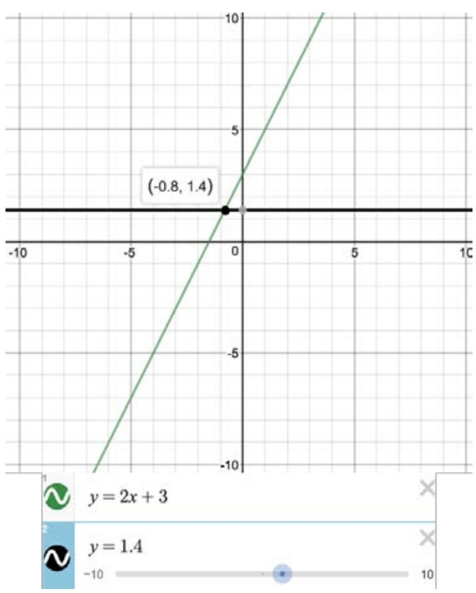


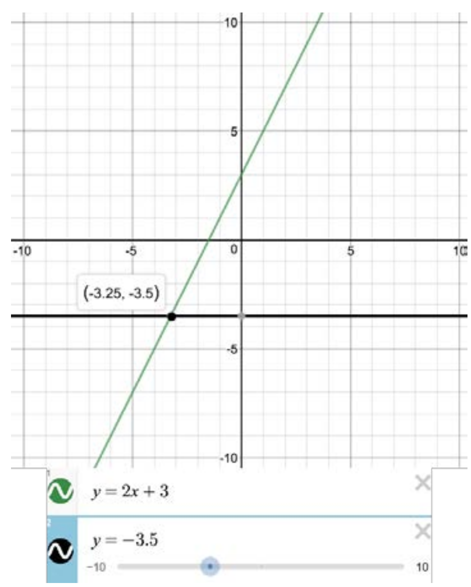
   As the value of that the expression is equal to increases, the solution increases. This will happen along the entire function because, as the line the expression intersects with rises, the function also moves further to the right.

2. **As the value that the expression is equal to decreases, how does the solution to the equation change?**

   <u>Solution</u>







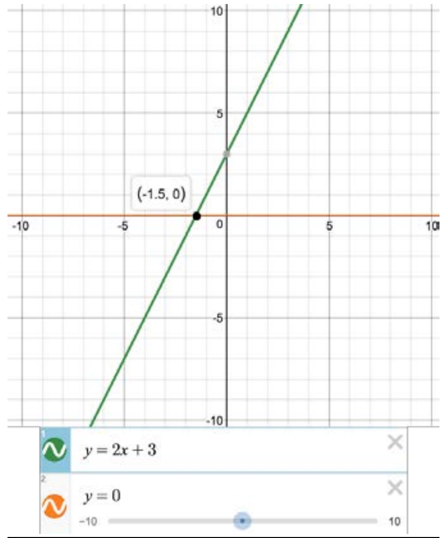
   As the value the expression is equal to decreases, the solution to the equation will decrease because the slope is positive.

Major Clusters | Supporting Clusters | Additional Clusters

3. **When the value of the expression equals zero, what key feature for the function $f(x) = 2x + 3$ is identified? Will that be true or *any* equation?**

<u>Solution</u>

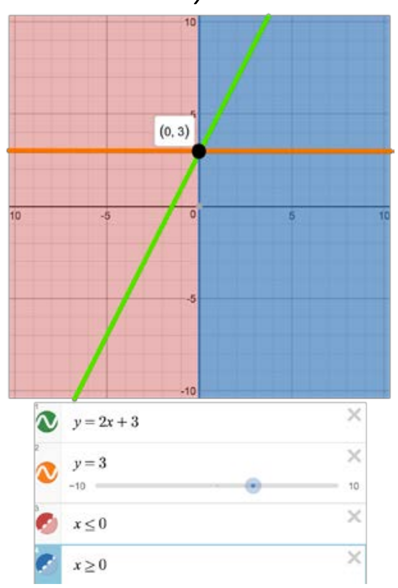
When the expression is equal to zero, the function graphed lies on the *x-axis* so the solution is the same as the *x*-intercept. This would be true for any equation.



4. **When does the solution become negative?**

<u>Solution</u>

When the *x* value (the solution) is negative, the graph will be on the left side of the *y*-axis. When the *x* value (the solution), is positive, the graph will be on the right side of the *y*-axis. Therefore, the solution changes from positive to negative when the graph intersects with the *y*-axis.



5. **For *any* equation, is that location where the solution will *always* change its sign?**

<u>Solution</u>

Yes because the *y*-axis will always be the place where the *x* value changes from positive to negative and the *x* value is the solution.

Resources/Tools:

- Desmos graphing calculator
- Desmos activity: Picture Perfect

High School – Functions

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation.*

Standard: F.IF.1 (all)

Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$. **(F.IF.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: Functions and Algebra Domain

Explanations and Examples:

The goal with the functions domain is teach students a *process* for analyzing functions, rather than teaching each function family in isolation, causing students to miss the larger structure holding them all together. For that reason, several of the Function standards were chosen as ALL standards and should be present throughout all mathematics instruction. In fact, the five ALL functions standards so clearly define that process and are so intertwined that it is difficult to separate them into individual standards to discuss here. As a result, this section of the flipbook will discuss the common process here and address any individual nuances in each additional standard.

1. Functions should be **analyzed qualitatively** (from a global perspective) for key features (F.IF.4)
   a) Direction of change (increasing/decreasing)
   b) Type of change (constant linear or non-constant nonlinear)
   c) Minimum and maximum values.
   d) Symmetry
   e) End behavior
   f) Periodicity
2. Functions should be **analyzed quantitatively** (from a local perspective) by
   a) Identifying the domain and range and relating it back to the relationship
   b) Representing the function across all representations and identifying key features.
   c) Understanding function notation in the context of the situation.
   d) Understanding how the graph of the function is related to the equation.

The first concepts for students to wrestle with are:

- Defining a function as the rule that *assigns* an *every* element from the input set to *exactly one* element of the output set
- Naming the input set domain and the output set range
- Introducing notation naming the input x and the output $f(x)$
- Defining the graph of f as the graph of the equation $y=f(x)$

Define the function relationship

- The standard uses an "assignment" definition for a function using a rule to assign an input to an output.
- Inputs and outputs are named domain and range and are a fundamental part of the definition for a function.
- The definition does **not** require a function to be an equation, graph, or even numerical.
- Assignment requires two elements**:** (1) an element from the domain is assigned (2) to exactly one element of the range.

Identifying functions from a table

- It is **not correct** to describe a procedure for determining if a table (or set of ordered pairs) is a function by saying **"the x value cannot repeat."** This neglects an essential requirement for a function: a correspondence between two values.
- In a data table the input value might repeat but, if it does, it must ALWAYS be assigned to the same output value.

Identifying a function from a graph

- Focus on the definition: that every element of the domain is assigned to exactly one element of the range.
- The vertical line test provides an easy procedure to instruct students but there are numerous reasons to **avoid this strategy.**
- Once students learn the vertical line test, they tend to blindly apply it to all graphs rather than having a procedure that can grow with them through geometric transformations, inverse function, polar functions, etc.
- Focus on clearly identifying the input (independent variable or domain) and matching it to a single output (dependent variable or range). This also provides essential practice the students need with domain and range.

Function notation

- Asking students to **interpret function notation** in non-quantitative situations can help reinforce what it means for *x* to be the input value in a function rule and how *f(x)*, the output value, is the associated output.
- x is a number and a specific input, *f(x)* is a number and the corresponding output. (See N.Q.1)
- The name of the function is *f*, for example, not *f(x).*
- Dr. Bill McCallum wrote about how to interpret the statement *y=f(x).*

  As for the $y = f(x)$ notation, when we say something like "the function $y = x^2$" we are using abbreviated language for "the function defined by the equation $y = x^2$, where *x* is the independent variable and *y* is the dependent variable." You can't say that every time, so we have a shortened form, which depends on certain conventions: the dependent variable occurs on the left and an expression in the independent variable occurs on the right."

Examples:

1. Determine which of the following tables represent a function and explain why.

A	
x	***f(x)***
0	1
1	2
2	2
3	4

B	
x	***f(x)***
0	0
1	2
1	3
4	5

Solution: Table A represents a function because for each element in the domain there is exactly one element in the range. Table B does not represent a function because when x = 1, there are two values for *f*(*x*): 2 and 3.

2. For the functions a. through f. below:

- List the algebraic operations in order of evaluation. What restrictions does each operation place on the domain of the function?
- Give the function's domain.

a. $y = \frac{2}{x-3}$

b. $y = \sqrt{x-5} + 1$

c. $y = 4 - (x-3)^2$

d. $y = \frac{7}{4-(x-3)^2}$

e. $y = 4 - (x-3)^{\frac{1}{2}}$

f. $y = \frac{7}{4-(x-3)^{\frac{1}{2}}}$

3. Is a geometric transformation an example of a function? If not, why? If so, how does that support its use in formal proofs?

Viewing transformations as functions is essential to proving congruence through rigid transformations. We have to know that if I do a translation, for example, that there is exactly one guaranteed output. Once we have a guaranteed output, we can apply geometric reasoning to the sequence of transformations and know, beyond all doubt, that the image is congruent.

Instructional Strategies:

Provide applied contexts in which to explore functions. For example, examine the amount of money earned when given the number of hours worked on a job, and contrast this with a situation in which a single fee is paid by the "carload" of people, regardless of whether 1, 2, or more people are in the car.

Use diagrams to help students visualize the idea of a function machine. Students can examine several pairs of input and output values and try to determine a simple rule for the function.

Rewrite sequences of numbers in tabular form, where the input represents the term number (the position or index) in the sequence, and the output represents the number in the sequence.

Help students to understand that the word "domain" implies the set of all possible input values and that the integers are a set of numbers made up of {…-2, -1, 0, 1, 2, …}.

Distinguish between relationships that are not functions and those that are functions (e.g., present a table in which one of the input values results in multiple outputs to contrast with a functional relationship). Examine graphs of functions and non-functions, recognizing that if a vertical line passes through at least two points in the graph, then *y* (or the quantity on the vertical axis) is not a function of x (or the quantity on the horizontal axis).

Common Misconceptions:

Students may believe that all relationships having an input and an output are functions, and therefore, misuse the function terminology.

Students may also believe that the notation *f(x)* means to multiply some value *f* times another value *x*. The notation alone can be confusing and needs careful development. For example, *f*(2) means the output value of the function *f* when the input value is 2.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.A
  - Interpreting the graph
- F-IF.A.1
  - The Parking Lot
  - Your Father
  - Parabolas and Inverse Functions
  - Using Function Notation I
  - The Customers
  - Points on a graph
  - Domains

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation*.

Standard: **F.IF.2 (all)**

Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. **(F.IF.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF.1

Explanations and Examples:

In addition to explanation about function notation discussed in N.Q.1 and F.IF.1, it is also important to analyze function notation across multiple contexts and representations.

Examples:

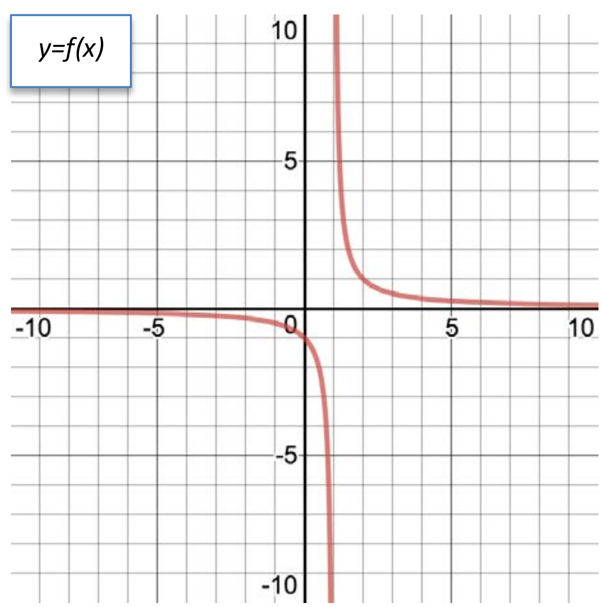
An Illustrated Math Task:

1. You put a yam in the oven. After 45 minutes, you take it out. Let f be the function that assigns to each minute after you placed the yam in the oven, its temperature in degrees Fahrenheit.
   a. Write a sentence explaining what $f(0) = 65$ means in everyday language
   b. Write a sentence explaining what $f(5) < f(10)$ means in everyday language.
   c. Write a sentence explaining what $f(40) = f(45)$ means in everyday language
   d. Write a sentence explaining what $f(45) > f(60)$ means in everyday language.

Use the table and graph to answer the questions below.

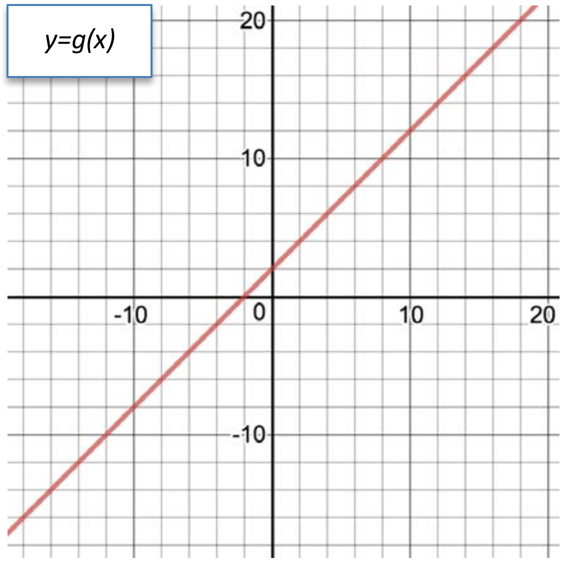
2. Find $f(4)$
3. If $f(x) = 2$, find x.

x	f(x)
-2	-4
-1	-1
0	2
1	5
2	8
3	11
4	14



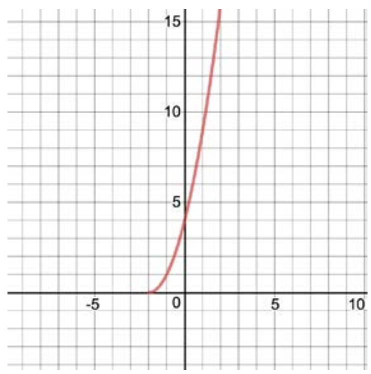
4. Use the table and/or the equation to perform the given function operation. Graph the result.

x	f(x)
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

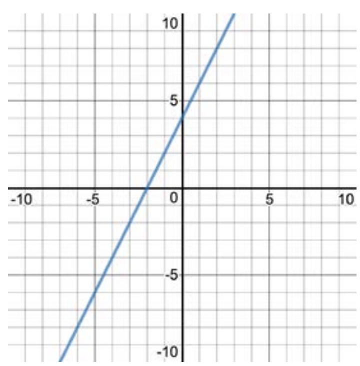


Solution:

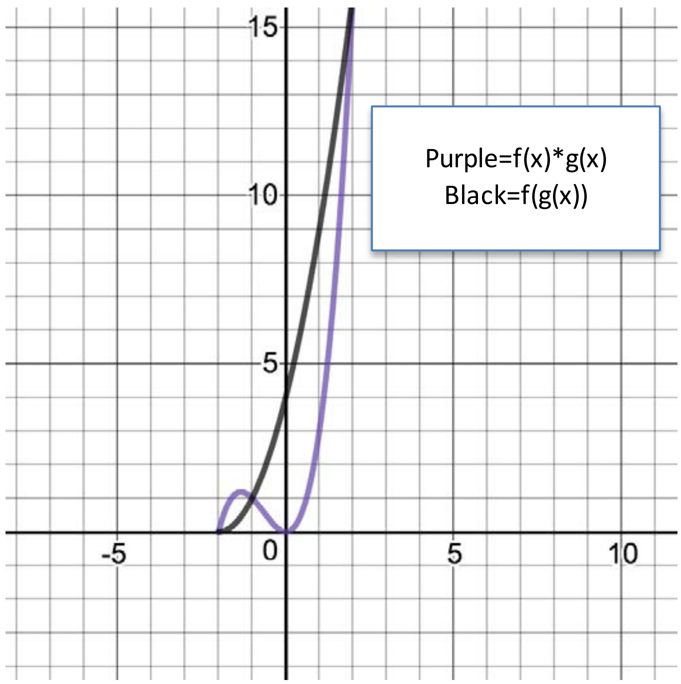
x	*f(x)*	*f(x+2)*
-2	4	*f(-2+2)=0*
-1	1	*f(-1+2)=1*
0	0	*f(0+2)=4*
1	1	*f(1+2)=9*
2	4	*f(2+2)=16*
3	9	
4	16	



x	*g(x)*	*2g(x)*
-2	0	2g(-2)=0
-1	1	2g(-1)=2
0	2	2g(0)=4
1	3	2g(1)=6
2	4	2g(2)=8
3	5	2g(3)=10
4	6	2g(4)=12



x	*f(x)*	*g(x)*	*f(x)*g(x)*	*f(g(x))*
-2	4	0	f(-2)*g(-2)=0	f(g(-2))=f(0)=0
-1	1	1	f(-1)*g(-1)=1	f(g(-1))=f(1)=1
0	0	2	f(0)*g(0)=0	f(g(0))=f(2)=4
1	1	3	f(1)*g(1)=3	f(g(1))=f(3)=9
2	4	4	f(2)*g(2)=16	f(g(2))=f(16)=2
3	9	5		
4	16	6		



Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.A.2
  - Using Function Notation II
  - Yarn in the Oven
  - The Random Walk
  - Cell phones
  - Random Walk II

Domain: Interpreting Functions (F.IF)

● **Cluster:** *Understand the concept of a function and use function notation.*

Standard: **F.IF.3**

(9/10/11) Recognize patterns in order to write functions whose domain is a subset of the integers. **(9/10)** Limited to linear and quadratic. *For example, find the function given* $\{(-1,4),(0,7),(1,10),(2,13)\}$. **(F.IF.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.IF.1-2, F.BF.1-2

Explanations and Examples:

This standard is not about sequences but is, more generally, about writing the function rule from a pattern. Previous pattern work asked students to describe the rule that generated the pattern recursively (i.e. "it's a plus 2 pattern" for 3, 5, 7, 9, 11, …). Now students are asked to number the terms in the pattern, with the number of the term the input and the term in the pattern the output, and create a function rule that generates the pattern. Students are not expected to write the recursive rule or identify the connection between the recursive description and the function rule but these concepts might be scaffolding for students struggling with writing the rule.

Examples:

Write the function rule that generates the following patterns:

1. 5, 8 11, 14, 17, …
2. -7, -7.5, -8, -8.5, …
3. 2, 8, 18, 32, 50, …
4. 0, 1, 4, 9, 16, 25, ….

Instructional Strategies:

Instruction for linear patterns will be very similar to instruction for arithmetic sequences but instruction for quadratic patterns might be new to many students. Students should be able to recognize the output values for the parent function of quadratic family. Comparing the output from the pattern to the parent function and analyzing the differences as a transformation can help students write a function to define the pattern.

Resources/Tools:

Generalizing Patterns: Table Tiles- This lesson unit is intended to help you assess how well students are able to identify linear and quadratic relationships in a realistic context: the number of tiles of different types needed for a range of square tabletops.

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI-B-4.b
  - Quadratic Sequence 1
  - Quadratic Sequence 2
  - Quadratic Sequence 3

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation.*

Standard: **F.IF.4 (all) ★**

For a function that models a relationship between two quantities, interpret key features of expressions, graphs and tables in terms of the quantities, and sketch graphs showing key features given a description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. **(F.IF.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.IF.7

Explanations and Examples:

This standard is often paired with F.IF.7 but F.IF.7 focuses on the key feature from the graph, while this standard focuses on identifying key features across all representations. Another key difference between the two standards is that F.IF.7 is not a modeling standard while F.IF.4 is a modeling standard. As a modeling standard, the focus should be on *interpreting the quantities* in context. This standard is an ALL standard because all the functions studied in high school have this standard applied and practice.

Examples:

- A rocket is launched from 180 feet above the ground at time *t* = 0. The function that models this situation is given by $h = -16t^2 + 96t + 180$, where *t* is measured in seconds and *h* is height above the ground measured in feet.
  - What is a reasonable domain restriction for *t* in this context?
  - Determine the height of the rocket two seconds after it was launched.
  - Determine the maximum height obtained by the rocket.
  - Determine the time when the rocket is 100 feet above the ground.
  - Determine the time at which the rocket hits the ground.
  - How would you refine your answer to the first question based on your response to the second and fifth questions?

- Compare the graphs of $y = 3x^2$ and $y = 3x^3$.
- Let $R(x) = \dfrac{2}{\sqrt{x-2}}$. Find the domain of *R*(*x*). Also find the range, zeros, and asymptotes of *R*(*x*).
- Let $f(x) = x^2 - 5x + 1$. Graph the function and identify end behavior and any intervals of constancy, increase, and decrease.
- It started raining lightly at 5am, then the rainfall became heavier at 7am. By 10am the storm was over, with a total rainfall of 3 inches. It didn't rain for the rest of the day. Sketch a possible graph for the number of inches of rain as a function of time, from midnight to midday.

Instructional Strategies:

Flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

Examine a table of related quantities and identify features in the table, such as intervals on which the function increases, decreases, or exhibits periodic behavior.

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5. However, if a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Given a table of values, such as the height of a plant over time, students can estimate the rate of plant growth. Also, if the relationship between time and height is expressed as a linear equation, students should explain the meaning of the slope of the line. Finally, if the relationship is illustrated as a linear or non-linear graph, the student should select points on the graph and use them to estimate the growth rate over a given interval.

Begin with simple, linear functions to describe features and representations, and then move to more advanced functions, including non-linear situations.

Provide students with many examples of functional relationships, both linear and non-linear. Use real-world examples, such as the growth of an investment fund over time, so that students can not only describe what they see in a table, equation, or graph, but also can relate the features to the real-life meanings.

Allow students to collect their own data sets, such as the falling temperature of a glass of hot water when removed from a flame versus the amount of time, to generate tables and graphs for discussion and interpretation.

Common Misconceptions:

Students may believe that it is reasonable to input any *x*-value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.

Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B
  - Pizza Place Promotion
- F-IF.B.4
  - Influenza epidemic
  - Warming and Cooling
  - How is the weather?
  - Telling a Story with Graphs
  - Logistic Growth Model , Abstract Version
  - Logistic Growth Model , Explicit Version

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation*..

Standard: F.IF.5 (all) ★

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★ **(F.IF.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

Connections: F.IF.1

Explanations and Examples:

Given the graph if a function, determine the practical domain of the function as it relates to the numerical relationship it describes.

Students may explain orally or in written format, the existing relationships.

Examples:

- If the function $h(n)$ gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function.
- A hotel has 10 stories above ground and 2 levels in its parking garage below ground. What is an appropriate domain for a function. $T(n)$ that gives the average number of times an elevator in the hotel stops at the n^{th} floor each day.
- Oakland Coliseum, home of the Oakland Raiders, is capable of seating 63,026 fans. For each game, the amount of money that the Raiders' organization brings in as revenue is a function of the number of people, n, in attendance. If each ticket costs \$30, find the domain and range of this function.

Sample Response:

Let *r* represent the revenue that the Raider's organization makes, so that $r = f(n)$. Since *n* represents a number of people, it must be a nonnegative whole number. Therefore, since 63,026 is the maximum number of people who can attend a game, we can describe the domain of *f* as follows: $\{Domain = n: 0 \le n \le 63{,}026 \textit{ and } n \textit{ is an integer}\}$.

The range of the function consists of all possible amounts of revenue that could be earned. To explore this question, note that $r = 0$ if nobody comes to the game, $r = 30$ if one person comes to the game, $r = 60$ if two people come to the game, etc. Therefore, *r* must be a multiple of 30 and cannot exceed $(30 \cdot 63{,}026) = 1{,}890{,}780$, so we see that $\{Range = r: 0 \le r \le 1{,}890{,}780 \textit{ and } r \textit{ is an integer multiple of } 30\}$.

Instructional Strategies:

The deceptively simple task above asks students to find the domain and range of a function from a given context. The function is linear and if simply looked at from a formulaic point of view, students might find the formula for the line and say that the domain and range are all real numbers. However, in the context of this problem, this answer does not make sense, as the context requires that all input and output values are non-negative integers, and imposes additional restrictions. This problem could serve different purposes. Its primary purpose is to illustrate that the domain of a function is a property of the function in a specific context and not a property of the formula that represents the function. Similarly, the range of a function arises from the domain by applying the function rule to the input values in the domain. A second purpose would be to illicit and clarify a common misconception, that the domain and range are properties of the formula that represent a function. Finally, the context of the task as written could be used to transition into a more involved modeling problem, finding the Raiders' profit after one takes into account overhead costs, costs per attendee, etc.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B.5
    - Oakland Coliseum
    - Average Cost

Domain: Interpreting Functions (F.IF)

◆ **Cluster:** *Interpret functions that arise in applications in terms of the context*

Standard: **F.IF.6 ★**

(9/10/11) Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★ **(9/10)** Limited to linear functions.★**(F.IF.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

Connections: F.IF.4, F.IF.7, S.ID.6, 8.EE

Explanations and Examples:

Consider: Is there a difference between "slope" and "rate of change"?

Slope is the description of a physical feature of the graph. It describes the *incline* of the line. Rate of change describes how the function changes and is not limited to any one representation. This standard focuses on calculating the rate of change across multiple representations. In grades 9/10, problems should be limited to linear functions.

Additionally, this standard is a modeling standard so students should be working problems from modeling situations. Collecting data, creating a scatterplot, and modeling the line of best fit is an opportunity for students to calculate and interpret the rate of change.

The wording "over a specified interval" can provide an opportunity for discussions about comparisons or how to rewrite the rate of change so that the description is more meaningful. For example, consider the following situation:

Example 1:

While planning for a vacation to Orlando, Florida during the month of June, you compare the cost of a taxi, an uber, and renting a car. Describe the conditions that would make each vehicle the best choice. Be sure to include any assumptions you made to create your predictive model and the rate of change for each transportation type. (My estimates are in red as an example of what a student might do.)

Estimate 2 trips every 5 miles. Cost shows maximum miles for that range.	Rental Vehicle	Taxi $4.50 base fee, 2.40/mile	Uber $9.11 base fee, $2.70 booking fee, $0.79/mile
1 mile	$255	$6.90	$12.60
10 miles	$255	$33.00	$31.52
20 miles	$255	$66.00	$63.04
30 miles	$255	$99.00	$94.56
60 miles	$255	$198.00	$189.12
100 miles	$255	$330.00	$315.20

- Rate of change for the rental vehicle: $\frac{255-255}{20-10} = \$0/mile$
- Rate of change for taxi: $\frac{66-33}{20-10} = \$3.30/mile$
- Rate of change for uber: $\frac{63.04-31.52}{20-10} = \$3.15/mile$

Even though the rental vehicle has the smallest rate of change, $0/mile, it is not the cheapest option unless I plan to travel more than 100 miles while I am in Orlando, the rental vehicle would not be the best option. Uber has the smallest rate of change at $3.15 and would be the best option for trips longer than 5 miles because the single booking fee and smaller mileage fee makes that the best option. If I will only take short trips, less than 5 miles an hour, than the taxi is the cheaper option because the lower base fee will offset the increased mileage cost.

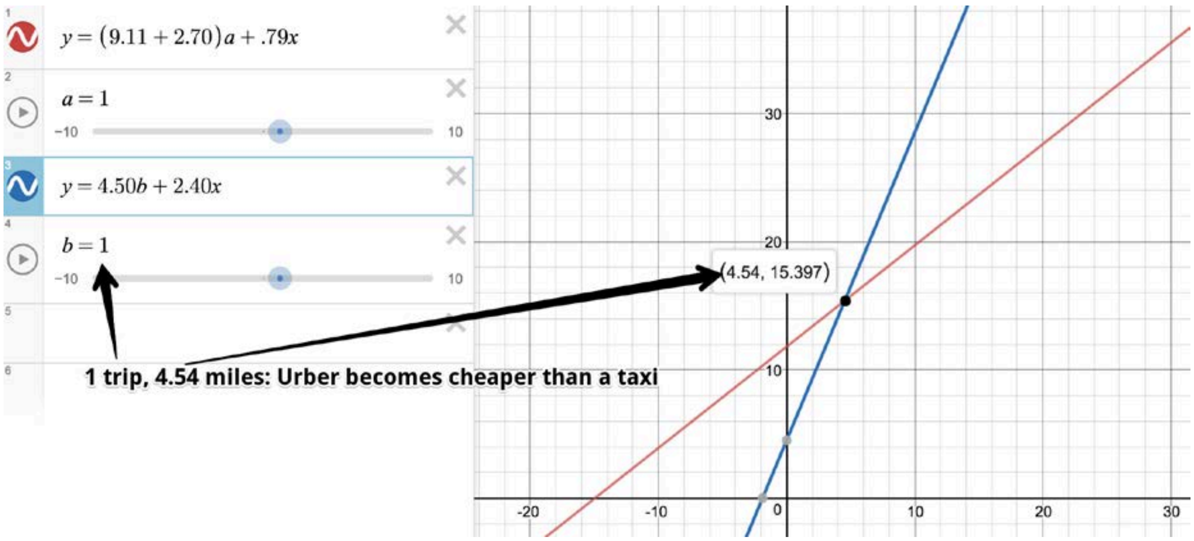
Note: Desmos can help you explore the multiple variables in this situation. This creates a more realistic model and would be a great extension, depending on student interest and ability.

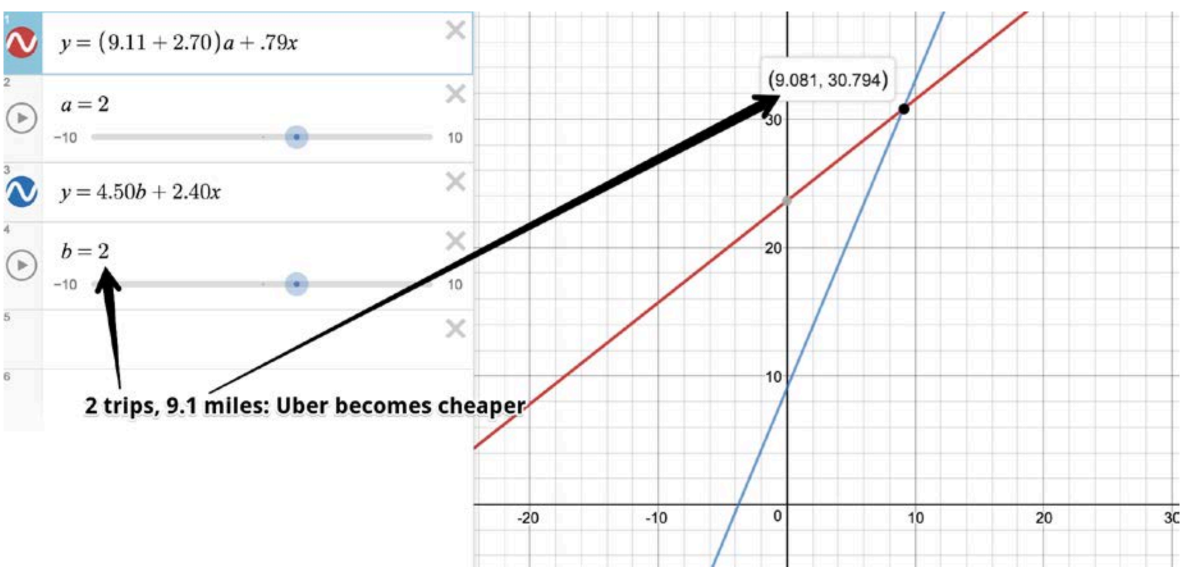
Function model for uber $f(x) = (9.11 + 2.70) * number\ of\ trips + 0.79 * number\ of\ miles$

$$f(x) = (9.11 + 2.70) * a + 0.79 * x$$

Function model for taxi $f(x) = 4.50 * number\ of\ trips + 2.40 * number\ of\ miles$

$$f(x) = 4.50 * b + 2.40x$$





Example 2:

Rewrite the interpretation for the given rate of change so that the interval of change is more easily understandable.

a. $0.003/mile
b. 0.2 bolts/lbs.
c. 0.5 people/seat

Possible solutions:

a. Change the interval from every 1 mile to every 1000 miles. $3 for every 1000 miles
b. Change the interval from every 1 lb. to every 10 lbs. 2 bolts for every 10 lbs.
c. Change the interval from every 1 seat to every 2 seats. 1 person for every 2 seats

Instructional Strategies:

The instructional strategies for this standard are not necessarily different than those used with slope, which was taught in 8th grade. The key difference is to connect the rate of change across representations and to connect the meaning to the context.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B.6
  - The High School Gym
  - Mathemafish Population
  - Temperature Change

Domain: Interpreting Functions F.IF

▶ **Cluster:** *Analyze functions using different representations.*

Standard: **F.IF.7 ★**

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

F.IF.7b. **(11)** Graph square root, cube root, and exponential functions. ★ **(F.IF.7b)**

F.IF.7c. **(11)** Graph logarithmic functions, emphasizing the inverse relationship with exponentials and showing intercepts and end behavior. ★ **(F.IF.7e)**

F.IF.7e. **(11)** Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. ★ **(F.IF.7c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

Connections: A.REI.9, F.BF.5, F.LQE,

Explanations and Examples:

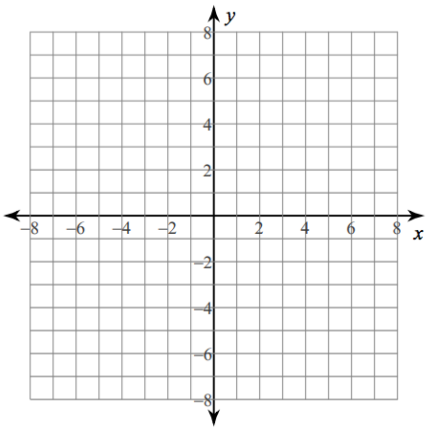
Graphing key features for the additional function families in Algebra 2 is similar to the graphing done in Algebra 1 (see F.IF.7 from 9/10). While this standard specifies graphing, the cluster statement says to analyze using different representations. Different representations can provide rich points of conversation when comparing the key features of the graph to the other representations.

Key characteristics include but are not limited to maxima, minima, intercepts, symmetry, end behavior, and asymptotes. Students may use graphing calculators or programs, spreadsheets, or computer algebra systems to graph functions.

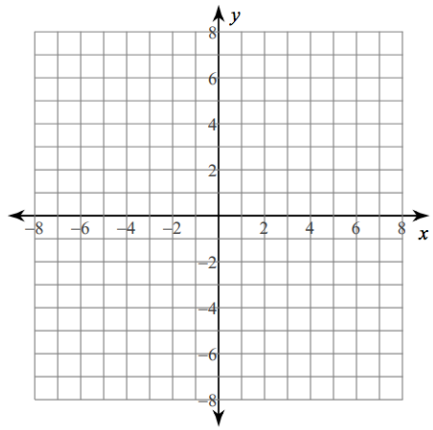
Examples:

State the maximum number of turns the graph of each function could make. Then sketch the graph. State the number of real zeros. Approximate each zero to the nearest tenth. Approximate the relative minima and relative maxima to the nearest tenth.

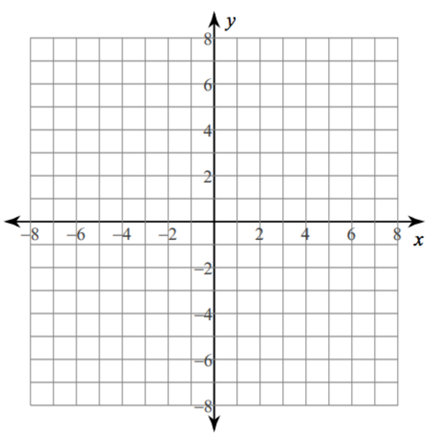
1) $f(x) = x^2 + 2x - 5$



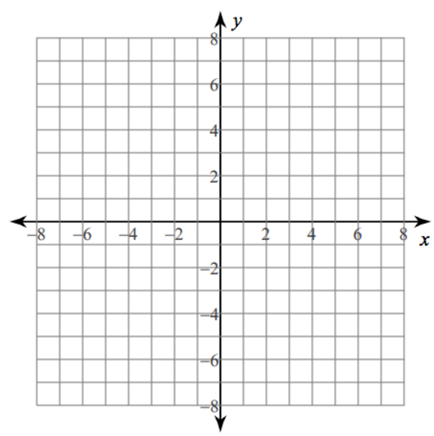
2) $f(x) = -x^4 + x^3 + 2x^2$



3) $f(x) = x^4 - 4x^3 + 2x^2 + x + 4$



4) $f(x) = x^3 + x^2 - x - 2$



Psychologists use an exponential model of the learning process $f(t) = c(1 - e^{-kt})$, where c is the total number of tasks to be learned, k is the rate of learning, t is the time, and f(t) is the number of tasks learned. Suppose you move to a new school and you want to learn the names of 30 classmates. If your learning rate for new tasks is 20% per day, how many complete names will you know after 2 days? How many will you know after 8 days? Use a graph to justify your answer.

Instructional Strategies:

Explore various families of functions and help students to make connections in terms of general features. For example, just as the function $y = (x + 3)^2 – 5$ represents a translation of the function $y = x$ by 3 units to the left and 5 units down, the same is true for the function $y = | x + 3 | - 5$ as a translation of the absolute value function $y = | x |$.

Discover that the factored form of a quadratic or polynomial equation can be used to determine the zeros, which in turn can be used to identify maxima, minima and end behaviors.

Use various representations of the same function to emphasize different characteristics of that function.
For example, the y-intercept of the function $y = x^2 -4x – 12$ is easy to recognize as (0, -12). However, rewriting the function as $y = (x – 6)(x + 2)$ reveals zeros at (6, 0) and at (-2, 0). Furthermore, completing the square allows the equation to be written as $y = (x – 2)^2 – 16$, which shows that the vertex (and minimum point) of the parabola is at (2, -16).

Examine multiple real-world examples of exponential functions so that students recognize that a base between 0 and 1 (such as an equation describing depreciation of an automobile $f(x) = 15{,}000(0.8)^x$ represents the value of a $15,000 automobile that depreciates 20% per year over the course of x years]) results in an exponential decay, while a base greater than 1 (such as the value of an investment over time $f(x) = 5{,}000(1.07)^x$ represents the value of an investment of $5,000 when increasing in value by 7% per year for x years]) illustrates growth.

Graphing utilities on a calculator and/or computer can be used to demonstrate the changes in behavior of a function as various parameters are varied.

Real-world problems, such as maximizing the area of a region bound by a fixed perimeter fence, can help to illustrate applied uses of families of functions.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.C.7
  - Identifying graphs of functions
  - Modeling London's Population
  - Graphs of Power Functions
- F-IF.C.7.c
  - Running Time
  - Graphing Rational Functions

Mathshell:

- Representing Functions in Everyday Situations

Domain: Interpreting Functions F.IF

Cluster: *Analyze functions using different representations.*

Standard: F.IF.8

Write a function in different but equivalent forms to reveal and explain different properties of the function.

F.IF.8b. **(11)** Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. **(F.IF.8a)**

F.IF.8c. **(11)** Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as*

$y = (1.02)^t, y = (0.97)^t, y = (1.01)^{12t}, y = (1.2)^{\frac{t}{10}}$, and classify them as representing exponential growth or decay. **(F.IF.8b)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.IF.7, A.REI.5

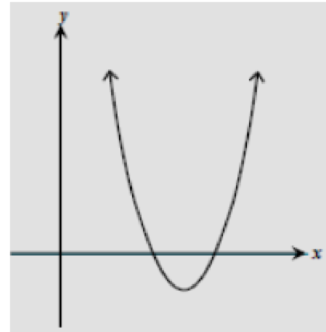
Explanations and Examples:

In Algebra 1 focus on this standard with linear, exponential and quadratic functions. In Algebra 2 students will extend their work to focus on applications and how key features relate to characteristics of a situation making selection of a particular type of function.

Examples:

1. Factor the following quadratic to identify its zeros: $x^2 + 2x - 8 = 0$
2. Complete the square for the quadratic and identify its vertex: $x^2 + 6x + 19 + 0$
3. Write the following function in a different form and explain what each form tells you about the function:
$$f(x) = x^3 - 6x^2 + 3x + 10$$
4. Write the function $y - 3 = \frac{2}{3}(x - 4)$ in the equivalent form most appropriate for identifying the slope and y-intercept of the function.
Solution: $y = \frac{2}{3}x + \frac{1}{3}$

5. Which of the following equations could describe the function of the given graph? Explain.



$f_1(x) = (x + 12)^2 + 4$
$f_2(x) = -(x - 2)^2 - 1$
$f_3(x) = (x + 18)^2 - 40$
$f_4(x) = (x - 10)^2 - 15$
$f_5(x) = -4(x + 2)(x + 3)$
$f_6(x) = (x + 4)(x - 6)$
$f_7(x) = (x - 12)(-x + 18)$
$f_8(x) = (20 - x)(30 - x)$

Solution: All of these equations describe quadratic functions. Since quadratic functions have graphs that are parabolas and the given graph appears to be a parabola, the given equations meet a minimum criteria for consideration.

The graph of $f_1(x) = (x+12)^2 + 4$ has a vertex of $(-12, 4)$ which is in the second quadrant, so it does not match the graph.

The graph of $f_2(x) = -(x-2)^2 - 1$ has maximum rather than a minimum value at $x = 2$ since the leading coefficient is negative (in other words, the graph opens downward), so it does not match the graph.

The graph of $f_3(x) = (x+18)^2 - 40$ has a vertex of $(-18, -40)$ which is in the third quadrant, so it does not match the graph.

The graph of $f_4(x) = (x-10)^2 - 15$ has a vertex of $(10, -15)$ which is in the fourth quadrant, and the leading coefficient is positive (so the graph would open upward) so this could describe the function whose graph is given.

The graph of $f_5(x) = -4(x+2)(x+3)$ has x-intercepts of $(-2, 0)$ and $(-3, 0)$. Since the x-intercepts are both positive for the given graph, they do not match.

The graph of $f_6(x) = (x+4)(x-6)$ has x-intercepts of $(-4, 0)$ and $(6, 0)$. The x-intercepts are both positive for the give graph, so they do not match.

The graph of $f_7(x) = (x-12)(-x+18)$ has a leading coefficient that is negative and so has a maximum rather than a minimum value (at $x = 15$) and thus cannot match the graph.

The graph of $f_8(x) = (x-20)(x-30)$ has x-intercepts of $(20, 0)$ and $(30, 0)$. Since the x-intercepts are both positive for the graph and the leading coefficient is positive (so the graph would open upward), this could possibly be the equation for this graph.

The functions f_4 and f_8 both have graphs of the approximate shape given, though we note that they certainly would not appear identical if plotted simultaneously. For example, the vertex of the graph of f_4 occurs at $x = 10$ whereas that of f_8 occurs at $x = 25$.

6. The projected population of Delroysville is given by the formula $(t) = 1500(1.08)$. You have been selected by the city council to help them plan for future growth. Explain what the formula $(t) = 1500(1.08)$ means to the city council members.

7. Which of the following functions will represent \$500 placed into a mutual fund yielding 10% per year for 4 years

a. $= 500(.10)^4$ **b.** $A = 500(1.1)^4$

c. $= 500(4)(.10)$ **d.** $A = 500(1.04)^{10}$

Instructional Strategies: See F.IF.7

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.C.8
  - Which Function?
- F-IF.C.8.a
  - Springboard Dive

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation*

Standard: **F.IF.9 (all)**

Compare properties of two functions using a variety of representations (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, a quantity increasing exponentially eventually exceeds a quantity increasing linearly.* **(F.IF.9)**

Suggested Standards for Mathematical Practice (MP):

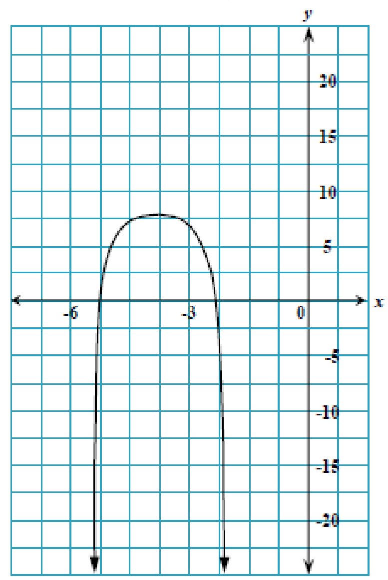
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF.4, F.IF.7

Explanations and Examples: See F.IF.1 and F.IF.4

Examples:

1. Examine the functions below. Which function has the larger maximum? How do you know?



$f(x) = 2x^2 - 8x + 20$

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.C.9
  - Throwing Baseballs

Domain: Building Functions F.BF

▶ **Cluster:** *Build a function that models a relationship between two quantities.*

Standard: F.BF.1

Use functions to model real-world relationships.

F.BF.1b. **(11)** Determine an **explicit expression** , a **recursive function**, or steps for calculation from a context. **(F.BF.1a)**

F.BF.1c. **(11)** Compose functions. *For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.* **(F.BF.1c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: A.CED, F.LQE

Explanations and Examples:

From context, write an explicit expression, define a recursive process, or describe the calculations need to model a function between tow quantities.

Students will analyze a given problem to determine the function expressed by identifying patterns in the function's rate of change. They will specify intervals of increase, decrease, constancy, and, if possible, relate them to the function's description in words or graphically. Students may use graphing calculators or programs, spreadsheets, or computer algebra systems to model functions.

Examples:

1. You buy a $10,000 car with an annual interest rate of 6% compounded annually and make monthly payments of $250. Express the amount remaining to be paid off as a function of the number of months, using a recursion equation.

2. A cup of coffee is initially at a temperature of 93º F. The difference between its temperature and the room temperature of 68º F decreases by 9% each minute.

   Write a function describing the temperature of the coffee as a function of time.

3. The radius of a circular oil slick after t hours is given in feet by $r = 10t^2 - 0.5t$, for $0 \leq t \leq 10$.
   Find the area of the oil slick as a function of time.

4. You are making an open box out of a rectangular piece of cardboard with dimensions 40 cm by 30 cm, by cutting equal squares out of the four corners and then folding up the sides.
   How big should the squares be to maximize the volume of the box? Draw a diagram to represent the problem and write an appropriate equation to solve.

5. A new social networking website was made available. The website had 10 members its first week. Beginning the second week, the creators of the website have a goal to triple the number of members every week.

6. For *Part A* and *Part B* below, select the appropriate expression for each blank region.
   To place an expression in a region, click on the expression, move the pointer over the region, and click again to place the expression in the region. Only one expression can be placed in each region. To return all expressions to their original positions, click the Reset button.

0	1	3	7	10
$3n+7$	$3n+10$	$30(n-1)$	$10(3^{n-1})$	$3(10^{n-1})$
$f(n-1)+2$	$f(n-1)+30$	$3f(n-1)$	$3f(n-1)+10$	$f(3n-1)$

Part A

Determine an explicit formula for $f(n)$, the number of members the creators have a goal of getting n weeks after the website is made available.

$f(n) =$

Part B

Determine a recursive formula for $f(n)$.

$f(n) =$	for $n >$
$f(1) =$	

Solution: *Part A:* $f(n) = 10(3^{n-1})$

Part B: $f(n) = 3f(n-1)$ for $n > 1$

$f(1) = 10$

Instructional Strategies:

Provide a real-world example (e.g., a table showing how far a car has driven after a given number of minutes, traveling at a uniform speed), and examine the table by looking "down" the table to describe a recursive relationship, as well as "across" the table to determine an explicit formula to find the distance traveled if the number of minutes is known.

Write out terms in a table in an expanded form to help students see what is happening. For example, if the *y*-values are 2, 4, 8, 16, they could be written as 2, 2(2), 2(2)(2), 2(2)(2)(2), etc., so that students recognize that 2 is being used multiple times as a factor.

Focus on one representation and its related language – recursive or explicit – at a time so that students are not confusing the formats.

Provide examples of when functions can be combined, such as determining a function describing the monthly cost for owning two vehicles when a function for the cost of each (given the number of miles driven) is known.

Using visual approaches (e.g., folding a piece of paper in half multiple times), use the visual models to generate sequences of numbers that can be explored and described with both recursive and explicit formulas. Emphasize that there are times when one form to describe the function is preferred over the other.

Hands-on materials (e.g., paper folding, building progressively larger shapes using pattern blocks, etc.) can be used as a visual source to build numerical tables for examination.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-BF.A.1
  - The Skeleton Tower
  - The Summer Intern
  - Kimi and Jordan
  - A Sum of Functions
  - Lake Algae
- F-BF.A.1.a
  - Susita's Account
  - Compounding with a 5% Interest Rate
  - Compounding with a 100% Interest Rate

Domain: Building Functions (F.BF)

● **Cluster:** *Build new functions from existing functions*

Standard: F.BF.3

(9/10/11) Transform parent functions $(f(x))$ by replacing $f(x)$ with $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. For **(9/10)** focus on linear, quadratic, and absolute value functions. **(F.BF.3)**

Suggested Standards for Mathematical Practice (MP):

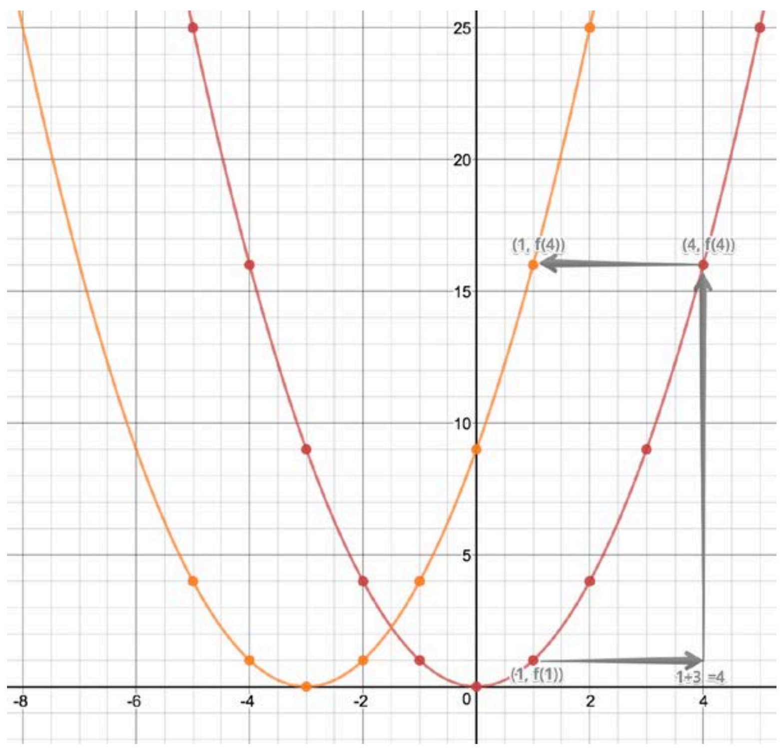
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.IF.1, F.IF.7

Explanations and Examples:

This standard defines transformations using reasoning function notation so students should investigate transformations by reasoning about functions using function notation. Students could use tables of values, equations, or graphs but are only asked to find the value of *k* when given the graph. Students should also be able to investigate and illustrate the effect on the graph using technology. The focus in 9/10 should be on linear, quadratic and absolute value functions. Because of this focus, identifying even and odd functions is reserved for 11th grade.

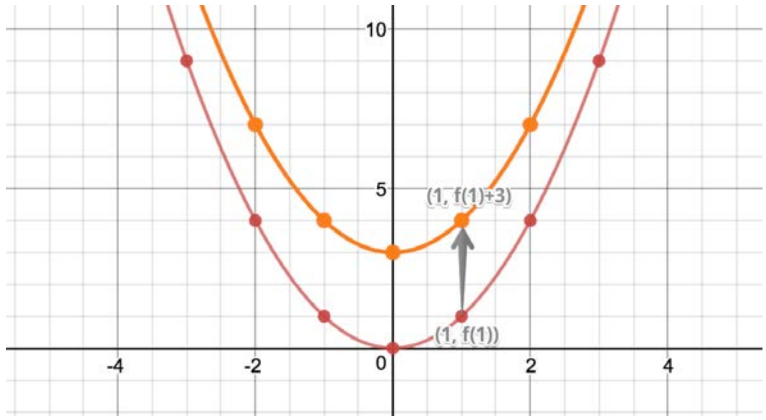
To investigate what it means to view transformations through the perspective of function notation, compare two transformations $f(x+k)$ and $f(x)+k$. Restating these transformations, the first adds the value *k* to the input while the second adds the value to the output. Students also need to know the parent function, which might be presented as an equation, table, or graph.



If *k=3* then the first transformation is $f(1+3) = f(4) = 16$. So if the input is *x=1* then the output is the output associated with $f(4)$=16.

If *x=-3*, then the output is $f(-3+3) = f(0) = 0$

After several examples, students should notice that the output shifts three units to the left.



The other function transformation, $f(x) + k$, adds the value k to the output instead of the input.
If *k=3* and x=1, then the transformation $f(1) = 1 + 3 = 4$.

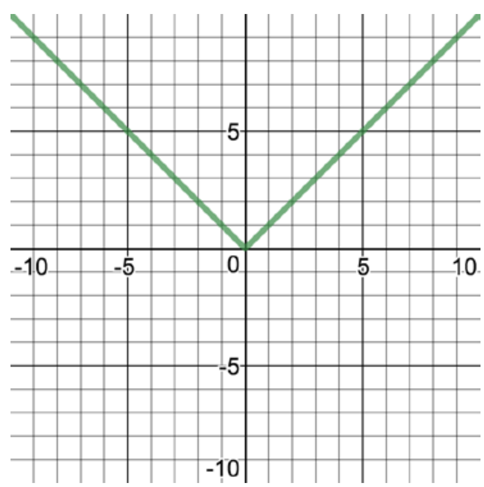
If *x=-3*, then the output is
$f(-3) = 9 + 3 = 12$

After a few examples, students can predict that changes to the input will affect the graph horizontally while changes to the output will affect the graph vertically. Students could then verify their conclusions using technology to make a prediction and then verify that their prediction was accurate.

Exploring transformations from an algebraic perspective provides another view of transformations and, in the case of linear and absolute value functions, explores different but equivalent forms of the same function. For example, if the parent function is $f(x) = 2x + 3$, justify that $f(x - 1) = f(x) - 2$ using at least two different representations. An example that might be even more confusing for students is when the parent function is $f(x) = x$. In this situation $f(x + 3) = f(x) + 3$ and $(-2x) = -2f(x)$. If these transformations are studied separate from quadratic functions, students might overgeneralize and conclude that it does not make a difference if the change happens to the input or output.

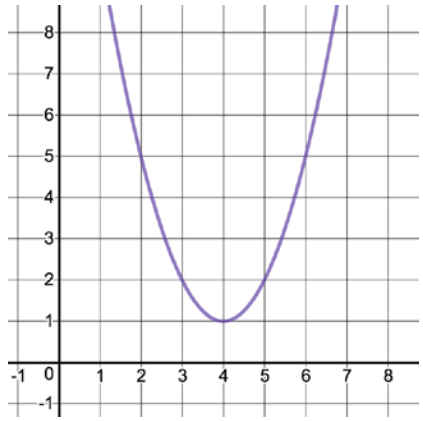
Examples:

1. Describe how to draw the parent function $y = |x|$ by hand, without technology.
2. Draw the following functions on the provided graph:
    a. $f(x - 1)$
    b. $f(x) + 3$
    c. $2f(x)$
    d. $f(-\frac{1}{3}x)$



3. Explain, algebraically why $f(x) = x^2$ and $f(x) = (-x)^2$ are equivalent by $f(x) = -x^2$ is not equivalent.

4. Write the function transformation, using function notation, that produces the following graph:



Instructional Strategies:

This standard provides a great opportunity to reinforce some ALL standards. Specifically, reinforcing the meaning of function notation can strengthen students understanding of both functions and function notation. Helping students focus on the key features for these three function families can help them find more efficient strategies for transforming the function. Using multiple representations will continue to help students think flexibly about functions. So while this standard can support F.IF.7, which focuses on graphing, students should experience transformations with a table of values and algebraically.

Desmos polygraph activities and marble slide activities are engaging opportunities to explore and practice transformation.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-BF.B.3
  - Medieval Archer
  - Building a Quadratic Function
  - Transforming the Graph of a Function

Desmos:

- Function Transformations: Practice with Symbols
- Function Transformations Bundle

Domain: Building Functions F.BF

Cluster: *Build new functions from existing functions.*

Standard: F.BF.4

Find inverse functions.

F.BF.4a. **(11)** Write an expression for the inverse of a function. **(F.BF.4a)**

F.BF.4b. **(11)** Read values of an inverse function from a graph or a table, given that the function has an inverse. **(F.BF.4c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: F.BF.3

Explanations and Examples:

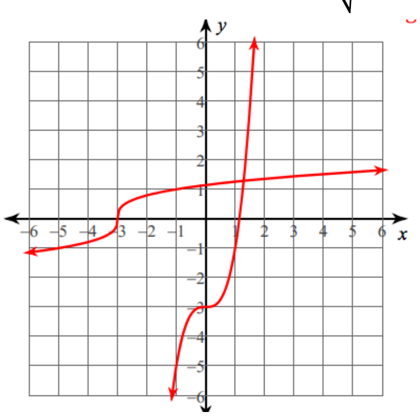
To write the inverse of a function the input and output are replaced. The inverse value can be found using the inverse function, the graph, or a table.

Examples:

1. Find the inverse of the function, if it exists.
   - $f(x) = -\frac{3}{-x-3} - 2$
   - $f(x) = \sqrt[5]{x+1} + 2$

2. Use the graph to find the value of the inverse.
   - Find $g^{-1}(4)$

$$g(n) = 2n^3 - 3, g^{-1}(n) = \sqrt[3]{\frac{n+3}{2}}$$



3. Use the table of values to find $f^{-1}(0)$.

x	*f(x)*
0	-1.5874
1	-1.4422
2	-1.2599
3	-1
4	0
5	1
6	1.2599

Instructional Strategies:

Provide examples of inverses that are not purely mathematical to introduce the idea. For example, given a function that names the capital of a state, *f*(Ohio) = Columbus. The inverse would be to input the capital city and have the state be the output, such that f^{-1}(Denver) = Colorado.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-BF.B.4.a
    - Temperature Conversions

Domain: Building Functions F.BF

Cluster: *Build new functions from existing functions.*

Standard: F.BF.5

(11) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents. **(F.BF.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF.7, A.REI.9, A.REI.3b

Explanations and Examples:

Understanding that a logarithm is the inverse of an exponential relationship can give students the ability to understand logarithms at a conceptual level.

Examples:

1. What is the inverse of the function $f(x) = 4^x$? Justify that the function is the inverse using a graph or table of values.

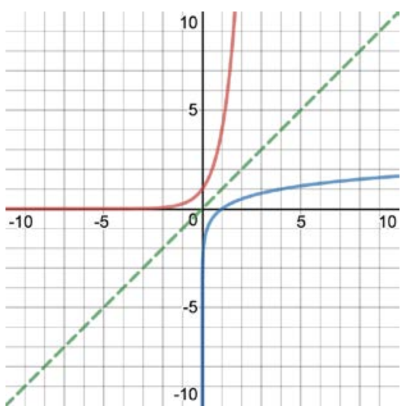
Solution:

$x = 4^y$

$\log_4 x = \log_4 4^y$

$\log_4 x = y$

$f^{-1}(x) = \log_4 x$



x	4^x
1	4
2	16
3	64
4	256
5	1024

x	$\log_4(x)$
4	1
16	2
64	3
256	4
1024	5

Since the graph is reflected across the line y=x, the two graphs are inverses. For the table, the input and output columns are reversed, illustrating that the two functions are inverses.

2. For the function, $f(x) = 3^{x-1}$, find x when $f(x) = 3$.

Solution:

$3 = 3^{x-1}$

$\log_3 3^1 = \log_3 3^{x-1}$

$1 = x - 1$

$x = 2$

Instructional Strategies:

Connecting the inverse relationship between exponential functions and logarithms to other inverse functions studied in Algebra 2 can help students build on their prior knowledge.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-BF.B.4.b
  - Exponentials and Logarithms 1
- F-BF.B.5
  - Exponential Kiss

Domain: Linear, Quadratic, and Exponential Models★ F.LQE

- **Cluster:** *Construct and compare linear, quadratic, and exponential models and solve problems.*

Standard: F.LQE.1 ★

Distinguish between situations that can be modeled with linear functions and with exponential functions. ★

F.LQE.1a. **(11)** Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. ★ **(F.LQE.1a)**

F.LQE.1b. **(11)** Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. ★ **(F.LQE.1b)**

F.LQE.1c. **(11)** Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another. ★ **(F.LQE.1c)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.LQE.1-4

Explanations and Examples:

Given a contextual situation, describe whether the situation in question has a linear pattern of change or an exponential pattern of change.

Show that linear functions change at the same rate over time and that exponential functions change by equal factors over time.

Describe situations where one quantity changes at a constant rate per unit interval as compared to another.

Describe situations where a quantity grows or decays at a constant percent rate per unit interval as compared to another.

Students may use graphing calculators or programs, spreadsheets, or computer algebra systems to model and compare linear and exponential functions.

Examples:

1. A cell phone company has three plans. Graph the equation for each plan, and analyze the change as the number of minutes used increases. When is it beneficial to enroll in Plan 1? Plan 2? Plan 3?
   - $59.95/month for 700 minutes and $0.25 for each additional minute,
   - $39.95/month for 400 minutes and $0.15 for each additional minute, and
   - $89.95/month for 1,400 minutes and $0.05 for each additional minute.
2. A computer store sells about 200 computers at the price of $1,000 per computer. For each $50 increase in price, about ten fewer computers are sold. How much should the computer store charge per computer in order to maximize their profit?

Students can investigate functions and graphs modeling different situations involving simple and compound interest. Students can compare interest rates with different periods of compounding (monthly, daily) and compare them with the corresponding annual percentage rate. Spreadsheets and applets can be used to explore and model different interest rates and loan terms.

3. A couple wants to buy a house in five years. They need to save a down payment of $8,000. They deposit $1,000 in a bank account earning 3.25% interest, compounded quarterly. How much will they need to save each month in order to meet their goal?

4. Sketch and analyze the graphs of the following two situations. What information can you conclude about the types of growth each type of interest has?
   - Lee borrows $9,000 from his mother to buy a car. His mom charges him 5% interest a year, but she does not compound the interest.
   - Lee borrows $9,000 from a bank to buy a car. The bank charges 5% interest compounded annually.

5. Calculate the future value of a given amount of money, with and without technology.

6. Calculate the present value of a certain amount of money for a given length of time in the future, with and without technology.

7. The data in the table was taken from Wikipedia.

   a. Based on the data in the table, would a linear function be appropriate to model the relationship between the world population and the year? Explain how you know.

   b. Using only the data from 1960 onward in the above table, would a linear function be appropriate to approximate the relationship between the world population and the year? Explain how you know.

Year	World Population (Estimate)
1804	1,000,000,000
1927	2,000,000,000
1960	3,000,000,000
1974	4,000,000,000
1987	5,000,000,000
1999	6,000,000,000
2012	7,000,000,000

   c. Based on your work in parts a. and b., would a linear function be appropriate to predict the world population in 2200? Explain.

8. Carbon 14 is a common form of carbon which decays exponentially over time. The half-life of Carbon 14, that is the amount of time it takes for half of any amount of Carbon 14 to decay, is approximately 5730 years. Suppose we have a plant fossil and that the plat, at the time it died, contained 10 micrograms of Carbon 14 (one microgram is equal to one millionth of a gram).

   a. Using this information, make a table to calculate how much Carbon 14 remains in the fossilized plant after $5730 \times n$ years for $n = 0, 1, 2, 3, 4$.
   b. What can you conclude from part a. about when there is one microgram of Carbon 14 remaining in the fossil?
   c. How much carbon remains in the fossilized plant after $2865 = \frac{5730}{2}$ years? Explain how you know.
   d. Using the information from part c., can you give a more precise response to when there is one microgram of Carbon 14 remaining in the fossilized plant?

Instructional Strategies:

Compare tabular representations of a variety of functions to show that linear functions have a first common difference (i.e., equal differences over equal intervals), while exponential functions do not (instead function values grow by equal factors over equal *x*-intervals).

Apply linear and exponential functions to real-world situations. For example, a person earning $10 per hour experiences a constant rate of change in salary given the number of hours worked, while the number of bacteria on a dish that doubles every hour will have equal factors over equal intervals.

Provide examples of arithmetic and geometric sequences in graphic, verbal, or tabular forms, and have students generate formulas and equations that describe the patterns.

Use a graphing calculator or computer program to compare tabular and graphic representations of exponential and polynomial functions to show how the *y* (output) values of the exponential function eventually exceed those of polynomial functions.

Have students draw the graphs of exponential and other polynomial functions on a graphing calculator or computer utility and examine the fact that the exponential curve will eventually get higher than the polynomial function's graph. A simple example would be to compare the graphs (and tables) of the functions $y = x^2$ and $y = 2x$ to find that the y values are greater for the exponential function when $x > 4$.

Help students to see that solving an equation such as $2x = 300$ can be accomplished by entering $y = 22$ and $y = 300$ into a graphing calculator and finding where the graphs intersect, by viewing the table to see where the function values are about the same, as well as by applying a logarithmic function to both sides of the equation.

Explore simple linear and exponential functions by engaging in hands-on experiments. For example, students can measure the diameters and related circumferences of several circles and determine a linear function that relates the diameter to the circumference – a linear function with a first common difference. They can then explore the value of an investment when told that the account will double in value every 12 years – an exponential function with a base of 2

Common Misconceptions:

Students may believe that all functions have a first common difference and need to explore to realize that, for example, a quadratic function will have equal second common differences in a table.

Students may also believe that the end behavior of all functions depends on the situation and not the fact that exponential function values will eventually get larger than those of any other polynomial functions.

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-LE.A
  - Comparing Exponentials
  - What functions do two graph points determine?
  - Rising Gas Prices
  - Extending the Definitions of Exponents

"Modeling Having Kittens" – Mathematics Assessment Project: This lesson unit is intended to help you assess how well students are able to:

- Interpret a situation and represent the constraints and variables mathematically.
- Select appropriate mathematical methods to use.
- Make sensible estimates and assumptions.
- Investigate an exponentially increasing sequence.
- Communicate their reasoning clearly

Domain: Linear, Quadratic, and Exponential Models★ F.LQE

▶ **Cluster:** *Construct and compare linear, quadratic, and exponential models and solve problems.*

Standard: **F.LQE.2 ★**

(11) Construct exponential functions, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). ★ **(F.LQE.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.BF.1-2

Explanations and Examples:

In constructing linear functions in F.LQE.2 draw on and consolidate previous work in Grade 8 on finding equations for lines and linear functions. In 8th grade students work on identifying slope and unit rates for linear functions given two points, a table or a graph.

Students may use graphing calculators or programs, spreadsheets, or computer algebra systems to model and compare linear and exponential functions.

Examples:

1. Determine an exponential function of the form $(x) = ab^2$ using data points from the table.
   Graph the function and identify the key characteristics of the graph.

x	*f*(*x*)
0	1
1	3
3	27

2. Sara's starting salary is $32,500. Each year she receives a $700 raise.
   Write a sequence in explicit form to describe the situation.

3. Solve the equation $2^x = 300$.

   Sample Response:
   Using a graphing calculator enter $y = 2^x$ and $y + 300$. Find where the graphs intersect by viewing the table to see where the function values are about the same.

4. Albuquerque boasts one of the longest aerial trams in the world. The tram transports people up to Sandia Peak. The table shows the elevation of the tram at various times during a particular ride.

Minutes into the Ride	2	5	9	14
Elevation in Feet	7069	7834	8854	10,129

a. Write an equation for a function (linear, quadratic, or exponential) that models the relationship between the elevation of the tram and the number of minutes into the ride. Justify your choice.

b. What was the elevation of the tram at the beginning of the ride?

c. If the ride took 15 minutes, what was the elevation of the tram at the end of the ride?

Solution:

a. The average rate of change in height with respect to time between each column in the table is the same:

$$\frac{7834-7069}{5-2}=\frac{8854-7834}{9-5}=\frac{10129-8854}{14-9}=255.$$

Therefore we choose a linear function to model the relationship. If y represents the elevation of the tram in feet and x represents the number of minutes into the trip, then $y - 7069 = 255(x-2)$ for each pair (x,y) in the table, so the linear function given by $y = 7069 + 255(x-2)$ works.

b. When $x = 0$ $y = 7069 + 255(0-2) = 6559$.

So, the elevation of the tram at the beginning of the ride is 6559 feet.

c. When $x = 15$ $y = 7069 + 255(15-2) = 10{,}384$.

So, the elevation of the tram at the end of the ride is 10,384 feet.

Instructional Strategies: See F.LQE.1

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-LE.A.2
  - Rumors
  - Sandia Aerial Tram
  - Basketball Rebounds
  - Points Determine an Exponential Function I
  - Boiling Water
  - Two Points Determine an Exponential Function II
  - Choosing an appropriate growth model
  - Basketball Bounces I
  - Finding Parabolas through Two Points

High School – Statistics & Probability

Domain: Interpret Categorical and Quantitative Data S.ID

● **Cluster:** *Interpret linear models*

Standard: **S.ID.7**

(11) Compute (using technology) and interpret the correlation coefficient of a linear fit. **(S.ID.8)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF

Explanations and Examples:

Explain that the correlation coefficient must be between –1 and 1 inclusive and explain what each of these values means. Determine whether the correlation coefficient shows a weak positive, strong positive, weak negative strong negative, or no correlation.
Use the correlation coefficient to determine if a linear model is a good fit for the data (significant).

Students may use spreadsheets, graphing calculators and statistical software to represent data, describe how the variables are related, fit functions to data, perform regressions, and calculate residuals and correlation coefficients.

Examples:

1. The correlation coefficient of a given data set is 0.97. List three specific things this tells you about the data.
2. Collect height, shoe-size, and wrist circumference data for each student. Determine the best way to display the data. Answer the following questions.
    1. Is there a correlation between any two of the three indicators?
    2. Is there a correlation between all three indicators?
    3. What patterns and trends are apparent in the data?
    4. What inferences can be made from the data?
3. Hypothesize the correlation between two sets of related data. Gather data to support or refute your hypothesis.

Instructional Strategies:

In this cluster, the key is that two quantitative variables are being measured on the same subject. The paired data should be listed and then displayed in a scatterplot. If time is one of the variables, it usually goes on the horizontal axis. That which is being predicted goes on the vertical; the predictor variable is on the horizontal axis.

Note that unlike a two-dimensional graph in mathematics, the scales of a scatterplot need not be the same, and even if they are similar (such as SAT Math and SAT Verbal), they still need not have the same spacing. So, visual rendering of slope makes no sense in most scatterplots, i.e., a 45 degree line on a scatterplot need not mean a slope of 1.

Using a number line between -1 and 1 to illustrate how the "cloud of points" changes across the possible correlations can help students see the pattern. Also, practice estimating the correlation can help check for understanding. Also, connecting correlation to other measures of spread such as MAD or SD can help students see that this is simply a measure of how far each point is away from the line of best fit.

Common Misconceptions:

That when two quantitative variables are related, i.e., correlated, that one causes the other to occur. Causation is not necessarily the case. For example, at a theme park, the daily temperature and number of bottles of water sold are demonstrably correlated, but an increase in the number of bottles of water sold does not cause the day's temperature to rise or fall.

Resources/Tools:

"Descriptive Statistics" - EngageNY Algebra I Module 2: (This Module includes lessons for standards S.ID. 1-3,5-9)

"Devising a Measure for Correlation" – Mathematics Assessment Project: This lesson unit is intended to help you assess how well students understand the notion of correlation. In particular this unit aims to identify and help students who have difficulty in:

- Understanding correlation as the degree of fit between two variables.
- Making a mathematical model of a situation.
- Testing and improving the model.
- Communicating their reasoning clearly.
- Evaluating alternative models of the situation.

Illustrative Mathematics High School Statistics & Probability tasks: **Scroll to the appropriate section to find named tasks.**

- S-ID.C.8
  - Coffee and Crime

Domain: Interpret Categorical and Quantitative Data S.ID

● **Cluster:** *Interpret linear models*

Standard: S.ID.8

(11) Distinguish between correlation and causation. **(S.ID.9)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.6 Attend to precision

Connections: See S.ID.7

Explanations and Examples:

Understand and explain the difference between correlation and causation.

Understand and explain that a strong correlation does not mean causation.

Determine if statements of causation seem reasonable or unreasonable and justify reasoning.

Choose two variables that could be correlated because one is the cause of the other; defend and justify selection of variables.

Choose two variables that could be correlated even though neither variable could reasonably be considered to be the cause of the other; defend and justify selection of variables.

Some data leads observers to believe that there is a cause and effect relationship when a strong relationship is observed. Students should be careful not to assume that correlation implies causation. The determination that one thing causes another requires a controlled randomized experiment.

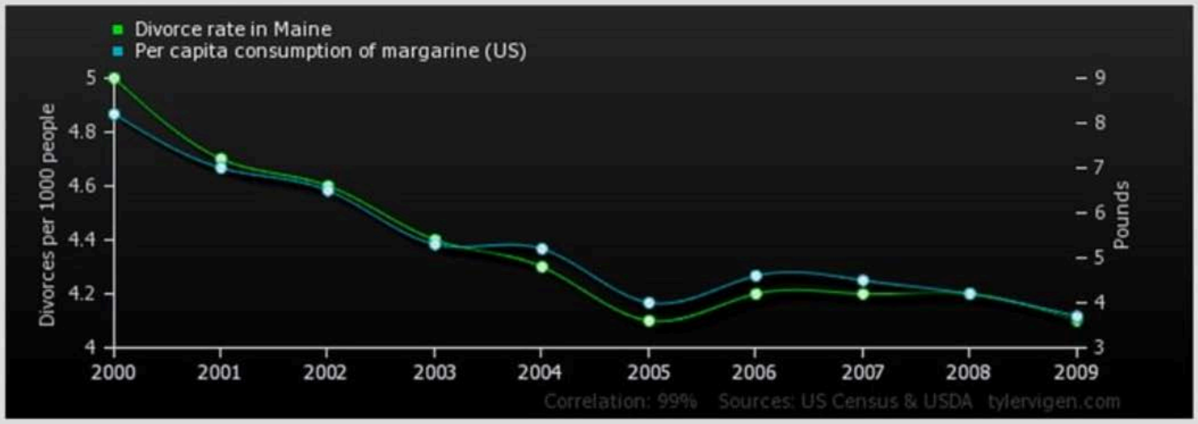
Examples:

1. Diane did a study for a health class about the effects of a student's end-of-year math test scores on height. Based on a graph of her data, she found that there was a direct relationship between students' math scores and height. She concluded that "doing well on your end-of-course math tests makes you tall."

   Is this conclusion justified? Explain any flaws in Diane's reasoning.

Instructional Strategies:

Discuss data that has correlation but no causation (height vs. foot length). There are also a plethora of resources listed spurious correlations that illustrate how absurd assuming correlation means causation. For example, from http://twentytwowords.com/funny-graphs-show-correlation-between-completely-unrelated-stats-9-pictures/



	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Divorce rate in Maine *Divorces per 1000 people (US Census)*	5	4.7	4.6	4.4	4.3	4.1	4.2	4.2	4.2	4.1
Per capita consumption of margarine (US) *Pounds (USDA)*	8.2	7	6.5	5.3	5.2	4	4.6	4.5	4.2	3.7
Correlation: 0.992558										

Note there is almost a perfect correlation between the divorce rate in Maine and the consumption of margarine.

Discuss data that has correlation and causation (number of M&Ms in a cup vs. weight of the cup).

Resources/Tools: See S.ID.7

APPENDIX: TABLE 1 *The Properties of Operations*

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c=a+(b+c)$	$(78+25)+75=78+(25+75)$
Commutative	$a+b=b+a$	$2+98=98+2$
Additive Identity	$a+0=a$ *and* $0+a=a$	$9875+0=9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a+-a=-a+a=0$	$-47+47=0$
Properties of Multiplication		
Associative	$(a\times b)\times c=a\times(b\times c)$	$(32\times 5)\times 2=32\times(5\times 2)$
Commutative	$a\times b=b\times a$	$10\times 38=38\times 10$
Multiplicative Identity	$a\times 1=a$ *and* $1\times a=a$	$387\times 1=387$
Multiplicative Inverse	*For every real number a,* $a\neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$\frac{8}{3}\times\frac{3}{8}=1$
Distributive Property of Multiplication over Addition		
Distributive	$a\times(b+c)=a\times b+a\times c$	$7\times(50+2)=7\times 50+7\times 2$

(Variables a, b, and c represent real numbers.)

Excerpt from NCTM's ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

TABLE 2. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for a in any expression containing a.	$If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 3. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.
$If\ a > b\ and\ b > c\ then\ a > c.$
$If\ a > b, then\ b < a.$
$If\ a > b, then\ -a < -b.$
$If\ a > b, then\ a \pm c > b \pm c.$
$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$
$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$
$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$
$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

Table 4. Cognitive Rigor Matrix/Depth of Knowledge (DOK)

Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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