Mathematical Mind Journeys: Awakening Minds to Computational Fluency

Have you ever wondered what your students are really thinking as they do mathematics? Do you wish that you could stimulate and excite your students while building the fundamental skills necessary for their future? Join us as we take our students on a Mathematical Mind Journey. This learning adventure does not require time-intensive planning or expensive props and materials. Mathematical Mind Journeys are "think aloud" strategies that demystify computation. Students use metacognition to explain the paths that their brains take when solving a problem and rely on mathematical memory rather than memorization. Whether you have a few minutes or a class period, a Mathematical Mind Journey will empower and engage every student in your class.

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Connecticut's New Canaan Public Schools has a mission of mathematical literacy for all children and has undergone an extensive curriculum review process over the past six years. Designing reformed curriculum and ensuring computational fluency have been shared visions for the district's mathematics educators, including the authors of this article. We are three teachers who have taught across levels K–12 and have led the mathematics initiative as mathematics resource teachers, mathematics coordinators, and building administrators. We coined the term "Mathematical Mind Journey," or "MMJ," to describe the adventure we take our students on each day. MMJs support the Process and Content Standards of the National Council of Teachers of Mathematics. Children grapple with and talk about their mathematical thinking in number, data, and position.

The MMJ aligns with NCTM's Principles and Standards, as well as any K–12 mathematics curriculum, and can be used throughout the year. Children learn to reason, problem-solve, communicate, and compute in a rich, meaningful way. Teachers pose the mathematical question or task, probe and question, assess understanding, support children during disequilibrium, and challenge students to reach new levels of understanding. This mathematics classroom is a community where everyone's ideas are respected and valued while students are empowered to work smarter, not just harder. For many students who have mastered the procedural part of computation, the process of mathematical thinking remains an elusive concept. The MMJ is not about a picture-perfect lesson. It is about discovery, disequilibrium, and discourse, and it can be messy. It is about students constructing meaning

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and using what they know to solve what they do not know.

Many people have memories of a very quiet mathematics classroom in which they sat alone with a worksheet filled with similar problems. As long as they could follow the teacher's directions and mimic her actions, they were successful. Now, teachers are moving beyond the use of flash cards and worksheets to practice facts. Our students are expected to think mathematically and not learn concepts and algorithms only by rote. Mathematical Mind Journeys, mathematics games, mathematics journals, daily routines, mathematics literature, and performance tasks help our students develop mathematical sense and fluency. Our students not only process the numerical operations but also think about number sense and how numbers connect to their everyday lives. They construct meaning from these numbers, and they often will process left to right in a way that is much more algebraic than "rote-algorithmic." Students use flexibility, fluency, and accuracy when solving problems. They communicate their reasoning in pictures, numbers, and words and orally, in writing, and by exhibition.

MMJ Strategies in Action

Mathematical Mind Journeys have become common language around our schools. In a second-grade classroom, students share their thinking about the problem 84 + 16 (see **fig. 1**). The teacher writes the problem on the overhead projector and asks students to think about the problem, find a reasonable estimate for an answer, then solve it mentally, making sure that they can share how they solved it. After a few minutes of quiet time to think

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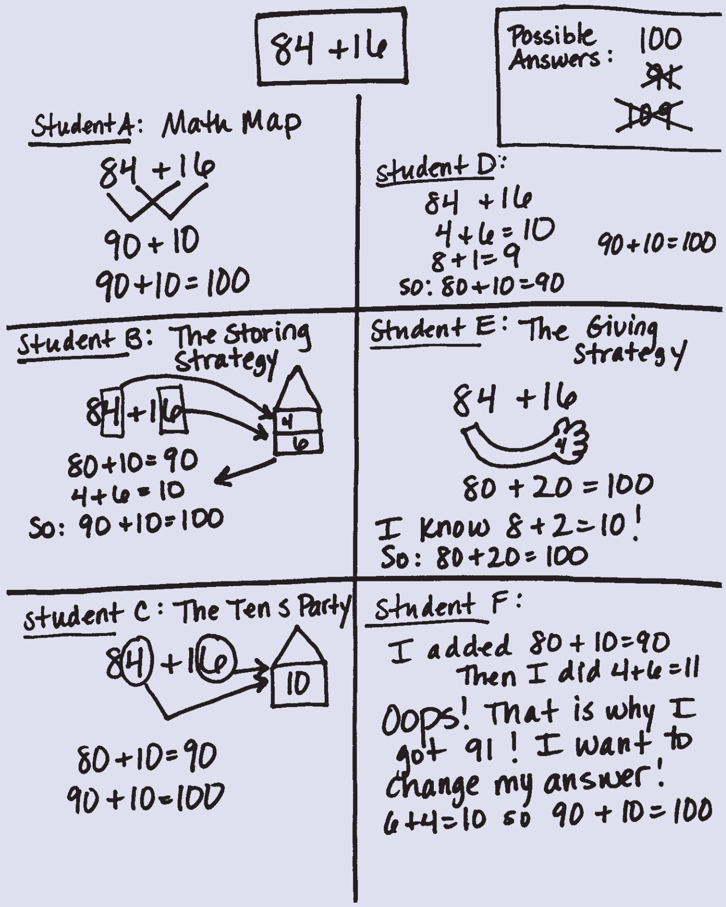
and reflect about the problem, the children indicate their readiness to share their answers with a "thumbs up" in front of their chest. The teacher then asks the students for possible answers. **Figure 1** shows what is recorded as the students share their ideas. Strategies that students named are posted around the room for students to refer to. Seeing children's eyes light up when they ask, "Should I think of a name for that strategy?" is always exciting. Of course, the answer is yes.

The storing strategy

A student who decided to take part of the number and "put it to the side of his brain" named this strategy. When asked, "What do you mean, you put it to the side of your head?" the student responded, "Oh, I take part of the number and put it in a little storage shed for later. I deal with part of the problem and then take those numbers out of storage again."

FIGURE 1

Students describe their thinking about 84 + 16.



The giving strategy

A student in this second-grade class said to the "Math Congress," a group of students who share ideas and strategies, "In my head I decided to take some of the number 84 and give it to the 16. I always look to make landmark numbers, like tens that are easy to work with, so 84 gives 4 to 16. This changes the problem to 80 + 20, which I know is 100. Eighty and 20 are easier numbers to work with. I think this strategy should be called the Giving Strategy because sometimes one number can 'give' some to another to make the problem easier."

The tens-party strategy

"I always look for tens in a problem," said one student. "Then we should call that the Tens-Party Strategy," exclaimed another student. These second graders constantly search for numbers that are easier to work with.

The math-map strategy

A second grader coined the term "Math Map" to describe his strategy: "Sometimes my brain looks like a road map. I often have arrows and carets connecting the tens and ones. This way I know what numbers I can join together."

Ownership of their strategies enables students to take hold of their mathematical minds and make decisions about what they should do. MMJs create student awareness of who they are as mathematical thinkers and learners. Students choose strategies that work for them and make sense, and that they know will work so they can be successful. The names of the strategies often are linked to broader mathematical terms and properties such as compensation and distribution.

Computational fluency is NCTM's answer to knowing the basic number facts and understanding them. To build computational fluency, students must be flexible, efficient, and accurate. The MMJ is a way to promote computational fluency with its rich discussion and discourse, problem-solving approach, and building blocks for mathematical memory. As Russell (1999b) states, "This kind of teaching leads students not to memorizing, but to the development of *mathematical memory*. Important mathematical procedures cannot be 'forgotten over the summer,' because they are based in a web of connected ideas about fundamental mathematical relationships."

MMJ as a Warm-Up

Miss G's third-grade class just came in from recess, and it is time for mathematics. She puts a problem on the overhead projector ($16 \times 4 = n$) and says that it is time for a ten-minute MMJ (see **fig. 2**). The students quickly scurry to their seats. The energy in the room is palpable. Miss G says to the class, "We may not have time for everyone to share, so please record your MMJ in your own journal. That way I can look at your strategies later." After about two minutes, the students are ready to share. One student explains that he used addition to solve the problem. Miss G records his strategy in numerical form. "Does someone have a different strategy?" she asks. Another student decides to break apart 16×4 into 16×2 and 16×2. Once students have shared their ideas, the energy spills over into the next mathematical task.

MMJ as a Whole Lesson

What follows is an example of a whole-class MMJ, using the problem $362 + 214 = n$. The teacher instructs students to take as much time as they need to solve the problem in their heads. The students are reminded that they can jot down landmark numbers or important steps to help them explain their thinking, and they also can change their minds at any time. If they find a strategy quickly, they should try another strategy.

Teacher. It looks like we are ready to start. Does anyone need some more time? Who thinks they know what the answer is?

John. I think the answer is 576.

Julie. I think the answer is 566.

Tom. I agree with John. The answer is definitely 576.

Teacher. Does anyone else have a different idea of what they think the answer is?

Caroline. I think the answer might be 586, but I am not sure. I will probably change that answer once I find my mistake in my brain.

Teacher. Who would like to defend an answer and share their strategy?

Chad. I would like to defend the answer of 576. Here is what my brain did: First, I looked at the problem and decided that the best thing to do was to break apart the two numbers. So I took the 2 and 4 and put them off to the side in my brain.

Teacher. Why did your brain put them off to the side?

Chad. Well, I wanted them out of the way for a minute so I could deal with the rest of the problem. I will bring them back later. After I put the 2 and 4 off to the side, I looked at how many hundreds I had. I added 300 + 200, and that equals 500. Then I took the 60 and 10 and added them together. That equals 70. Finally, I added 2 and 4, and that equals 6.

Teacher. So your answer is what?

Chad. Well, while I was solving the problem I was keeping a note of the numbers and their value in my head as well. When I got 500, I put a 5 in the hundreds place holder because 5 is not 5 in this case; it is 500. So I put the 5 in the hundreds, the 7 in the tens, and the 6 in the ones; so my answer is 576. And that is my final answer.

Julie. I think we should name Chad's strategy and put it on the wall.

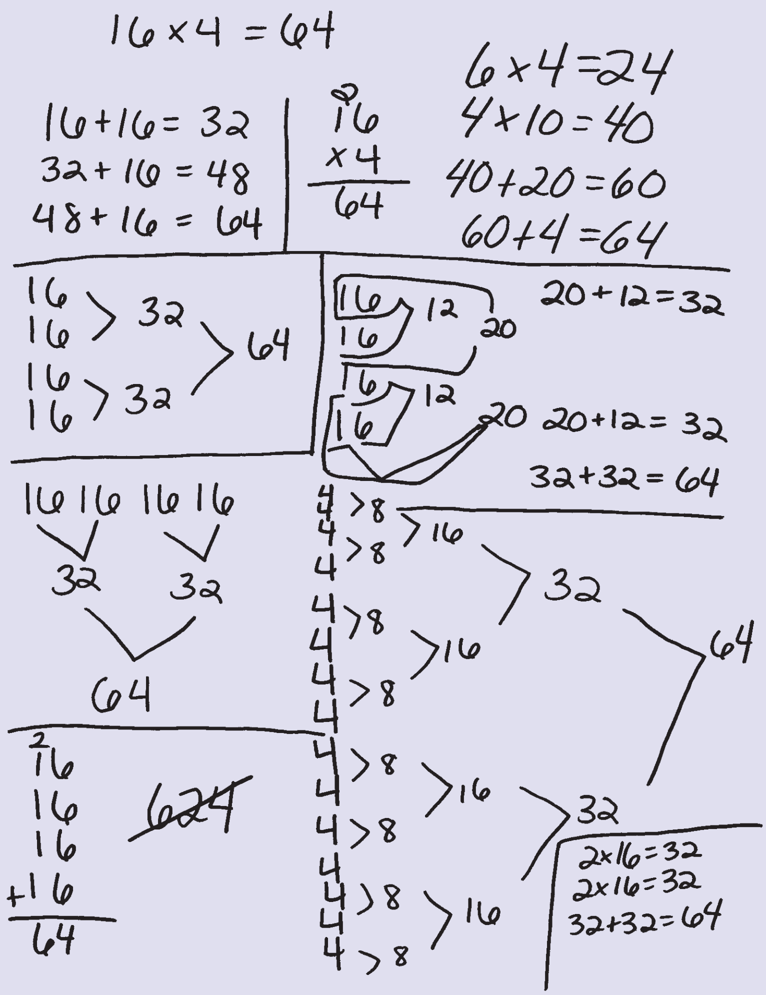
Tara. Me too. I think we should name it Chad's Off-to-the-Side Strategy.

Teacher. Does the class agree? If we agree, let's write it down and put it on the wall right now. [The teacher takes a fluorescent, jagged-edged sheet of card stock and writes out the strategy. It is taped on the wall near several others.] Does anyone have a different strategy or want to defend a different answer?

Caroline. I do! I found my mistake. I used Chad's Off-to-the-Side Strategy too. I put 200 and 300 off to the side and 60 and 10 off to the side. I added them up and got 580. That is where I made

FIGURE 2

Examples of MMJs for 16 × 4 = *n*



my mistake. 60 + 10 equals 70, not 80. I was over by 10 in my final answer, so I would like to change my answer from 586 to 576.

Julie. I would like to change my answer too. I did the same thing, except when I added 60 and 10, I got 70 but recorded 60. I was under by 10 but I know the answer is 576.

Ashley. I have another way to solve this problem. I think the answer is 576 too. I used the "break apart" strategy. I broke apart 362 into 300 + 60 + 2 and I broke apart 214 into 200 + 10 + 4. Then I added each up at a time. First, 300 + 200 = 500, then 500 + 60 = 560, then 560 +10 = 570, then 570 + 2 = 572, and finally, 572 + 4 = 576. That's what my brain did [see **fig. 3**].

During a whole-class MMJ, teachers must have a clear mathematical agenda for each class discussion. This often means that they should choose a strategy to discuss before class begins. During the

FIGURE 3

Ashley solved the problem by using a "break apart" strategy.

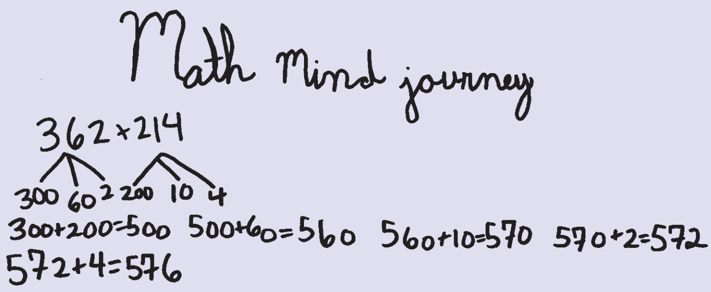
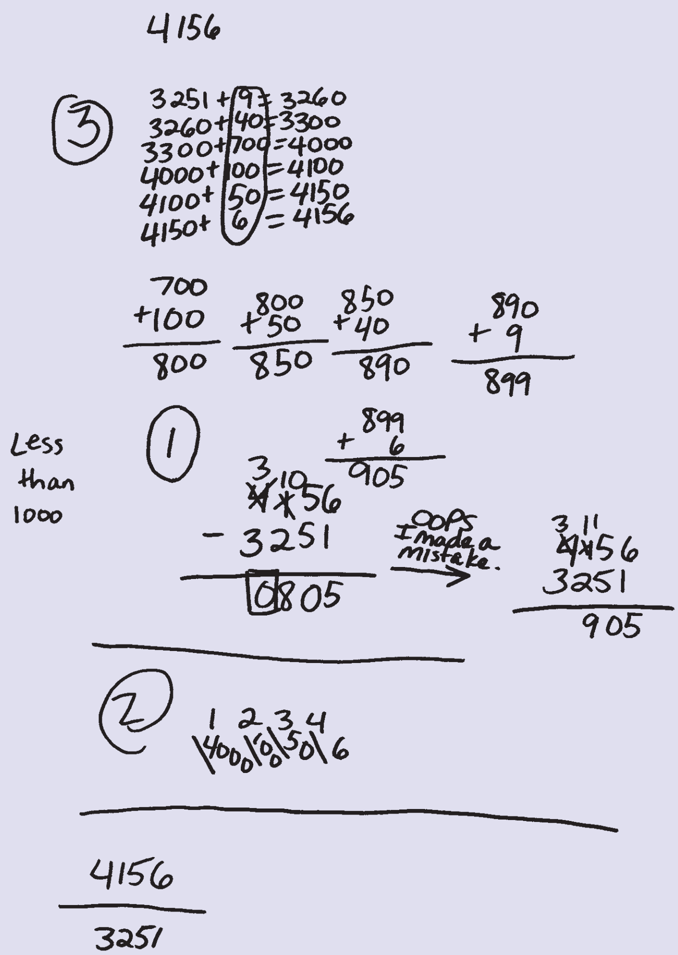


FIGURE 4

Emmy tries to find the difference between her target number and the number she makes with her numeral cards.



discussion, they might ask the class questions such as the following:

- Which strategy would you choose that would work for many problems?
- Which strategy is clearly written and easy to follow?
- How could you change this strategy to meet the needs of this situation?
- Which strategies are most efficient and why?

By asking these questions during an MMJ, teachers help students see what is important about computational fluency: efficiency, a clearly written explanation; flexibility, a demonstration of conceptual understanding; and accuracy, the right answer (Russell 1999a).

Children will choose a strategy or take a novel path that we teachers have not seen before, which causes us to explore a child's thinking and reasoning. It is not up to us to decide which pathways our students will use. However, we need to understand the mathematical landscape and ideas that we will encounter. In *Young Mathematicians at Work*, Dolk and Fosnot (2001) address the importance of teachers understanding the mathematical landscape that their students will traverse:

> Teaching is a planned activity. Madeline [the teacher] did not walk into her classroom in the morning wondering what to do. She had planned her lesson, and she knew what she expected her students to do. As the children responded, she acknowledged the differences in their thinking and in their strategies, and she adjusted her course accordingly. While she honored divergence, development, and individual differences, she also had identified landmarks along the way that grew out of her knowledge of mathematics and mathematical development. These helped her plan, question, and decide what to do next.

MMJ as One to One

Sometimes, during a mathematics lesson, you may want to pull a small group of children or an individual student aside and go on an MMJ "one to one." Anne, a first-grade teacher, always uses observation of her students during a mathematical task or game as a valid indicator of how they are thinking and where they are in the mathematical landscape. When a colleague asks her to observe a younger student and assess the child's mathematical thinking, she asks the child how she would show nine peas and carrots, using green and orange cubes. The child takes one green cube and one orange cube and counts 1, 2, and so on, until she reaches 9. She tells the teacher, "I have 9 now." Anne probes further: "How many peas and how many carrots do you have?" The student replies, "I have 5 peas and 4 carrots." Anne continues, "Can you think of another way to show 9 peas and carrots in all?" The student repeats the same steps, using one-to-one correspondence, and arrives at the same solution. With different questioning and continuous probing, the teacher is able to use the MMJ as an individualized diagnostic tool to assess the child's mathematical thinking and development.

Similarly, students in a fourth-grade classroom are playing a game called "The Digits Game" (Russell 1999a). In this activity, students have a target number and the goal is to make another number, from five numeral cards, that will get them as close to their target number as possible. Emmy, a fourth grader, has a target number of 3,251 (see **fig. 4**). She looks at her five numeral cards and decides to go with the number 4,156.

Teacher. Do you think that is pretty close?

Emmy. I think I am less than 1,000 away from my target number.

As Emmy works to find the difference between the two numbers, she first attempts a traditional "borrowing" algorithm.

Teacher. Do you think you could check your answer another way to see if you are right?

Emmy. Sure. I can show you another way.

Emmy starts with her target number—3,251—and adds on to it until she gets to her other number.

Teacher. What made your brain start with the target number and not the greater number?

Emmy. I know that I have to find how many numbers are between these two numbers. I can use addition or subtraction to find that out. I like to use addition because it is easier on my brain. It's better to go forward rather than backward.

Teacher. So what do you think about your answer now?

Emmy. I crossed out the 1 and made it a 10, but it should be 11, I think. I know I made a mistake with the subtraction rule. Next time, I will definitely think about what strategy to use before I start. Using addition was more efficient for this problem.

Both of these MMJs are in the context of a lesson or activity. Both students are able to share what they are thinking, and the teacher is able to assess where the students are on the horizon of mathematical understanding.

MMJ as Metacognition

Metacognition is often described as thinking about your own thinking. According to the ERIC Digest on "Metacognition and Reading to Learn," metacognition involves "both the conscious awareness and the conscious control of one's learning" (Collins 1994). Metacognition often is used in reading education circles as a strategy to improve reading comprehension. But it is just as relevant for improving students' mathematical problem solv-

ing. Schoenfeld (1987) asserts that creating a "mathematics culture" in a classroom is the best way to develop metacognition. Such a culture involves solving unfamiliar problems with your students, putting the problems on the board, and working on them together. He continues, "Students participate with the teacher, sometimes making mistakes and having to rethink where they have been. Such an approach exposes them to the process of thinking about the way a problem is being/could be solved. When they reflect on or talk about the process of problem solving, this is metacognition."

In an MMJ, students are constantly encouraged to take responsibility for their own learning. Students reflect on their own thinking in an authentic and meaningful way. A Mathematical Mind Journey allows students and teachers to practice the skill of asking themselves what they already know about a problem that might help them solve what they do not know. MMJs provide many opportunities for developing this skill. Students often say, "Well, I know that 3 + 2 = 5, so 30 + 20 must be 50." Students constantly rely on their mathematical memory and build on what they already know.

Simon (1995) describes the educational journey as follows:

> You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you may constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature

FIGURE 5

A student explains a new strategy.

$27 \times 8 =$

Mind Map

$27 \times 8 = \underline{216}$

20 7

10 10

$10 \times 8 = 80$
$10 \times 8 = 80$
$8 \times 7 = +56$
216

Persuasive Explanation

Did you know that your brain can solve intricate, complex, and mamoth problems succesfuly with ease? This is an example of one. I took 27 in 27×8 and broke apart 27 into 20 and 7. Then I broke 20 into 10 and 10. and then I knew 10×8=80. Then I used the other 10 and 10×8=80. And I can't forget the 7. So I knew 7×8=56. So 80+80=160. 160+56=216. My problem is so simple that when you close your eyes to think you will see a glimpse of the answer.

FIGURE 6

Another persuasive explanation

$27 \times 8 = 216$

Mind Map

way #1

$27 + 27 = 54$ + $54 + 54 = 108$ + $108 + 108 = 216$

way #2

$27 \times 8 = 216$

Persuasive Explanation

I think you should use my strategy because it is not confusing and it is easy enough to do in your head. Here is what I did: I took the 8 and split it down to its smallest double which is 2. I then added 27 two times which equals 54. Then I doubled 54 (that is 4-27's) that equals 108. I then doubled 108 which equals 216 (8-27's).

of your visits as a result of interactions with people along the way. You add destinations that, prior to the trip, were unknown to you. The path that you travel is your (actual) trajectory. The path that you anticipate at any point is your hypothetical trajectory.

MMJ: A Path to Understanding

The most exciting part of a Mathematical Mind Journey is watching students' understanding develop. Students, parents, teachers, and administrators come to understand mathematics as they never have before. Children are excited about numbers and teachers cannot believe how their students are thinking.

As a culture, a conversation, and an integral part of the mathematics community, the MMJ is one way that teachers can move their children along the landscape of learning. Dolk and Fosnot (2001) state,

> The big ideas, strategies, and models are important landmarks for . . . [the teacher] as she journeys with her students across the landscape of learning. The paths to these landmarks are not linear. They twist and turn and are not in an ordered sequence. It is not up to us, as teachers, to decide which pathways our students will use. Often, to our surprise, children will use a path we have not yet encountered. That challenges us to understand the child's thinking. What is important, though, is that we help all students reach the horizon.

More than ever before, children are finding ways to make sense of mathematics for themselves and internalizing the information with less risk of losing these concepts over time. As we see with our fourth graders, mathematics becomes persuasive speech in the MMJ Challenge (see **figs. 5** and **6**). Students defend their strategies and persuade others to use them.

Empowered and excited by their own constructed mathematical ideas, students' minds are awakening to a new world of computational fluency and understanding. The MMJ format encourages students and their teachers to reflect on the importance of the work at hand, to understand the purpose of instruction, and to demonstrate the landscape of mathematics. As a group of second graders stated (see **fig. 7**), "Sometimes classmates have different strategies and other classmates have the same strategies. And that's OK—my teacher just says, 'Great minds think alike.'"

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FIGURE 7

Students describe MMJs in an article for the school newspaper.

Page 2 The East School Times

Strategy Rollercoaster

Mrs. Iwasaki writes down a subtraction problem on the chart paper and we show how we got the answer or a way to get the answer. Sometimes we tell two ways to get the answer. On the chart paper, Mrs. Iwasaki writes our names in the problem:

Kyra and Jake were eating peanuts. There were 39 peanuts in the bag. Kyra ate 18 of them and Jake ate the rest. How many peanuts did Jake eat?

Jake's way to figure it out:

18 + 10 = 28
28 + 10 = 38
38 + 1 = 39
10 + 10 + 1 = 21
It proves that Jake ate 21

Peter
18+20=38
38+1=39
Answer 21
=or
18+__=39
or
39−18=___

peanuts. You should use both subtraction and addition.

We do our problems during math time. Sometimes classmates have different strategies and other classmates have the same strategies. And that's OK—my teacher just says, "Great minds think alike."

Subtraction is fun if you do it the right way. Here is a problem for you to solve:

Jon-Luke and Jake were eating watermelon. There were 60 pieces in a bowl. Jake ate 30 slices. Jon-Luke ate the rest. How many slices did Jon-Luke eat?

By Peter, Jon-Luke, Jake, and Sean