# 2017 Kansas Mathematics Standards

## Flip Book

## High School- Algebra I

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

# About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

# Planning Advice - Focus on the Clusters

*The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)*

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope"; 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

*"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path; if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)*

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size." In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

*A caution--Grain size is important, but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.*

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

## Recommendations for Cluster-Level Priorities

***Appropriate Use:***

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

***Things to Avoid:***

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

# Mathematics Teaching Practices

## (High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move “toward improved instructional practice” and support “one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students” (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students’ reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

# *Standards for Mathematical Practice in High School*

The State Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students Grades K-12. Below are a few examples of how these Practices may be integrated into tasks that High School students complete.

| Practice | Explanation and Example |
|---|---|
| **1) Make sense of problems and persevere in solving them.** | Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. High School students make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. They might transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. They can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They constantly check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They understand the approaches of others to solving complex problems and identify correspondence between different approaches. |
| **2) Reason abstractly and quantitatively.** | Mathematically proficient students make sense of quantities and their relationships in problem situations. They represent a wide variety of real world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*—to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. |
| **3) Construct viable arguments and critique the reasoning of others.** | Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. High School students reason inductively about data, making plausible arguments that take into account the context from which the data arose. They compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High School students determine domains to which an argument applied, they listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. |

| | |
|---|---|
| **4) Model with mathematics.** | Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. High School students might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximation to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. |
| **5) Use appropriate tools strategically.** | Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil/paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software. They are sufficiently familiar with tools appropriate for their grade to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts. |
| **6) Attend to precision.** | Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussions with others and in their *own* reasoning. They state the meaning of the symbols they choose. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. High school students have learned to examine claims and make explicit use of definitions. |
| **7) Look for and make use of structure.** | Mathematically proficient students look closely to discern a pattern or structure. For example, high school students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. |
| **8) Look for and express regularity in repeated reasoning.** | Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1,2) with slope 3, they might abstract the equation $\frac{(y-2)}{(x-1)} = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)\,(x\ +1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2\ +x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results. |

# Implementing Standards for Mathematical Practice

This guide was created to help educators implement these standards into their classroom instruction. These are the practices for the **students,** and the teacher can assist students in using them efficiently and effectively.

## #1 – Make sense of problems and persevere in solving them.

### Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

| Student Actions | Teacher Actions |
|---|---|
| • Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding).<br>• Relate current "situation" to concepts or skills previously learned, and checking answers using different methods.<br>• Monitor and evaluate their own progress and change course when necessary.<br>• Always ask, "Does this make sense?" as they are solving problems. | • Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway.<br>• Constantly ask students if their plans and solutions make sense.<br>• Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem.<br>• Consistently ask students to defend and justify their solution(s) by comparing solution paths. |

### What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

### What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

# #2 – Reason abstractly and quantitatively.

**Summary of this Practice:**

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

| Student Actions | Teacher Actions |
| --- | --- |
| • Use varied representations and approaches when solving problems.<br>• Represent situations symbolically and manipulating those symbols easily.<br>• Give meaning to quantities (not just computing them) and making sense of the relationships within problems. | • Ask students to explain the meaning of the symbols in the problem and in their solution.<br>• Expect students to give meaning to all quantities in the task.<br>• Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood. |

**What questions develop this Practice?**

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

**What are the characteristics of a good math task for this Practice?**

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

# #3 – Construct viable arguments and critique the reasoning of others.

**Summary of this Practice:**

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

| Student Actions | Teacher Actions |
|---|---|
| • Make conjectures and exploring the truth of those conjectures.<br>• Recognize and use counter examples.<br>• Justify and defend all conclusions and using data within those conclusions.<br>• Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions. | • Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning.<br>• Question students so they can tell the difference between assumptions and logical conjectures.<br>• Ask questions that require students to justify their solution and their solution pathway.<br>• Prompt students to respectfully evaluate peer arguments when solutions are shared.<br>• Ask students to compare and contrast various solution methods<br>• Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.) |

**What questions develop this Practice?**

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn’t it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

**What are the characteristics of a good math task for this Practice?**

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others’ solutions.

# #4 – Model with mathematics.

**Summary of this Practice:**

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

| Student Actions | Teacher Actions |
|---|---|
| • Apply mathematics to everyday life.<br>• Write equations to describe situations.<br>• Illustrate mathematical relationships using diagrams, data displays, and/or formulas.<br>• Identify important quantities and analyzing relationships to draw conclusions. | • Demonstrate and provide students experiences with the use of various mathematical models.<br>• Question students to justify their choice of model and the thinking behind the model.<br>• Ask students about the appropriateness of the model chosen.<br>• Assist students in seeing and making connections among models. |

**What questions develop this Practice?**

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line..., chart..., table...?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

**What are the characteristics of a good math task for this Practice?**

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

# #5 – Use appropriate tools strategically.

**Summary of this Practice:**

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information

| Student Actions | Teacher Actions |
|---|---|
| • Choose tools that are appropriate for the task.<br>• Know when to use estimates and exact answers.<br>• Use tools to pose or solve problems to be most effective and efficient. | • Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available.<br>• Question students as to why they chose the tools they used to solve the problem.<br>• Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations.<br>• Ask student to explain their mathematical thinking with the chosen tool.<br>• Ask students to explore other options when some tools are not available. |

**What questions develop this practice?**

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

**What are the characteristics of a good math task for this Practice?**

- Lends itself to multiple learning tools. (Tools may include; concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
    - a task when there is no need to have an exact answer
    - a task when there is not enough information to get an exact answer
    - a task to check if the answer from a calculation is reasonable

# #6 – Attend to precision.

**Summary of this Practice:**

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

| Student Actions | Teacher Actions |
|---|---|
| • Use mathematical terms, both orally and in written form, appropriately.<br>• Use and understanding the meanings of math symbols that are used in tasks.<br>• Calculate accurately and efficiently.<br>• Understand the importance of the unit in quantities. | • Consistently use and model correct content terminology.<br>• Expect students to use precise mathematical vocabulary during mathematical conversations.<br>• Question students to identify symbols, quantities and units in a clear manner. |

**What questions develop this Practice?**

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

**What are the characteristics of a good math task for this Practice?**

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

# #7 – Look for and make use of structure.

**Summary of this Practice:**

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

| Student Actions | Teacher Actions |
|---|---|
| • Look closely at patterns in numbers and their relationships to solve problems.<br>• Associate patterns with the properties of operations and their relationships.<br>• Compose and decompose numbers and number sentences/expressions. | • Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.)<br>• Expect students to explain the overall structure of the problem and the big math idea used to solve the problem. |

**What questions develop this Practice?**

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

**What are the characteristics of a good math task for this Practice?**

- Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value; working with properties; etc.)
- Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm. When "unit size" cannot be equally distributed, it is necessary to break down into a smaller "unit size". (example below)

4 )351
-32
31
-28
3

3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.

There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.

This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation. i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

# #8 – Look for and express regularity in repeated reasoning.

**Summary of this Practice:**

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

| Student Actions | Teacher Actions |
|---|---|
| • Notice if processes are repeated and look for both general methods and shortcuts.<br>• Evaluate the reasonableness of intermediate results while solving.<br>• Make generalizations based on discoveries and constructing formulas when appropriate. | • Ask what math relationships or patterns can be used to assist in making sense of the problem.<br>• Ask for predictions about solutions at midpoints throughout the solution process.<br>• Question students to assist them in creating generalizations based on repetition in thinking and procedures. |

**What questions develop this Practice?**

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

**What are the characteristics of a good math task for this Practice?**

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

# Critical Areas for Mathematics in Algebra I

The fundamental purpose of the Algebra I course is to formalize and extend the mathematics that students learned in the middle grades. For the high school Algebra I course, instructional time should focus on four critical areas:

1. In Algebra I, students learn to solve linear equations in one variable and apply graphical and algebraic methods to analyze and solve systems of linear equations in two variables. They analyze and explain the process of solving an equation and justify the process used in solving a system of equations. Students develop fluency writing, interpreting, and translating among various forms of linear equations and inequalities, and use them to solve problems. They master the solution of linear equations and apply related solution techniques and the laws of exponents to the creation and solution of simple exponential equations.

2. In earlier grades, students define, evaluate, and compare functions, and use them to model relationships between quantities. In Algebra I, students learn function notation and develop the concepts of domain and range. They focus on linear, quadratic, and absolute value functions, including patterns; they interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations. Students learn and apply properties of exponents to generate equivalent numerical and algebraic expressions. Students explore systems of linear and/or quadratic equations and linear inequalities, and they find and interpret their solutions.

3. Students apply the laws of exponents to integer exponents and recognize that squaring and cubing a number are inverse operations of taking the square root and cube root of a number; they strengthen their ability to see structure in and create quadratic expressions. They create and solve equations, inequalities, and systems of equations involving quadratic expressions. Students become facile with algebraic manipulation, including rearranging and collecting terms and factoring. Students consider quadratic functions, comparing the key characteristics of quadratic functions to those of linear functions. They select from among these functions to model phenomena. Students learn to anticipate the graph of a quadratic function by interpreting various forms of quadratic expressions. In particular, they identify the real solutions of a quadratic equation as the zeros of a related quadratic function. Students expand their experience with functions to include more specialized functions such as absolute value.

4. Building upon their prior experiences with data, students explore a more formal means of assessing how a model fits data. Students use regression techniques to describe approximately linear relationships between quantities. They use graphical representations and knowledge of context to make judgments about the appropriateness of linear models. With linear models, they look at residuals to analyze best fit.

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

# Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

# Growth Mindset


The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.

youcubed at Stanford University

**Building a Mathematical Mindset Community**

| | |
|---|---|
| **Teachers and students believe *everyone* can learn maths at HIGH LEVELS.**<br>• Students are not tracked or grouped by achievement<br>• All students are offered high level work<br>• "I know you can do this" "I believe in you"<br>• Praise effort and ideas, not the person<br>• Students vocalize self-belief and confidence | **Communication and *connections* are valued.**<br>• Students work in groups sharing ideas and visuals.<br>• Students relate ideas to previous lessons or topics<br>• Students connect their ideas to their peers' ideas, visuals, and representations.<br>• Teachers create opportunities for students to see connections.<br>• Students relate ideas to events in their lives and the world. |
| **The maths is VISUAL.**<br>• Teachers ask students to draw their ideas<br>• Tasks are posed with a visual component<br>• Students draw for each other when they explain<br>• Students gesture to illustrate their thinking | **The maths is OPEN.**<br>• Students are invited to see maths differently<br>• Students are encouraged to use and share different ideas, methods, and perspectives<br>• Creativity is valued and modeled.<br>• Students' work looks different from each other<br>• Students use ownership words - "my method", "my idea" |
| **The environment is filled with *WONDER* and *CURIOSITY*.**<br>• Students extend their work and investigate<br>• Teacher invites curiosity when posing tasks<br>• Students see maths as an unexplored puzzle<br>• Students freely ask and pose questions<br>• Students seek important information<br>• "I've never thought of it like that before." | **The classroom is a risk-taking, *MISTAKE VALUING* environment**<br>• Students share ideas even when they are wrong<br>• Peers seek to understand rather than correct<br>• Students feel comfortable when they are stuck or wrong<br>• Teachers and students work together when stuck<br>• Tasks are low floor/high ceiling<br>• Students disagree with each other and the teacher |

*Developed by Jo Boaler/Youcubed.org and Tulare County Office of Education*

# High School Notation

(★) **Modeling Standards:** Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol. The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

## Grade Level Classifications

To assist with the organization of high school mathematics courses, the standards have grade level classifications to identify the appropriate grade at which they should be taught. The classifications were designed with the following framework in mind:

| Year of School | Traditional Course Sequence | Integrated Course Sequence |
|---|---|---|
| 9th Grade | Algebra I | Mathematics 1 |
| 10th Grade | Geometry | Mathematics 2 |
| 11th Grade | Algebra II | Mathematics 3 |

There will be variation with student placement in the courses listed above. At the present time, the "gateway" math class in Kansas for postsecondary schooling is College Algebra. The standards committee used this as a guide when identifying grade level classifications.

The grade level classifications are as follows

| | |
|---|---|
| **(9/10)** | These standards are required for all students by the end of their first two years of high school math courses. |
| **(11)** | These standards are required for all students by the end of their third year math course. |
| **(9/10/11)** | These standards are required for all students in their first three years of high school math courses. These standards are often further divided to (9/10) and (11) to identify specific concepts and their appropriate grade level. *(9/10) should primarily accomplish the standards described as linear, quadratic and absolute value while (11) should primarily accomplish the standards described as logarithmic, square root, cube root, and exponential .* |
| **(all)** | These standards should be taught throughout every high school math course, and often represent over-arching themes or key features of the mathematical concept. These standards should be taught in conjunction with the appropriate grade level standards. |
| **(+)** | These standards should be taught as **extensions** to grade level standards when possible, or in a 4th year math course. These standards prepare students to take advanced courses in high school such as college algebra, calculus, advanced statistics, or discrete mathematics. |

# High School- Algebra I Overview

## High School—Modeling

Modeling

## High School – Number and Quantity

**The Real Number System (N.RN)**

A. Use properties of rational and irrational numbers.
N.RN.1

**Quantities (★) (N.Q)**

A. Reason quantitatively and use units to solve problems.
N.Q.1 (★) N.Q.2 (★) N.Q.3 (★)

## High School – Algebra

**Seeing Structure in Expressions (A.SSE)**

A. Interpret the structure of expressions.
A.SSE.1 (★) A.SSE.2

B. Write expressions in equivalent forms to solve problems.
A.SSE.3a (★)

**Arithmetic with Polynomials and Rational Expressions (A.APR)**

A. Perform arithmetic operations on polynomials.
A.APR.1

B. Use polynomial identities to solve problems.
A.APR.4

**Creating Equations (★) (A.CED)**

A. Create equations that describe numbers or relationships.
A.CED.1 (★) A.CED.2 (★) A.CED.3 (★)
A.CED.4 (★)

**Reasoning with Equations and Inequalities (A.REI)**

A. Understand solving equations as a process of reasoning and explain the reasoning.
A.REI.1

B. Solve equations and inequalities in one variable.
A.REI.2 A.REI.3a A.REI.5a

C. Solve systems of equations.
A.REI.6

D. Represent and solve equations and inequalities graphically.
A.REI.8 A.REI.9 (★) A.REI.10

## High School – Functions

**Interpreting Functions (F.IF)**

A. Understand the concept of a function and use function notation.
F.IF.1 F.IF.2 F.IF.3

B. Interpret functions that arise in applications in terms of the context.
F.IF.4 (★) F.IF.5 (★) F.IF.6 (★)

C. Analyze functions using different representations.
F.IF.7a (★) F.IF.8a F.IF.9

**Building Functions (F.BF)**

A. Build a function that models a relationship between two quantities.
F.BF.1a

B. Build new functions from existing functions.
F.BF.3

## High School – Statistics & Probability

**Interpreting Categorical and Quantitative Data (S.ID)**

A. Summarize, represent, and interpret data on a single count or measurement variable.
S.ID.1 S.ID.2

B. Summarize, represent, and interpret data on two categorical and quantitative variables.
S.ID.4 S.ID.5a-b

C. Interpret linear models.
S.ID.6

# High School – Modeling

## Domain: Modeling (★)

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: See ★ standards on the overview page.

**Explanations and Examples:**

The goal for this section is to expand on the information in the Modeling section of the standards by adding information from research using an article summarizing our current knowledge base “Quality Teaching of Mathematical Modeling: What Do We Know, What Can we Do?” from Werner Blum.

The word “modeling” is a word that is difficult to define because it is used to describe both a process and a product. The process of *modeling* creates a product called a *model*. The standards expect students can successfully use the process to create a model and that, given a model; they can successfully interpret and understand how the math model is related to the real world situation. But what *exactly* is a model? Niss (2007) defines a model as “a deliberately simplified and formalized image of some part of the real world, formally speaking: a triple (D, M, f) consisting of a domain D of the real world, a subset M of the mathematical world and a mapping from D to M (Niss et al. 2007).”

The standards describe a six step modeling cycle:

1. Identify the variables in the situation and select those that represent essential features.
2. Formulate a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
3. Analyze and perform operations on these relationships to draw conclusions.
4. Interpret the results of the mathematics in terms of the original situation.
5. Validate the conclusions by comparing them with the situation and then either improve the model or, if is acceptable, move to step 6.
6. Report the conclusions and the reasoning behind them.

Throughout the cycle, students will make choices, assumptions, and approximations.

Blum, in his research summary, identifies an important first step that is not explicitly described in the modeling process- to construct a mental model of the situation. This requires understanding the situation, being able to mentally imagine all the parts of the situation.  Research has found that many students get stuck here, in this “pre-step.” The reason some students don’t gain entry into the process is because they have been *taught* to ignore the context, find the numbers, and apply a familiar operation. This has been labeled by researchers as the "suspension of sense-making" and occurs whenever students are processing any word problem. Robert Kaplinksy created a video illustrating this phenomenon. He asked 32 8th grade students the following question:


"There are 125 sheep and 5 dogs in a flock. How old is the shepherd?"

Sadly 75% of students performed math operations and provided a numerical answer. This question has been replicated across a variety of settings since 1993 with the same consistent results.

After students have created a mental model of the situation, they are ready to begin the modeling process. The first step is to simplify the mental model down to the critical elements. This requires making assumptions and estimating any missing information. This is another source of difficulties for students- they are afraid to make assumptions. For example, one PISA task that asks students to make assumptions to solve the problem had low success rates across multiple countries.

**Fig. 6** PISA task "Rock Concert"

**ROCK CONCERT**

For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

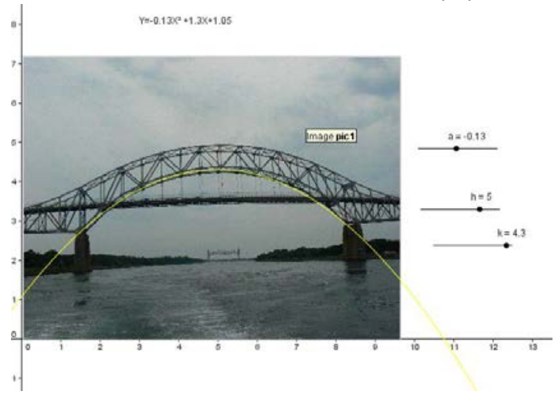
A 2 000
B 5 000
C 20 000
D 50 000
E 100 000

For this PISA task, given to 15 year olds, the success rates were:

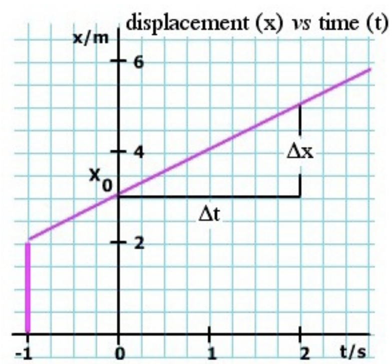
- Finland- 37%
- Korea- 21%
- USA- 26%

This question required students to estimate about how many people could fit in a square meter and an assumption that each square meter had the same density. The students should have realized that A and B did not match the scenario of a full stadium because it would be one or fewer persons per square meter.  Choices D and E were also poor estimates because those choices require 10 or 20 people per square meter. Leaving the only reasonable estimate choice C. Even countries more familiar with metric measurement than the USA struggled with this type of estimation.

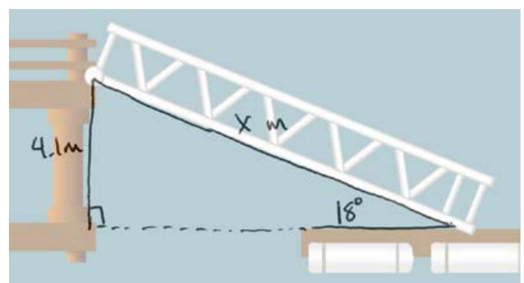
The real world is messy, filled with irrelevant data and partial information. If students are only presented problems that have been simplified and all the assumptions are made, then they do not get practice with this critical step. Developmentally, once given the freedom to find holes or irrelevant information in a problem, adolescents are often excited to explore a problem in this way.



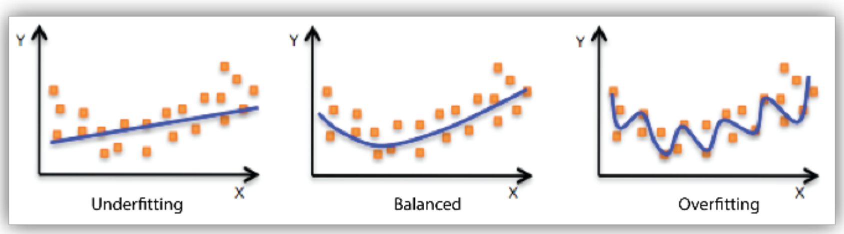

Step 2 is to mathematize the problem by creating geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables. This requires students recognize the general structure of a problem, to understand how the rate of change identifies the function family, the parameters which describe the situation, and possibly the representation that best demonstrates the relationship and can be used to find the needed information.




Step 3 and 4 are the steps we ask students to complete in a typical word problem. Unfortunately these are often the only two steps the students are regularly asked to work. The problem has been sufficiently mathematized and structured so that there are few questions about the correct structure for the problem. Students must "do the math" and interpret the results.




Step 5- validating the conclusions- involves more than interpreting the results and is another step often skipped by students. This step involves determining if the model is suitable in the real world. For example, F.LQE asks students to compare linear, quadratic, and exponential models and use the model to solve problems. If a student selects a model inappropriate to the situation, they will still be able to complete steps 3 and 4. It is once they have reached step 5 to validate the conclusions that they are given the opportunity to re-evaluate the model. Or the validation might decide the function family is correct but that the parameters chosen could be modified to better represent the situation. This step is when, in statistics, the researcher must try to fit the model to the data but also be careful to not over fit the data so that it can't be generalized to other similar situations.




The final step is to have students write up the process and conclusions. Asking students to convince a friend that they are correct can help students structure their persuasive and descriptive argument. Another reason that writing up the process, the assumptions, the simplified structure, etc. is difficult is because the problems we provide are not truly modeling problems- they are word problems. There is one solution path and it isn't messy. Providing problems with multiple viewpoints and different conclusions will help students have something to talk about. For example, when analyzing data do not clean the data for the students. Let them decide how to approach incorrect data and outliers. Using Dan Meyer's Three-Act Math Videos can provide common input data for the students with multiple paths to the solution.

These six steps are cognitively demanding and difficult because they require math knowledge, non-math knowledge, and a specific set of beliefs and attitudes about ones ability to do mathematics. So *why* do we take valuable class time to go through this entire process? Research has identified there are four different justifications and perspectives which drive the modeling process, depending on the type of problem presented to students. Understanding these justifications and perspectives can help the teacher present a wide variety of problem types and to be more intentional about highlighting the purpose for the chosen problem.

1. Applied Math: Applied mathematics justifies the modeling process because the mathematics will help the learner understand the real world. The other three justifications use the situation to support math understanding so applied mathematics is the only justification with the purpose of supporting a deeper theoretical understanding about the world. When working these modeling problems, students are practicing sensemaking through understanding the real-world.
2. Educational Modeling: Another reason to practice the modeling process is to formatively assess the thinking of students. For these problems, the examples are concrete and authentic. They are cognitively rich and include time for students to reflect on their process. When the purpose is educational modeling, students are making sense of the problem through the lens of their own growth.
3. Cultural Modeling: Modeling has the ability to connect the outside world to the math classroom, to allow students to see how math can *help* the world around them. The problems that students work will be authentic and will show how math shapes the world around them or will allow the student to see that mathematics is a science. Students will make sense of these problems by seeing the role of math in the real-world.
4. Pedagogical Modeling: Psychologically modeling problems have the ability to spark interest, motivate students, and increase retention in STEM fields. These problems are interesting, illustrating how math will benefit the student, or are rich enough to deepen students understanding of a mathematics concept (sometimes called conceptual modeling). Students will make sense by finding joy in mathematics or puzzling through a math concept.

It is clear that modeling is an important process in mathematics but also that modeling is demanding. There must be significant efforts to make this process accessible for all learners. There are many resources available by performing an internet search for STEM problems. Below are four examples to start you on your journey.

**Resources/Tools:**

Quantamagazine:

- https://www.quantamagazine.org/the-math-of-causation-puzzle-20180530/
- https://www.quantamagazine.org/puzzle-solution-bongard-problems-20170628/

- https://www.nytimes.com/spotlight/learning-stem

NRICH Math:

- https://nrich.maths.org/6458

Dan Meyer's Three-Act Math Tasks:

- https://docs.google.com/spreadsheets/d/1jXSt_CoDzyDFeJimZxnhgwOVsWkTQEsfqouLWNNC6Z4/edit#gid=0

# High School – Number and Quantity

## Domain: The Real Number System (N.RN)

**Cluster:** *Use properties of rational numbers and irrational numbers*

**Standard**: **N.RN.1**

**(9/10)** Know and apply the properties of integer exponents to generate equivalent numerical and algebraic expressions. **(8.EE.1)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

**Connections**: 6.EE, 8.EE

**Explanations and Examples:**

Integer (positive and negative) exponents are further used to generate equivalent numerical expressions when multiplying, dividing or raising a power to a power. Using numerical and algebraic bases with the laws of exponents, students generate equivalent expressions.

Use A.SSE.2 to have students use the structure of the expression to identify equivalent ways to rewrite it.

**Examples:**

1. $\frac{4^3}{5^2} = \frac{64}{25}$

$$\frac{4^3}{4^7} = 4^{3-7} = 4^{-4} = \frac{1}{4^4} = \frac{1}{256}$$

$$\frac{4^{-3}}{5^2} = 4^{-3} \times \frac{1}{5^2} = \frac{1}{4^3} \times \frac{1}{5^2} = \frac{1}{64} \times \frac{1}{25} = \frac{1}{1{,}600}$$

2. Select all of the expressions that have a value between 0 and 1.

   A. $8^7 \cdot 8^{-12}$

   B. $\frac{7^4}{7^3}$

   C. $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^9$

   D. $\frac{(-5)^6}{(-5)^{10}}$

   *Solution: A, C, D*

3. $(3x^{-2}y)(-2xy^{-3}) = -6x^{-1}y^{-2} = \frac{-6}{xy^2}$

4. $\left(\frac{2x}{3y^2}\right)^3 = \frac{(2x)^3}{(3y^2)^3} = \frac{2^3x^3}{3^3(y^2)^3} = \frac{8x^3}{27y^6}$

5. $(7a^3b^{-1})^0 = 1$

**Instructional Strategies:**

Although students begin using whole-number exponents in Grades 5, 6, and Grade 8, this standard is when students are first expected to know and use the properties of exponents and to extend the meaning beyond counting-number exponents. It is no accident that these expectations are simultaneous, because it is the properties of counting-number exponents that provide the rationale for the properties of integer exponents. In other words, *students should not be told these properties but rather should derive them through experience and reason.*

For natural number exponents (and for nonzero bases), the following properties follow directly from the meaning of exponents.

1. $a^n a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$
3. $a^n b^n = (ab)^n$

Students should have experience simplifying expressions with exponents so that these properties become natural and obvious. For example,

$$2^3 \cdot 2^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2^8$$
$$(x^3)^4 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) \cdot (x \cdot x \cdot x) = x^{12}$$
$$(3 \cdot 7)^4 = (3 \cdot 7) \cdot (3 \cdot 7) \cdot (3 \cdot 7) \cdot (3 \cdot 7) \cdot= (3 \cdot 3 \cdot 3 \cdot 3) \cdot (7 \cdot 7 \cdot 7 \cdot 7) = 3^4 \cdot 7^4$$

If students reason about these examples, they begin to articulate the properties. For example, "I see that 3 twos is being multiplied by 5 twos, and the results is 8 twos being multiplied together, where the 8 is the sum of 3 and 5, the number of twos in each of the original factors. That would work for a base other than two (as long as the bases are the same)."

Note: When talking about the meaning of an exponential expression, it is easy to say (incorrectly) that "$3^5$ means 3 multiplied by itself 5 times." ‖But by writing out the meaning, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$, students can see that there are only 4 multiplications. So a better description is "$3^5$ means 5 3s multiplied together."‖

Students also need to realize that these simple descriptions work only for natural number exponents. When extending the meaning of exponents to include 0 and negative exponents, these descriptions are limiting: Is it sensible to say "$3^0$ means 0 3s multiplied together"‖ or that "$3^{-2}$ means -2 3s"?

For example, Property 1 can be used to reason what $3^0$ should be. Consider the following expression and simplification: $3^0 \cdot 3^5 = 3^{0+5} = 3^5$. This computation shows that the when $3^0$ is multiplied by $3^5$, the result (following Property 1) should be $3^5$,which implies that $3^0$ must be 1.

Because this reasoning holds for any base other than 0, we can reason that $a^0 = 1$ for any nonzero number $a$. To make a judgment about the meaning of $3^{-4}$, the approach is similar: $3^{-4} \cdot 3^4 = 3^{-4+4} = 3^0 = 1$. This computation shows that $3^{-4}$ should be the reciprocal of $3^4$, because their product is 1. And again, this reasoning holds for any nonzero base. Thus,

**Properties of Integer Exponents**

For any nonzero real numbers *a* and *b* and integers *n* and *m*:

1. $a^n a^m = a^{n+m}$
2. $(a^n)^m = a^{nm}$
3. $a^n a^n = (ab)^n$
4. $a^0 = 1$
5. $a^{-n} = \frac{1}{a^n}$

we can reason that $a^{-n} = \frac{1}{a^n}$.

Putting all of these results together, we now have the properties of integer exponents, shown in the chart.
A supplemental strategy for developing meaning for integer exponents is to make use of patterns, as shown below:

**Patterns in Exponents**

| | |
|---|---|
| ⋮ | ⋮ |
| $5^4$ | 625 |
| $5^3$ | 125 |
| $5^2$ | 25 |
| $5^1$ | 5 |
| $5^0$ | 1 |
| $5^{-1}$ | 1/5 |
| $5^{-2}$ | 1/25 |
| $5^{-3}$ | 1/125 |
| ⋮ | ⋮ |

As the exponent decreases by 1, the value of the expression is divided by 5, which is the base. Continue that pattern to 0 and negative exponents.

The meanings of 0 and negative-integer exponents can be further explored in a place-value chart:

| *thousands* | *hundreds* | *tens* | *ones* | | *tenths* | *hundredths* | *thousandths* |
|---|---|---|---|---|---|---|---|
| $10^3$ | $10^2$ | $10^1$ | $10^0$ | | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ |
| 3 | 2 | 4 | 7 | . | 5 | 6 | 8 |

Thus, integer exponents support writing any decimal in expanded form like the following:

$$3247.568 = 3 \cdot 10^3 + 2 \cdot 10^2 + 4 \cdot 10^1 + 7 \cdot 10^0 + 5 \cdot 10^{-1} + 6 \cdot 10^{-2} + 8 \cdot 10^{-3}$$

Expanded form and the connection to place value is important for helping students make sense of scientific notation, which allows very large and very small numbers to be written concisely, enabling easy comparison. To develop familiarity, go back and forth between standard notation and scientific notation for numbers near, for example, $10^{12}$ or $10^{-9}$. Compare numbers, where one is given in scientific notation and the other is given in standard notation. Real-world problems can help students compare quantities and make sense about their relationship.

Provide practical opportunities for students to flexibly move between forms of squared and cubed numbers. For example, If $3^2 = 9$ then $\sqrt{9} = 3$. This flexibility should be experienced symbolically and verbally.

Opportunities for conceptually understanding irrational numbers should be developed. One way is for students to draw a square that is one unit by one unit and find the diagonal using the Pythagorean Theorem. The diagonal drawn has an irrational length of $\sqrt{2}$. Other irrational lengths can be found using the same strategy by finding diagonal lengths of rectangles with various side lengths.

**Resources/Tools:**

Also see engageNY Modules: https://www.engageny.org/resource/grade-8-mathematics

Illustrative Mathematics **Grade 8** tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.A.1
  - Extending the Definitions of Exponents, Variation 1
  - Ants versus humans
  - Raising to the zero and negative powers

# Domain: Quantities ★ (N.Q)

**Cluster:** *Reason quantitatively and use units to solve problems.*

**Standard**: **N.Q.1 ★ (all)**

Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. **(N.Q.1)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

**Connections**: Algebra and Functions

**Explanations and Examples:**

Across a wide variety of problems and applications, units can and should be used a way to understand a problem and make an effective problem solving tool, guiding the student to the relevant measurements and operations. Interpreting units consistently could be as simple as interpreting the meaning of the y-intercept to as complicated as using the units to support the selection of the appropriate regression model.

Particular attention should be paid to creating graphs that follow standard mathematical and scientific conventions for graphing or discussing decisions made when graphing in cases of no consensus.

- Graphs must be partitioned into equal intervals.
- Intervals should be chosen so that the area interest is easily visible (for example, small enough to see an intersection or large enough to view the vertex of a parabola).
- Intervals should allow global analysis of direction of change, maximum/minimum, end behavior, etc. For example, it is possible to zoom in or out so much that a nonlinear graph appears linear.

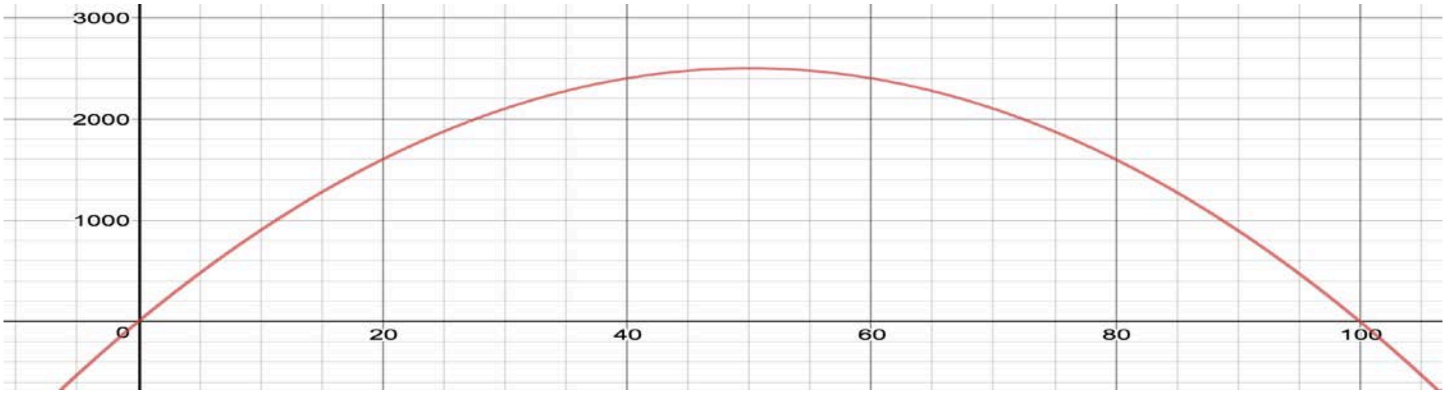
Things to consider:

- Is it more important for the graph to take up the majority of the graphing space or should the intervals on the domain and range be the same? Taking up more space might make it easier to see the key features of interest but can distort the appearance of rate of change. Keeping the intervals the same helps create a visual of the rate of change but might not make sense if the domain is $0 \leq x \leq 2$ while the range is $150 \leq y \leq 500$.
- Should the graph include the origin or use a "compressed scale" to begin the scale at a higher number? Compressing the y-axis has the benefit of using more of the graphable space but might create a false y-intercept.

This is an "all" standard because there is no one right answer to most of these questions. Fluency and skill with making these decisions and interpreting the decision of others comes only after consistent and explicit discussion during learning.

**Algebra 1 Examples:**

1. Your math teacher has hired you to put up a fence to make a protected rectangular garden area. The only thing she provides you is 200 feet of fencing material and a graph relating the length of the fence to the area. She tells you she will pay you a bonus if you construct the largest garden possible with the materials she has provided. What will be the dimensions and area for her garden?
   a. The graph she provided is not labeled with the name of the variable or units. How should the graph be labeled? How do you know?




This task has been adapted from Illustrative Mathematics and is intended as an example of F.LE.5- Interpret the parameters in context. Part *c)* added a question about the units. This question will add scaffolding for students who might struggle with finding the meaning of the numbers.

Part *d)* was added to the question and will allow the teacher to assess if the student understands the similarities and differences between the function and the situation.

**Task:**
Lauren keeps records of the distances she travels in a taxi and what she pays:

| Distance, d, in miles | Fare, F, in dollars |
|---|---|
| 3 | 8.25 |
| 5 | 12.75 |
| 11 | 26.25 |

a) If you graph the ordered pairs (d, F) from the table, they lie on a line. How can you tell this without graphing them?
b) Show that the linear function in part (a) has equation $F = 2.25d + 1.5$.
c) What are the units for the numbers 2.25 and 1.5 in the formula? What do these number represent in terms of taxi rides?
d) How is the domain and range for the function different than the domain and range for the situation? Be sure to include units as appropriate.

**Instructional Strategies:**
As you think about this the standard, the first few words of the standard should guide you "Use units *as a way to understand problems…"* This standard should be highlighted when it will enhance student understanding and not as part of a procedural checklist or as an addition to question that confuses more than it supports. Remember that this is an ALL standard because mastery is developed *over time*. Initially the conversations will be difficult but students should progress

in sophistication throughout their time in high school mathematics. Think instructionally about how you can monitor, assess, and provide feedback to students on growth in this area, as well as all areas in the "ALL" category.
This standard also provides an opportunity to ensure alignment with other departments. The science department likely has some criteria for the graphs that they draw, which might be discussed during class. Sharing the outcome for this standard with other departments might give them some ideas for supporting mathematics within their class.

Some contextual problems may require an understanding of derived measurements and capability in unit analysis. Keeping track of derived units during computations and making reasonable estimates and rational conclusions about accuracy and the precision of the answers help in the problem-solving process.

**Common Misconceptions:**
Students may not realize the importance of the units' conversions in conjunction with the computation when solving problems involving measurements. Students often have difficulty understanding how ratios expressed in different units can be equal to one**.** For example, $\frac{5280\ ft}{1\ mile}$ is simply one, and it is permissible to multiply by that ratio.

Students need to make sure to put the quantities in the numerator or denominator so that the terms can cancel appropriately. Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than the required.

**Resources/Tools:**
(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

**N.Q.1-3**
"Relationships Between Quantities & Reasoning with Equations & Their Graphs" - EngageNY Algebra 1 Module 1:
In this module students analyze and explain precisely the process of solving an equation. Through repeated reasoning, students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and make conjectures about the form that a linear equation might take in a solution to a problem. They reason abstractly and quantitatively by choosing and interpreting units in the context of creating equations in two variables to represent relationships between quantities. They master the solution of linear equations and apply related solution techniques and the properties of exponents to the creation and solution of simple exponential equations.

Illustrative Mathematics tasks:
- Traffic Jam
- Weed Killer

**N.Q.1**
Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**
- N-Q.A.1
    - Ice Cream Van
    - How Much is a Penny Worth?
    - Selling Fuel Oil at a Loss
    - Fuel Efficiency
    - Runners' World

# Domain: Number and Quantities ★ (N.Q)

**Cluster:** *Reason quantitatively and use units to solve problems.*

**Standard: N.Q.2 ★ (all)**

Define appropriate quantities for the purpose of descriptive modeling. **(N.Q.2)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

**Connections**:

The Modeling domain provides a list of standards connected to this standard.

**Explanations and Examples:**

This standard focuses on critical aspect of modeling and these three words are each essential for students to use consistently: define, appropriate, and quantities.

First, students must clearly define the meaning for the variable. This requires them to attend to precision. For example, $t = time$. Time on the clock, time since the event started, time till the event ends?

It is critical to clearly define the variables to ensure that everyone understands what kind of input is expected and acceptable.

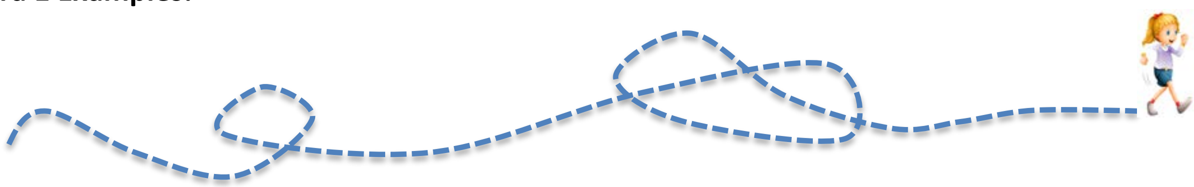
Second, the variable assignment must be appropriate. This means that students should be able to define the independent and dependent variables correctly. It also means they should be able to sift through extra information to identify the required information to answer the question. A research study from Dr. Marilyn Carlson and her team found that students struggle with identifying the appropriate variables. Students were asked to draw a graph showing the height of the fluid given the amount of fluid in the bottle. They found most misidentified the independent variable as the height and the dependent variable as the volume. Even more surprising, the students felt like *time* was an additional variable (i.e. if the water was poured in faster, the rate of change would be greater, if the water was poured and then stopped and then poured again, the graph would reflect those changes). This study illustrates the value in not only clearly defining variables but ensuring that students have made appropriate identifications and are not distracted by incorrect ideas about rate.

Finally, students *must* define the variable as quantities and use the variables as quantities. On Bill McCallum's forum about the standards, a question was asked about function notation which illustrates the importance of this concept:

> …"it can't literally be true that *f(x)* is a function, because it's a number, and a number is not a function. The letter *x* refers to a specific but unspecified number in the domain of *f*, and *f(x)* refers to the corresponding output. That's the way function notation works. I would worry that not being precise in this usage leads to confusion and misconceptions later on. I think your desire to use *f(x)* to refer to the function comes from a sense that *x* in some way represents all the input values at once. But this itself is dangerous, I think: **a lot of the trouble students have with algebra comes from a feeling that *x*** (or

whatever letter you are using) **isn't really a number** but is some vague mystical thing they have to perform mysterious rites on. So the more we can **keep students anchored** in the idea that the letters in algebraic expressions and equations are **just numbers,** and that the things you do to expressions and equations are just the things you can do to numerical expressions, the better.

**Algebra 1 Examples:**



1. When a student left the classroom with a bathroom pass, she took the path drawn above. Define the independent and dependent variable so that the situation can be modeled by a function. Define the variables so that the model is not a function.

The following is an Illustrated Math Task:

2. Jane wants to sell her Subaru Forester and does research online to find other cars for sale in her area. She checks on craigslist.com and finds 22 Subaru Foresters recently listed, along with their mileage (in miles), age (in years), and listed price (in dollars). (Collected on June 6th, 2012 for the San Francisco Bay Area.)

   She examines the scatterplot of price versus age and determines that a linear model is appropriate. She finds the equation of the least squares regression equation:
   $$predicted\ price = 24{,}247.56 - 1482.06\ age$$

   a) What variable is the explanatory (independent) variable and what are the units it is measured in? What variable is the response (dependent) variable and what are the units it is measured in?
   b) What is the slope of the least squares regression line and what are its units?
   c) Interpret the slope of the least squares regression line in the context of the problem, discussing what the slope tells you about how price and age are related. Use appropriate units in your answer.
   d) What is the y-intercept of the least squares regression line? Interpret the y-intercept in the context of the problem, including appropriate units.

**Algebra 2 Example:**

The following is an Illustrated Math Task:

1. On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way that the area covered by the algae doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.
   a) When will the lake be covered half-way?
   b) On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?
   c) On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?

d) Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

**Instructional Strategies:** See N.Q.1

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.2
  - Harvesting the Fields

## Domain: Quantities ★ (N.Q)

**Cluster:** *Reason quantitatively and use units to solve problems.*

**Standard**: **N.Q.3 ★ (all)**

Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. **(N.Q.3)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

**Connections**: See N.Q.1

**Explanations and Examples:**

Determine the accuracy of values based on their limitations in the context of the situation.

The margin of error and tolerance limit varies according to the measure, tool used, and context.

**Examples:**

- Determining price of gas by estimating to the nearest cent is appropriate because you will not pay in fractions of a cent but the cost of gas is $\frac{\$3.479}{gallon}$.
- A liquid weed-killer comes in four different bottles, all with the same active ingredient. The accompanying table gives information about the concentration of active ingredient in the bottles, the size of the bottles, and the price of the bottles. Each bottle's contents is made up of active ingredient and water.

| | Concentration | Amount in Bottle | Price of Bottle |
|---|---|---|---|
| A | 1.04% | 64 fl oz. | $12.99 |
| B | 18.00% | 32 fl oz. | $22.99 |
| C | 41.00% | 32 fl oz. | $39.99 |
| D | 1.04% | 24 fl oz. | $5.99 |

The margin of error and tolerance limit varies according to the measure, tool used, and context.

a) You need to apply a 1% solution of the weed killer to your lawn. Rank the four bottles in order of best to worst buy. How did you decide what made a bottle a better buy than another?

b) The size of your lawn requires a total of 14 fl. oz. of active ingredient. Approximately how much would you need to spend if you bought only the A bottles? Only the B bottles? Only the C bottles? Only the D bottles?

Supposing you can only buy one type of bottle, which type should you buy so that the total cost to you is the least for this particular application of weed killer?

The principal purpose of the task is to explore a real-world application problem with algebra, working with units and maintaining reasonable levels of accuracy throughout. Of particular interest is that the optimal solution for long-term purchasing of the active ingredient is achieved by purchasing bottle C, whereas minimizing total cost for a *particular* application comes from purchasing bottle B. Students might need the instructor's aid to see that this is just the observation that buying in bulk may not be a better deal if the extra bulk will go unused.

*Solution:*

a) All of the bottles have the same active ingredient, and all can be diluted down to a 1% solution, so all that matters in determining value is the cost per fl. oz. of active ingredient. We estimate this in the following table:

| | Amount active in Bottle | Price of bottle | Cost per ounce |
|---|---|---|---|
| A | 1.04% × 64 ≈ 0.64 fl oz | \$12.99 ≈ \$13 | $\frac{13}{0.64} \approx \$20\ per\ fl\ oz$ |
| B | 18.00% × 32 ≈ 6 fl oz | \$22.99 ≈ \$23 | $\frac{23}{6} \approx \$4\ per\ fl\ oz$ |
| C | 41.00% × 32 ≈ 13 fl oz | \$39.99 ≈ \$40 | $\frac{40}{13} \approx \$3\ per\ fl\ oz$ |
| D | 1.04% × 24 ≈ 0.24 fl oz | \$5.99 ≈ \$6 | $\frac{6}{0.24} \approx \$24\ per\ fl\ oz$ |

If we assume that receiving more active ingredient per dollar is a better buy than less active ingredient per dollar, the ranking in order of best-to-worst buy is C,B,A,D.

b) The A bottles have about 0.64 fl. oz. of active ingredient per bottle so to get 14 fl. oz. we need

c) $\frac{14\ fl.\ oz.}{0.64\ fl.\ oz./bottle} \approx 22\ bottles.$

Purchasing 22 A bottles at about \$13 each will cost about \$286.

The B bottles have a little less than 6 fl. oz. of active ingredient per bottle so to get 14 fl. oz. we need 3 bottles. Purchasing 3 B bottles at about \$23 each will cost about \$69.

The C bottles have a little more than 13 fl. oz. of active ingredient per bottle, so we need 2 bottles. Purchasing 2 C bottles at about \$40 each will cost about \$80.

The D bottles have only 0.24 fl. oz. of active ingredient per bottle, so to get 14 fl. oz. we need

$$\frac{14\ fl.\ oz.}{0.24\ fl.\ oz./bottle} \approx 58\ bottles.$$

Purchasing 58 D bottles at about \$6 each will cost about \$348.

Thus, although the C bottle is the cheapest when measured in dollars/fl. oz., the B bottles are the best deal for this job because there is too much unused when you buy C bottles.

**Instructional Strategies:** See N.Q.1

**Resources/Tools:**
Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.3
  - Felicia's Drive
  - Calories in a sports drink
  - Dinosaur Bones
  - Bus and Car

# High School – Algebra

## Domain: Seeing Structure in Expressions A.SSE

**Cluster:** *Interpret the structure of expressions.*

**Standard:** **A.SSE.1 ★ (all)**

Interpret expressions that represent a quantity in terms of its context.

A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients. **(A.SSE.1a)**

A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and $(1+r)^n$.* **(A.SSE.1b)**

**Suggested Standards for Mathematical Practice (MP):**

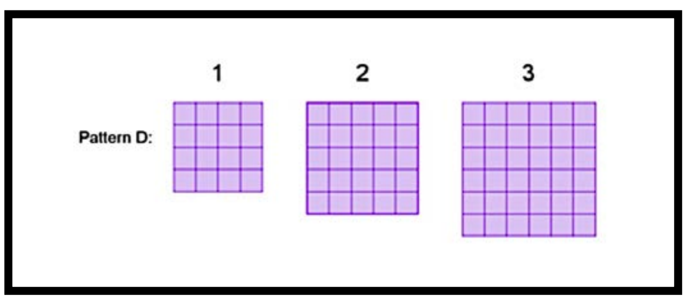
- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

**Connections**: A.SSE.2

**Explanations and Examples:**

Viewing this standard as part of the Modeling Domain helps clarify that the purpose of this standard is to interpret in the context of the situation. Students are asked to reflect on the interaction between the situation and the equation. Students should be able to explain the individual parts and, as part of A.SSE.2, explain how the parts of an equivalent expression written in a different form continues to describe the same situation. A question asking students to describe a pattern algebraically provides a great opportunity for students to explain how they see the pattern algebraically.

For example, the following Illustrated Math task could be extended or adapted to demonstrate this standard. The number of tiles in step *n* of Pattern D is defined by $d(n) = (n+3)^2$.



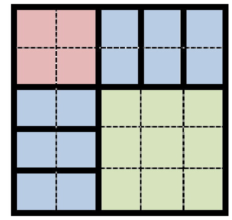

a) Explain "n+3" in the context of the situation?

The length of each side is three more than the number of the step.

b) Explain why the formula has an exponent of 2?

The pattern is describing the area of a square. The formula for the area of the square is $A = s^2$ with $s = n+3$. This is an example of viewing a one or more of their parts as a single entity. Being able to see the similarity between the area formula and this pattern helps students write their own equations to model a situation.

c) Expanding the pattern into standard form, explain how each component can still be seen the pattern. $d(n) = n^2 + 6n + 9$



It is unlikely that someone would notice the connection between this equation and the pattern and generate this model independently. But asking students to connect an equation they did not generate to a pattern is a good way to assess students' ability to connect an equation to its geometric representation. Students might see different parts "growing" in different ways. One possible interpretation is that the red squares in the corner represent $n^2$ and that square grows by one row and column each time. The green square is a constant 9 units each time. The blue represents the *6n*, meaning I have 6 groups of size 2. The next iteration will have 6 groups of size 3.

The analysis of this question illustrates how A.SSE 1 and 2 can work together to reveal new information about the problem. Explaining each part in context can create reach conversations with students, as well as reinforce the meaning behind the mathematics.

**Algebra 1 Example:**

Adapted from an [Illustrated Math Task] simply by asking students to justify their reasoning in part a.

A fisherman illegally introduces some fish into a lake and they quickly propagate. The growth of the population of this new species (within a period of a few years) is modeled by $P(x) = 5b^x$, where *x* is the time in weeks following the introduction and *b* is a positive unknown base.

a) Exactly how many fish did the fisherman release into the lake? Justify your reasoning.
b) Find *b* if you know the lake contains 33 fish after eight weeks. Show step-by-step work. (Notice that substitution into a formula without being told how is an example of this standard because it requires interpretation of the formula and then acting on that information.)
c) Instead, now suppose that $P(x) = 5b^x$ and *b=2*. What is the weekly percent growth rate in this case? What does this mean in every-day language?

**Instructional Strategies:**

Using visuals to highlight the connection between the situation, the problem, and the equation is a great strategy to help students not get lost in the problem. Highlighting one piece of information, say time, the same color throughout the problem can help show where this piece of information goes throughout the problem. Another strategy is to use post-it notes to physically cover larger pieces of information with "a single entity" to help students see it as one large chunk. Flipping back and forth between the "big" piece written on one side and the "small" piece written on the other can help students view it as a chunk.

Without going into more detail than you might need here, let me briefly name what we asking students to do in part b of the standard so that you have the concept on hand for further research.

> "**The theory of reification** (Sfard, 1987, 1988, 1991, 1992, 1994; Sfard & Linchevski, 1994) describes how concepts come into existence from a cognitive perspective. The theory is based on the fact that many **mathematical concepts** are **conceived in two complementary ways, operationally and structurally**. Operational conceptions are "**about processes, algorithms, and actions rather than about objects**" (emphasis in original, Sfard, 1991, p. 4), in contrast to **structural conceptions** where mathematical entities are conceived as **objects, wholes, or as the result of a process instead of the process itself...** Reification is 'an ontological shift- a sudden ability to see something familiar in a totally new light" (Sfard, 1991, p. 19); what was previously only a process can now be seen as an object also."

Reification is difficult to achieve, thus, its placement as an ALL standard. It will require consistent practice.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

"Polynomial and Quadratic Expressions, Equations, and Functions" - EngageNY Algebra I Module 4:

In this module, students continue to interpret expressions, create equations, rewrite equations and functions in different but equivalent forms, and graph and interpret functions, but this time using polynomial functions, and more specifically quadratic functions, as well as square root and cube root functions.

"Interpreting Algebraic Expressions" – Mathematics Assessment Project:

This lesson unit is intended to help you assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It will help you to identify and support students who have difficulty in: Recognizing the order of algebraic operations. / Recognizing equivalent expressions. or Understanding the distributive laws of multiplication and division over addition (expansion of parentheses).

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSSE.A.1
    - Mixing Fertilizer
    - Increasing or Decreasing? Variation 1
    - Throwing Horseshoes
    - Quadrupling Leads to Halving
    - Kitchen Floor Tiles
    - Mixing Candles
    - The Bank Account
    - Delivery Trucks
    - Radius of a Cylinder
    - The Physics Professor

## Domain: Seeing Structure in Expressions ★ A.SSE

**Cluster:** *Interpret the structure of expressions.*

**Standard**: **A.SSE.2 ★ (all)**

Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* **(A.SSE.2)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.

**Connections**: A.SSE.1

**Explanations and Examples:**

This standard partners well with A.SSE.1 to support explorations when modeling (see A.SSE.1 for additional information) but the standard can also stand alone as an algebraic standard. There are many standards that focus on the specifics of re-writing an expression (i.e. factoring, completing the square, laws of exponents, trig identities, etc.) but this standard is not focused on typical algebraic manipulation. The goal is for students to take a step back and *see the structure* and connect the structure to the procedures. As with most ALL standards, there is not a specific procedure to teach students; rather this is a skill that develops over time and through intentional questions. For example, in Algebra 1 students learn the formula for slope ($m = \frac{y_2 - y_1}{x_2 - x_1}$). After developing an understanding of slope, they move into linear functions and look for the general form for a linear function. This standard supports the student development for the forms of a linear function because, through the structure, they can manipulate the equation into a more familiar form.

**Point Slope Form:** What equations could be written if we know the slope, *m,* and one point $(x_1, y_1)$?

$m = \frac{y - y_1}{x - x_1}$ ---We can multiply both sides by $x - x_1$

$m(x - x_1) = y - y_1$ **Point slope form**

$y = m(x - x_1) + y_1$ Not a typical form but it is structurally the same as transformations of functions. How can a linear function be

**Slope Intercept Form:** What equations could be written if we know the slope, *m,* and the y-intercept (0, b)?

$m = \frac{y - b}{x - 0}$ ---We can multiply both sides by $x - 0$, simply *x.*

$mx = y - b$

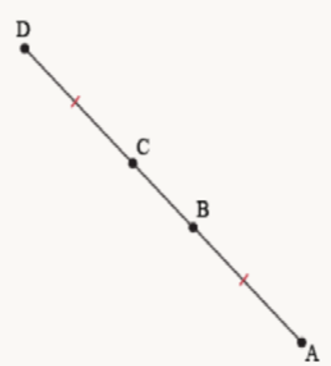
$y = mx + b$

There are many ways that these three examples rely on the structure of the equation.

- First, many students do not realize that understand that $(x_1, y_1)$ is the convention for writing a specific point or value when generalizing an equation but that an equation must still have variables, such as *x* and *y.*
- They also forget that the equation must have an equal sign. This seems like basic understanding but, when working a new type of problem such as finding the general form of an equation, they tend to forget these basic structural requirements.
- Teachers tend to direct students toward the end result that we see (Point-Slope Form or Slope-Intercept Form) rather than letting students play with the equation to see what results they arrive at. As math teachers, if provided with an unknown equation to rewrite into a known form, we would naturally eliminate fractions. Students need to see this same structure and know that eliminating fractions is often valuable. Notice that neither result in the Point-Slope Form example distributed the *m* to $(x - x_1)$ but students would likely think this is a good next step. They need to learn that, structurally, there is often more information gained from the factored form than the distributed form.
- Another common missed opportunity is the ability to perform arithmetic to make an equation less complicated. Students need to recognize the structure of adding or subtracting 0 and multiplying or dividing by 1 and make use of these properties whenever possible.

Student's weakness with the structure of expressions is especially apparent in Geometry class. When students are asked to prove something by performing algebraic operations on geometric statements, they often fail to see that the structure of this equation is the same as the structure learned in Algebra. For example, see the two geometry problems below, which use the structure of the expression to identify ways to rewrite it.

Given the diagram below, show that $\overline{AC} \cong \overline{BD}$.




| Statements | Reasons |
|---|---|
| 1. $\overline{AB} \cong \overline{CD}$ | Given (in figure) |
| 2. $AB = CD$ | Definition of congruence (1) |
| 3. $AB + BC = BC + CD$ | Addition of $BC$ to (2) |
| 4. $AC = BD$ | Segment addition postulate (3) |
| 5. $\overline{AC} \cong \overline{BD}$ | Definition of congruence (4) |

Many of the steps that rely on the structure of the equation are not written explicitly in the proof. How could additional steps be added to highlight the structure of the equation in relation to the geometric shape?

Show that if two angles are complementary to the same angle, then they're congruent.

| Statements | Reasons |
|---|---|
| 1. ∠1 and ∠3 are complementary | Given |
| 2. ∠2 and ∠3 are complementary | Given |
| 3. m∠1 + m∠3 = 90 | Definition of complementary angles |
| 4. m∠2 + m∠3 = 90 | Definition of complementary angles |
| 5. m∠1 = 90 − m∠3 | Subtract m∠3 from (3) |
| 6. m∠2 = 90 − m∠3 | Subtract m∠3 from (4) |
| 7. m∠1 = m∠2 | Transitive property of equality (5 and 6) |
| 8. ∠1 ≅ ∠2 | Definition of congruence (7) |

Similarly, students must recognize that that one solution path is to create the same expression in both equations in order to apply the transitive property. Another strategy could have been to solve for $m\angle 3$, set the resulting expressions equal and use the algebraic structure to solve. Both of these approaches rely on students' ability to recognize the structure of the expressions in order to create a strategy that will arrive at the needed proof statement. These types of problems are a struggle for students because there isn't a predictable algorithm and they find it difficult to see how the structure of the expression helps them identify a strategy. The ability to continually reinforce this standard, along with the need to develop this skill slowly over years of instruction is why this standard is an "ALL" standard.

**Instructional Strategies:**

Strategies from A.SSE.1 will also be useful here. In addition, teachers should use "think alouds" to focus on decision the decision they, as an expert, made as a result of the structure. Comparing and contrasting multiple correct solution strategies is another way to highlight how the structure of an expression can help the student rewrite it.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.A.2
  - Equivalent Expressions
  - Sum of Even and Odd
  - A Cubic Identity
  - Seeing Dots
  - Animal Populations

## Domain: Seeing Structure in Expressions (A.SSE)

- **Cluster:** *Use properties of rational numbers and irrational numbers*

**Standard**: **A.SSE.3**

Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. 

A.SSE.3a **(9/10)** Factor a quadratic expressions to reveal the zeros of the function it defines.  **(A.SSE.3a)**

### Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

**Connections**: A.SSE.1-2

### Explanations and Examples:

Write expressions in equivalent forms by factoring to find the zeros of a quadratic function and explain the meaning of the zeros. As a modeling standard, students should answer questions in the context of a situation.

**Examples:**

1. Given a quadratic function explain the meaning of the zeros of the function. That is if $f(x) = (x - c)(x - a)$ then $f(a) = 0$ and $f(c) = 0$.

2. Given a quadratic expression, explain the meaning of the zeros graphically. That is for an expression; $(x - a)(x - c)$, *a* and *c* correspond to the *x*-intercepts (if *a* and *c* are real).

3. Express $2(x^3 - 3x^2 + x - 6) - (x - 3)(x + 4)$in factored form and use your answer to say for what values of *x* the expression is zero.

4. The profit that a company makes selling an item (in thousands of dollars) depends on the price of the item (in dollars). If p is the price of the item, then three equivalent forms for the profit are:

Standard Form: $-2p^2 + 24p - 54$

Factored Form: $-2(p - 3)(p - 9)$

Vertex Form: $-2(p - 6)^2 + 18$

Which form is most useful for finding:

- The prices that give a profit of zero dollars?
- The profit when the price is zero?
- The price that gives the maximum profit?

**Instructional Strategies:**

Factoring is a difficult skill for students to master. There are three strategies which seem to be helpful for students. Two of these strategies extend the area model for multiplication that students learned in elementary school. Algebra Tiles are a fantastic kinesthetic tool that helps students visualize the factoring process. There are two common complaints given against using Algebra Tiles but both have simple solutions. The first is that classroom management is too difficult with all the little pieces. There are now multiple online technology tools that can be used to model with Algebra Tiles without using the physical tiles. Another common concern is that it is confusing to factor with Algebra Tiles. Experience has shown that, if students have enough practice with multiplication, that they will connect factoring as the inverse of multiplication and they will understand the "rules" that govern factoring.

Another factoring strategy uses the same area model as Algebra Tiles but at a more abstract level- the box method. This seems to be a preferred method amongst many Algebra teachers but will not convert to completing the square, as Algebra Tiles will.

Finally, many teachers use factoring by grouping. Each of these methods still requires a certain element of "guess and check" in order to find the pair of numbers that will multiply and add to get the desired results.

**Resources/Tools:**

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.B
  - Taxes and Sales
- A-SSE.B.3
  - Increasing or Decreasing?, Variation 2
  - Ice Cream
  - Profit of a Company
  - Profit of a Company, assessment variation
- A-SSE.B.3.c
  - Forms of exponential expressions

# Domain: Arithmetic with Polynomials with Rational Expressions (A.APR)

◆ **Cluster:** *Perform arithmetic operations on polynomials*

**Standard: A.APR.1**

**(9/10)** Add, subtract, and multiply polynomials. **(A.APR.1)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.

**Connections**: N.RN.1, A.SEE.1-2, F.BF.1a

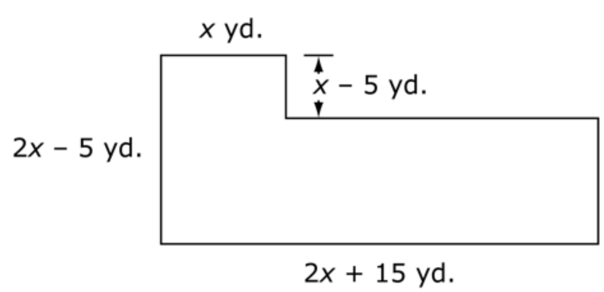
**Explanations and Examples:**

This standard is fairly procedural but the procedures are necessary because they are required for success with other standards and provide additional application of others. First, this procedure is a review and application of the laws of exponents. When practicing these problems, students will need to move between addition/subtraction and multiplication, applying two different exponent rules within the same problem. F.BF.1a asks students to combine functions to create new functions. These combinations might require these operations, connecting these two standards.

Finally, this standard connects to the ALL standard A.SSE.1-2 by asking students to interpret parts of an expressions and seeing the individual parts as a whole single number that we can perform arithmetic operations on.

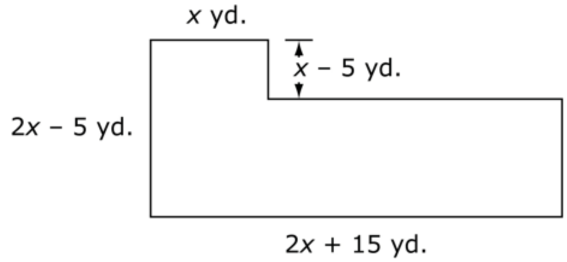
**Examples:**

1. A town council plans to build a public parking lot. The outline below represents the proposed shape of the parking lot.

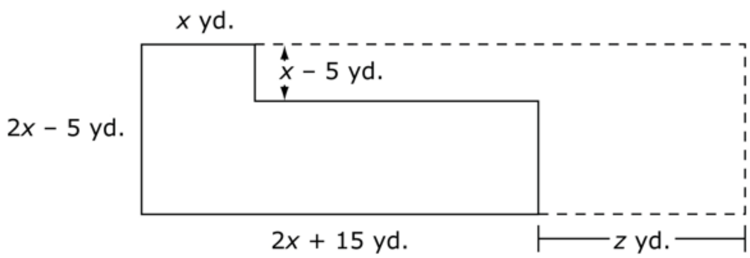



   Write an expression for the area, in square feet, of this proposed parking lot. Explain the reasoning you used to find the expression.

2. A town council plans to build a public parking lot. The outline below represents the proposed shape of the parking lot.




Write an expression for the area, in square feet, of this proposed parking lot. Explain the reasoning you used to find the expression.
The town council's second plan to double the area changes the shape of the parking lot to a rectangle, as shown in the diagram below.




Can the value of *z* be represented as a polynomial with integer coefficients? Justify your reasoning.

**Instructional Strategies:**

The primary strategy for this cluster is to make connections between arithmetic of integers and arithmetic of polynomials. In order to understand this standard, students need to work toward both understanding and fluency with polynomial arithmetic. Furthermore, to talk about their work, students will need to use correct vocabulary, such as integer, monomial, polynomial, factor, and term.

In arithmetic of polynomials, a central idea is the distributive property, because it is fundamental not only in polynomial multiplication but also in polynomial addition and subtraction. With the distributive property, there is little need to emphasize *misleading mnemonics*, such as FOIL, which is relevant only when multiplying two binomials, and the procedural reminder to "collect like terms" as a consequence of the distributive property. For example, when adding the polynomials 3x and 2x, the result can be explained with the distributive property as follows:
$3x + 2x = (3 + 2)x = 5x$.

An important connection between the arithmetic of integers and the arithmetic of polynomials can be seen by considering whole numbers in base ten place value to be polynomials in the base *b* = 10. For two-digit whole numbers and linear binomials, this connection can be illustrated with area models and algebra tiles. But the connections between methods of multiplication can be generalized further. For example, compare the product 213 x 47 with the product $(2b^2 + 1b + 3)(4b + 7)$.

$$
\begin{array}{rrrr}
 & 2b^2 + & 1b + & 3 \\
\times & & 4b + & 7 \\
\hline
 & 14b^2 + & 7b + & 21 \\
8b^3 + & 4b^2 + & 12b & \\
\hline
8b^3 + & 18b^2 + & 19b + & 21
\end{array}
\qquad
\begin{array}{rrrr}
 & 200 + & 10 + & 3 \\
\times & & 40 + & 7 \\
\hline
 & 1400 + & 70 + & 21 \\
8000 + & 400 + & 120 & \\
\hline
8000 + & 1800 + & 190 + & 21
\end{array}
\qquad
\begin{array}{r}
213 \\
\times \quad 47 \\
\hline
1491 \\
8520 \\
\hline
10011
\end{array}
$$

Note how the distributive property is in play in each of these examples: In the left-most computation, each term in the factor $(4b + 7)$ must be multiplied by each term in the other factor, $(2b^2 + 1b + 3)$, just like the value of each digit in 47 must be multiplied by the value of each digit in 213, as in the middle computation, which is similar to “partial products methods” that some students may have used for multiplication in the elementary grades. The common algorithm on the right is merely an abbreviation of the partial products method.

### Common Misconceptions

Some students will apply the distributive property inappropriately. Emphasize that it is the *distributive property of multiplication over addition*. For example, the distributive property can be used to rewrite $2(x + y)$ as $2x + 2y$, because in this product the second factor is a sum (i.e., involving addition). But in the product 2(*xy*), the second factor,(*xy*), is itself a product, not a sum.

Some students will still struggle with the arithmetic of negative numbers. Consider the expression $(-3) \cdot \big(2 + (-2)\big)$. On the one hand, $(-3) \cdot \big((2) + (-2)\big) = (-3) \cdot (0) = 0$. But using the distributive property, $(-3) \cdot \big((2) + (-2)\big) = (-3) \cdot (2) + (-3) \cdot (-2)$. Because the first calculation gave 0, the two terms on the right in the second calculation must be opposite in sign.

Thus, if we agree that $(-3) \cdot (2) = -6$, then it must follow that $(-3) \cdot (-2) = 6$.

Students often forget to distribute the subtraction to terms other than the first one. For example, students will write $(4x + 3) - (2x + 1) = 4x + 3 - 2x + 1 = 2x + 4$ rather than $4x + 3 - 2x - 1 = 2x + 2$.

Students will change the degree of the variable when adding/subtracting like terms. For example, $2x + 3x = 5x^2$ rather than 5*x*.

Students may not distribute the multiplication of polynomials correctly and only multiply like terms. For example, they will write $(x + 3)(x - 2) = x^2 - 6$ rather than $x^2 - 2x + 3x - 6$.

**Resources/Tools:**

"Manipulating Polynomials" – Mathematics Assessment Project :

This lesson unit is intended to help you assess how well students are able to manipulate and calculate with polynomials. In particular, it aims to identify and help students who have difficulties in switching between visual and algebraic representations of polynomial expressions and performing arithmetic operations on algebraic representations of polynomials, factorizing and expanding appropriately when it helps to make the operations easier.

## Domain: Arithmetic with Polynomials with Rational Expressions (A.APR)

**Cluster:** *Use polynomial identities to solve problems.*

**Standard: A.APR.4**

**(9/10/11)** Generate polynomial identities from a pattern. *For example, difference of squares, perfect square trinomials, (emphasize sum and difference of cubes in grade **11**)*. **(A.APR.4)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: A.APR.1, A.SSE.1-2

**Explanations and Examples:**

In Grade 6, students began using the properties of operations to rewrite expressions in equivalent forms. When two expressions are equivalent, an equation relating the two is called an *identity* because it is true for all values of the variables. This cluster is an opportunity to highlight polynomial identities that are commonly used in solving problems. To learn these identities, students need experience using them to solve problems.

This standard is both a 9/10 standard and an 11 standard because it is done when factoring is introduced with polynomials (which might be 9th for a traditionally sequenced Algebra 1 or 10th, course 2 for an Integrated Math sequence).In 9/10, students could explore difference of two squares, perfect square binomials, and others(list might not be complete):

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $(x-y)^2 = x^2 - 2xy + y^2$
3. $(x+y)(x-y) = x^2 - y^2$
4. $(x+a)(x+b) = x^2 + x(a+b) + ab$

Eleventh grade will address polynomial identities from cubes.

**Examples:**

Use the distributive law to explain why $x^2 - y^2 = (x-y)(x+y)$ for any two numbers *x* and *y*.

Derive the identity $(x-y)^2 = x^2 - 2xy + y^2$ from $(x+y)^2 = x^2 + 2xy + y^2$ by replacing $y$ by $-y$.

Use an identity to explain the pattern

$2^2 - 1^2 = 3$
$3^2 - 2^2 = 5$
$4^2 - 3^2 = 7$
$5^2 - 4^2 = 9$

*Solution:* $(n+1)^2 - n^2 = 2n + 1$ for any whole number *n*.

**Instructional Strategies:**

Students have been taught the rules for these identities and sometimes forget that the rules simply provide a shortcut. They forget (or don't realize) that they can factor the problems using the same procedures they use with other quadratics.

Students should be able to explain any of these identities. Furthermore, they should develop sufficient fluency that they can recognize expressions of the form on either side of these identities in order to replace that expression with an equivalent expression in the form of the other side of the identity.

With identities such as these, students can discover and explain facts about the number system. For example, in the multiplication table, the perfect squares appear on the diagonal. Diagonally, next to the perfect squares are "near squares," which are one less than the perfect square. Why?

Why is the sum of consecutive odd numbers beginning with 1 always a perfect square?

**Resources/Tools:**

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-APR.C.4
    - Trina'a Triangles

## Domain: Creating Equations A.CED

**Cluster:** *Create equations that describe numbers or relationships.*

**Standard**: **A.CED.1 ★ (all)**

Apply and extend previous understanding to create equations and inequalities in one variable and use them to solve problems. **(A.CED.1)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

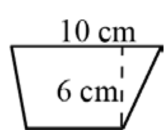
**Connections**: Modeling and Functions

**Explanations and Examples:**

Every year, in every course, students should be creating equations and inequalities. There isn't a time when we say, "The students have mastered it! They no longer need to develop this skill." For that reason, this standard is selected as an ALL standard. Algebra 1 focuses on creating equation and inequalities that are linear, quadratic, or exponential. Algebra 2 continues to increase in sophistication with linear, quadratic, and exponential but adds new function families such as rational, square root, logarithmic, and polynomial. Geometry reinforces algebraic skills while learning geometric properties by asking students to solve geometry problems using algebra skills.

**Examples:**

- Given that the following trapezoid has area 54 $cm^2$, set up an equation to find the length of the unknown base, and solve the equation.




- Lava coming from the eruption of a volcano follows a parabolic path. The height *h* in feet of a piece of lava *t* seconds after it is ejected from the volcano is given by $h(t) = -16t^2 + 64t + 936$. $h(t) = -t^2 + 16t + 936$. After how many seconds does the lava reach its maximum height of 1000 feet?

- The value of an investment over time is given by the equation $A(t) = 10{,}000(1.03)^t$. What does each part of the equation represent?

  *Solution:* The $10,000 represents the initial value of the investment. The 1.03 means that the investment will grow exponentially at a rate of 3% per year for *t* years.

- You bought a car at a cost of $20,000. Each year that you own the car the value of the car will decrease at a rate of 25%. Write an equation that can be used to find the value of the car after *t* years.

  *Solution:* $C(t) = \$20{,}000(0.75)^t$. The base is $1 - 0.25 = 0.75$ and is between 0 and 1, representing exponential decay. The value of $20,000 represents the initial cost of the car.

- Suppose a friend tells you she paid a total of $16,368 for a car, and you'd like to know the car's list price (the price before taxes) so that you can compare prices at various dealers. Find the list price of the car if your friend bought the car in:
  a) Arizona, where the sales tax is 5.6%.
  b) New York, where the sales tax is 8.25%.
  c) A state where the sales tax is *r*.

  *Solution:*
  a) If *p* is the list price in dollars then the tax on the purchase is 0.056*p*. The total amount paid is $p + 0.056p$, so

$$p + 0.056p = 16{,}368$$

$$(1 + 0.056)p = 16{,}368$$

$$p = \frac{16{,}368}{1 + 0.056} = \$15{,}500$$

$$p = \$15{,}500$$

  b) The total amount paid is $p + 0.0825p$ so

$$p + 0.0825p = 16{,}368$$

$$(1 + 0.0825)p = 16{,}368$$

$$p = \frac{16{,}368}{1 + 0.0825} =$$

$$p = \$15{,}120.55$$

  c) The total amount paid is $p + rp$ so

$$p + rp = 16{,}368$$

$$(1 + r)p = 16{,}368$$

$$\frac{16{,}368}{1 + r}$$

$p = \frac{16{,}368}{1+r}$ dollars.

**Instructional Strategies:**

Reading and comprehension strategies such as highlighting and annotating will help students make meaning from the problem. It is also important that the students understand the context and can visualize what is happening in the problem. These general strategies are well-known and can be effective literacy strategies that support writing equations.

A math specific strategy is to try and identify a general structure to math problems what can be applied to a variety of different situations. For example, there are many types of situations that fit into a part/part/whole or start/change/unknown situation in Algebra 1. Mixture problems is another general problem type with a fairly consistent problem structure. Identifying additional general problem types will help students see the larger structure present in Algebra problems. The appendix at the back of the Standards and at the back of each flip book provides a general structure that is useful scaffolding for students. Many situations will fit within these computation situations and can help students see the pattern across a wide array of problems.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.1
  - Planes and wheat
  - Paying the rent
  - Buying a car
  - Sum of angles in a polygon
- A-CED.A.2
  - Throwing a Ball

**Common Misconceptions:**

Students may believe that equations of linear, quadratic and other functions are abstract and exist only "in a math book," without seeing the usefulness of these functions as modeling real-world phenomena.

Additionally, they believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.

Students may interchange slope and $y$-intercept when creating equations. For example, a taxi cab costs $4 for a dropped flag and charges $2 per mile. Students may fail to see that $2 is a rate of change and is slope while the $4 is the starting cost and incorrectly write the equation as $y = 4x + 2$ instead of $y = 2x + 4$.

Given a graph of a line, students use the x-intercept for b instead of the y-intercept.

Given a graph, students incorrectly compute slope as run over rise rather than rise over run. For example, they will compute slope with the change in $x$ over the change in $y$.

Students do not know when to include the "or equal to" bar when translating the graph of an inequality.

Students do not correctly identify whether a situation should be represented by a linear, quadratic, or exponential function.

Students often do not understand what the variables represent. For example, if the height $h$ in feet of a piece of lava $t$ seconds after it is ejected from a volcano is given by $h(t) = -16t^2 + 64t + 936$ and the student is asked to find the time it takes for the piece of lava to hit the ground, the student will have difficulties understanding that $h = 0$ at the ground and that they need to solve for $t$.

## Domain: Creating Equations A.CED

**Cluster:** *Create equations that describe numbers or relationships.*

**Standard**: **A.CED.2 ★ (all)**

Apply and extend previous understanding to create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **(A.CED.2)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

**Connections**: See A.CED.1

**Explanations and Examples:**

See A.CED.1

**Examples:**

- The formula for the surface area of a cylinder is given by $A = 2\pi rh + 2\pi r^2$, where *r* represents the radius of the circular cross-section of the cylinder and *h* represents the height. Choose a fixed value for *h* and graph *V* vs. *r*. Then pick a fixed value for *r* and graph *V* vs. *h*. Compare the graphs.

  What is the appropriate domain for *r* and *h*? Be sure to label your graphs and use an appropriate scale.
- Gold is alloyed with different metals to make it hard enough to be used in jewelry. The amount of gold present in a gold ally is measured in 24ths called karats. 24-karat gold is 100% gold. Similarly, 18-karat gold is 75% gold.

  How many ounces of 18-karat gold should be added to an amount of 12-karat gold to make 4 ounces of 14-karat gold? Graph equations on coordinate axes with labels and scales.

- A metal alloy is 25% copper. How much of each alloy should be used to make 1000 grams of a metal alloy that is 45% copper?

- Find a formula for the volume of a single-scoop ice cream cone in terms of the radius and height of the cone. Rewrite your formula to express the height in terms of the radius and volume. Graph the height as a function of radius when the volume is held constant.

- David compares the sizes and costs of photo books offered at an online store. The table below shows the cost for each size photo book.

| Book Size | Base Price | Cost for Each Additional Page |
|---|---|---|
| 7-in. by 9-in. | \$20 | \$1.00 |
| 8-in. by 11-in. | \$25 | \$1.00 |
| 12-in. by 12-in. | \$45 | \$1.50 |

1. Write an equation to represent the relationship between the cost, *y*, in dollars, and the number of pages, *x*, for each book size. Be sure to place each equation next to the appropriate book size. Assume that *x* is at least 20 pages.

| Book Size | Equation |
|---|---|
| 7-in. by 9-in. | |
| 8-in. by 11-in. | |
| 12-in. by 12-in. | |

2. What is the cost of a 12-in. by 12-in. book with 28 pages?

3. How many pages are in an 8-in. by 11-in. book that costs \$49?

*Solution*:

1. 7-in. by 9-in. $y = x$
   8-in. by 11-in. $y = x + 5$
   12-in. by 12-in. $y = 1.50x + 15$

2. \$57
3. 44 pages

**Instructional Strategies:** See A.CED.1

**Resources/Tools:**
Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.2
  - Clea on an escalator
  - Silver rectangle

## Domain: Creating Equations A.CED

**Cluster:** *Create equations that describe numbers or relationships.*

**Standard**: **A.CED.3 ★ (all)**

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★ **(A.CED.3)**

**Suggested Standards for Mathematical Practice (MP):**

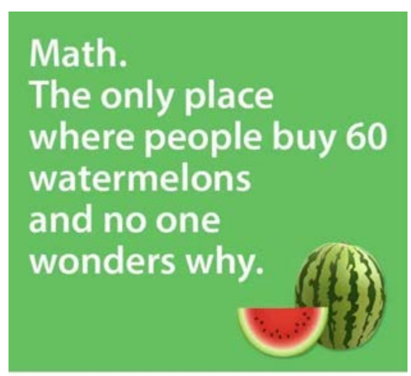
- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

**Connections**: See A.CED.1

**Explanations and Examples:**

**Part 1:** Represent constraints by equations or inequalities, and by systems of equations and/or inequalities.
A constraint is often thought about (by math teachers) as a linear programing problem. While that type of problem certainly fits within this standard, it would not be categorized as an ALL standard if that were the entire scope. The word "constrained" can apply to a wide variety of problem types. For example, if child tickets cost $3 and an adult's ticket cost $5, we can purchase tickets in any quantity- unconstrained by additional information. However if additional information were provided about how much money could be spent, then the solution set would be *constrained* to the values that add to no more than that amount. This situation also has a constrained domain; which is constrained to discrete values and that must be between zero and the maximum number of a single variable able to purchase tickets.

**Part 2:** and interpret solutions as viable or non-viable options in a modeling context.
This part of the standard has frequently been interpreted to address extraneous solutions but the phrase "in a modeling context" should redirect instruction back to thinking about the constraints on a problem, which could include constraints on the solution. For example, in the situation described above, it isn't possible to purchase half a ticket. Therefore, viable solutions would be whole numbers. Negative numbers are also not possible. The viable has broader meaning than simply possible or not possible. Is the solution "viable" or feasible? If we were working a rate problem, it might be possible for the speed of a motorcycle to be 350 mph according to all the constraints given in the situation but is it a viable solution? Thinking about the asymptote on an exponential function, as the value gets infinitesimally close to zero, does it remain viable?



desmos.com

As seen in many of the ALL standards, there is not a simple procedure or algorithm that will always produce the correct answer. A classroom discussion about the variety of factors that might constrain the equation or solution provides a rich opportunity for students to learn how to think about the world mathematically, in a truly real life situation. The statement above, posted frequently on social media, reflects the memes that result from a lack of attention to viability in a modeling context.

**Examples:**

A club is selling hats and jackets as a fundraiser. Their budget is $1500 and they want to order at least 250 items. They must buy at least as many hats as they buy jackets. Each hat costs $5 and each jacket costs $8.

- Write a system of inequalities to represent the situation.
- Graph the inequalities.
- If the club buys 150 hats and 100 jackets, will the conditions be satisfied?
- What is the maximum number of jackets they can buy and still meet the conditions?

Represent inequalities describing nutritional and cost constraints o combinations of different foods.

The coffee variety *Arabica* yields about 750 kg of coffee beans per hectare, while *Robusta* yields about 1200 kg per hectare. Suppose that a plantation has *a* hectares of *Arabica* and *r* hectares of *Robusta*.

a) Write an equation relating *a* and *r* if the plantation yields 1,000,000 kg of coffee.
b) On August 14, 2003, the world market price of coffee was$1.42 per kg of *Arabica* and $0.73 per kg of *Robusta*. Write an equation relating *a* and *r* if the plantation produces coffee worth $1,000,000.

This task is designed to make students think about the meaning of the quantities presented in the context and choose which ones are appropriate for the two different constraints presented. The purpose of the task is to have students *generate* the constraint equations for each part (though the problem and not to have students *solve* said equations. If desired, instructors could also use this task to touch on such solutions by finding and interpreting solutions to the system of equations created in parts (a) and (b).

*Solution:*

a) We see that *a* hectares of *Arabica* will yield 750*a* kg of beans, and that *r* hectares of *Robusta* will yield 1200r kg of beans. So the constraint equation is

$$750a + 1{,}200r = 1{,}000{,}000.$$

b) We know that a hectares of *Arabica* yield 750*a* kg of beans worth $1.42/kg for a total dollar value of $1.42(750a) = 1065a$. Likewise, *r* hectares of *Robusta* yield 1200*r* kg of beans worth $0.73/kg for a total dollar value of $0.73(1200r) = 876r$. So the equation governing the possible values of *a* and *r* coming from the total market value of the coffee is

$$1065a + 876r = 1{,}000{,}000$$

**Instructional Strategies:**

While this standard represents an exciting opportunity to open the math world up to big, wide, real world; that ambiguity can be intimidating and/or distracting. Students will enjoy thinking about ways the problem will be constrained by the context of the problem. Teaching students Talk Moves such as revoicing or adding on can provide some structure to guide the conversation. Another strategy is to restrict the types of constraints the students can discuss either by number or type. Finally, as with each of other ALL standards, the goal is to *enhance* student understanding of functions and how quantities vary together.. Not every solution is possible. Infinite possibilities doesn't mean infinite viable solutions. Do not let the discussion drag you down rabbit hole or cause confusion.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.3
  - Dimes and Quarters
  - Writing Constraints
  - Growing Coffee
  - How Much Folate
  - Bernardo and Sylvia Play a Game

## Domain: Creating Equations A.CED

**Cluster:** *Create equations that describe numbers or relationships.*

**Standard**: **A.CED.4 ★ (all)**

Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance R.*★ **(A.CED.4)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

**Connections**: A.SSE.2, A.REI.2

**Explanations and Examples:**

Rearranging formulas is a critical skill in many applications such as computer programming or spreadsheet formulas. But, often, in the process of solving for the quantity of interest, student misunderstandings about the steps for solving equations will come to the forefront. Mistakes that they wouldn't make when solving other equations will begin to appear. The algebraic reasoning required to correct the flaw causes significant struggle, even when the same problem with numbers would have an immediate and correct answer. Usiskin provided some insight into this area of student difficulty- quoted below.

Quoting from Teaching Mathematics in Grades 6-12:Developing Research Based Instructional Practices:

**Figure 8.2** Examples to illustrate the complexity of literal symbols (Usiskin, 1988, p. 40).

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot (\frac{1}{n})$
5. $y = kx$

"Usiskin (1988) provided a poignant example of the complexity involved in understanding the meanings of literal symbols by asking readers to consider the equations shown in Figure 8.2. Although all five symbol strings in Figure 8.2 are equations, each uses literal symbols in different ways. In equation 2, for example, x is often referred to as an unknown whose value can be determined by dividing each side by five. In equation 5, however, we usually think of x as being free to vary and take on numerous different values, making x feel more like a variable than a specific unknown. In equation 5, k is often thought of as a constant that specifies the slope of a line. Therefore, this set of five equations illustrates at least three different ways literal symbols can be used in algebra— to represent unknowns, variables, and constants. In addition, the equations themselves can be used for different purposes. To illustrate this, observe that equation 1 is commonly referred to as a formula, equation 3 as an identity, and equation 4 as a property."

Recognizing that students struggle with the meaning of the variable in a literal equation does not change the reasoning required. It does indicate that shifting back and forth between different meanings for a variable can be confusing for students and addressing this confusion directly might help some students.

So one goal for this standard is for students to become comfortable with different uses of the variable in the equation and to surface any flaws with their algebraic reasoning. Another goal is for students to develop the ability to think ahead to their goal and plan a path to get there. For example, if the variable of interest appears in several different terms, with different exponents, then factoring or completing the square might be required. If the variable of interest is in the exponent, then logarithms might be the strategy. As has been stated before, ALL standards require practice and the development of sophistication over time. With each new function family, an effort to should be made to solve a formula for a given variable.

**Examples:**

1. Solve for *h:* $A = \frac{1}{2}bh$

   An example like the problem above can highlight how students will move the $\frac{1}{2}$. Dividing by $\frac{1}{2}$ would work if the student remembers how to divide fractions. A more robust solution path would be to multiply by the multiplicative inverse of $\frac{1}{2}$ or 2. A student who writes the answer $h = \frac{A}{\frac{1}{2}b}$ is correctly reasoning about equations but does not see that the fraction divided by a fraction is unnecessary. Even more concerning is the student whose answer is $h = \frac{A}{2b}$ because they incorrectly divided by $\frac{1}{2}$.

2. Solve $A = P + Prt$ for *r.*

   When working this problem, some students arrive at the correct solution $r = \frac{A-P}{Pt}$ but will go one step too far and "cancel" the *P*. These students struggle with the reasoning necessary in A.SSE.2, seeing part of an expression as a single entity. The numerator cannot be separated in this way and should be viewed as a whole piece.

3. Solve $A = P + Prt$ for *P.*

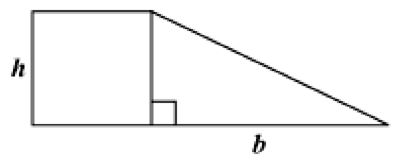
   This kind of problem is particularly challenging for students because they do not see how connection between factoring and distributing. With numbers, they understand that a problem like $2(x + y)$ will "give" the *x* and *y* a two but will not necessarily observe that *2x+2y* can be "undone" by factoring the two. A problem like the one above can highlight that students have missed this connection.

An Illustrated Math Task:

4. A bacteria population *P* is modeled by the equation $P = P_0 10^{kt}$ where time *t* is measured in hours, *k* is a positive constant, and $P_0$ is the bacteria population at the beginning of the experiment. Rewrite this equation to find *t* in terms of *P.*

   This equation will assess if students understand *when* to use logarithms.

   The figure below is made up of a square with height, *h* units, and a right triangle with height, *h* units, and base length, *b* units.




   The area of this figure is 80 square units.

   Write an equation that solves for the height, *h*, in terms of *b*.
   Show all work necessary to justify your answer.

   *Sample Response*:

$$h^2 + \frac{1}{2}bh = 80$$
$$h^2 + \frac{1}{2}bh + \frac{1}{16}b^2 = 80 + \frac{1}{16}b^2$$
$$\left(h + \frac{1}{4}b\right)^2 = 80 + \frac{1}{16}b^2$$
$$h + \frac{1}{4}b = \sqrt{80 + \frac{1}{16}b^2}$$
$$h = \sqrt{80 + \frac{1}{16}b^2} - \frac{1}{4}b$$

**Instructional Strategies:**

Substituting numbers in for the variables and solving it side by side with the literal equation can help scaffold the abstract thinking required.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.4
  - Equations and Formulas

## Domain: Reasoning with Equations and Inequalities A.REI

**Cluster:** *Understand solving equations as a process of reasoning and explain the reasoning.*

**Standard**: **A.REI.1 (all)**

Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. **(A.REI.1)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.

**Connections**: Algebra standards

**Explanations and Examples:**

In Algebra 1 students should focus on and master A.REI.1 for linear equations and be able to extend and apply their reasoning to other types of equations in future courses. In Algebra 2, extend to simple rational and radical equations.

Assuming an equation has a solution, construct a convincing argument that justifies each step in the solution process. Justifications may include the associative, commutative, and division properties, combining like terms, multiplication by 1, etc.

Properties of operations can be used to change expressions on either side of the equation to equivalent expressions. In addition, adding the same term to both sides of an equation or multiplying both sides by a non-zero constant produces an equation with the same solutions. Other operations, such as squaring both sides, may produce equations that have extraneous solutions.

**Examples:**

1. Explain why the equation $\frac{x}{2}+\frac{7}{3}=5$ has the same solutions as the equation $3x+14=30$.
   Does this mean that $\frac{x}{2}+\frac{7}{3}$ is equal to $3x+14$?

2. Show that $x=2$ and $x=-3$ are solutions to the equation $x^2+x=6$.
   Write the equation in a form that shows these are the only solutions, explaining each step in your reasoning.

3. Transform $2x-5=7$ to $2x=12$ and tell what property of equality was used.
   *Solution:*

$$\begin{aligned} 2x-5&=7 \\ 2x-5+5&=7+5 \quad \text{Addition property of equality .} \\ 2x&=12 \end{aligned}$$

**Instructional Strategies:**

Challenge students to justify each step of solving an equation. Transforming $2x - 5 = 7$ to $2x = 12$ is possible because $5 = 5$, so adding the same quantity to both sides of an equation makes the resulting equation true as well. Each step of solving an equation can be defended, much like providing evidence for steps of a geometric proof.

Provide examples for how the same equation might be solved in a variety of ways as long as equivalent quantities are added or subtracted to both sides of the equation, the order of steps taken will not matter.

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ -\quad 2 &=& -2 \\ \hline 3n &=& n - 12 \\ -n &=& -n \\ \hline 2n &=& -12 \\ n &=& -6 \end{array}$$

**OR**

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ +\quad 10 &=& +10 \\ \hline 3n + 12 &=& n \\ -3n &=& -3n \\ \hline 12 &=& -2n \\ n &=& -6 \end{array}$$

**OR**

$$\begin{array}{rcl} 3n + 2 &=& n - 10 \\ -n &=& -n \\ \hline 2n + 2 &=& -10 \\ -2 &=& -2 \\ \hline 2n &=& -12 \\ n &=& -6 \end{array}$$

Connect the idea of adding two equations together as a means of justifying steps of solving a simple equation to the process of solving a system of equations. A system consisting of two linear functions such as $2x + 3y = 8$ and $x - 3y = 1$ can be solved by adding the equations together, and can be justified by exactly the same reason that solving the equation $2x - 4 = 5$ can begin by adding the equation $4 = 4$.

Investigate the solutions to equations such as $3 = x + \sqrt{2x - 3}$. By graphing the two functions, $y = 3$ and $y = x + \sqrt{2x - 3}$ students can visualize that graphs of the functions only intersect at one point. However, subtracting $x = x$ from the original equation yields $3 - x = \sqrt{2x - 3}$ which when both sides are squared produces a quadratic equation that has two roots $x = 2$ and $x = 6$. Students should recognize that there is only one solution $(x = 2)$ and that $x = 6$ is generated when a quadratic equation results from squaring both sides; $x = 6$ is extraneous to the original equation. Some rational equations, such as $\frac{x}{(x-2)} = \frac{2}{(x-2)} + \frac{5}{x}$ result in extraneous solutions as well.

Begin with simple, one-step equations and require students to write out a justification for each step used to solve the equation.

Ensure that students are proficient with solving simple rational and radical equations that have no extraneous solutions before moving on to equations that result in quadratics and possible solutions that need to be eliminated.

Provide visual examples of radical and rational equations with technology so that students can see the solution as the intersection of two functions and further understand how extraneous solutions do not fit the model.

It is very important that students are able to reason how and why extraneous solutions arise.
Computer software that generates graphs for visually examining solutions to equations, particularly rational and radical. Examples of radical equations that do and do not result in the generation of extraneous solutions should be prepared for exploration.

**Common Misconception:**

Students may believe that solving an equation such as $3x + 1 = 7$ involves "only removing the 1," failing to realize that the equation $1 = 1$ is being subtracted to produce the next step.

Additionally, students may believe that all solutions to radical and rational equations are viable, without recognizing that there are times when extraneous solutions are generated and have to be eliminated.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Mathematics Assessment Project:

- "Building and Solving Equations 2"

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.A
  - Same Solutions?
  - How does the solution change?

# Domain: Reasoning with Equations and Inequalities A.REI

**Cluster:** *Solve equations and Inequalities in one variable.*

**Standard**: **A.REI.2 (all)**

Apply and extend previous understanding to solve equations, inequalities, and compound inequalities in one variable, including **literal equations** and inequalities. **(A.REI.3)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: A.CED.4

**Explanations and Examples:**

See A.CED.4

**Examples:**

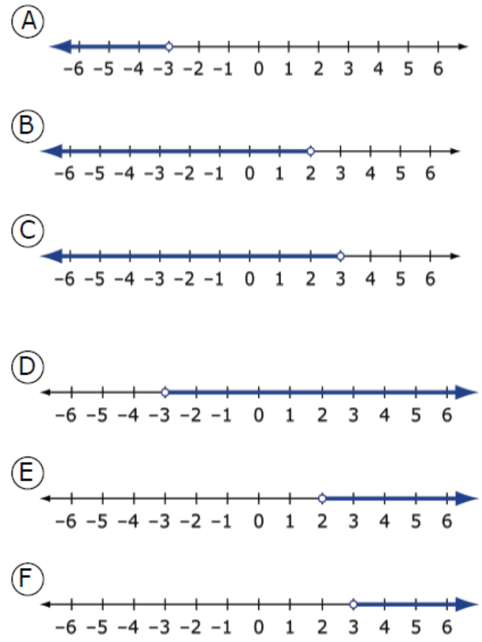
1. Solve for the variable:
   - $\frac{7}{3}y - 8 = 111$
   - $3x > 9$
   - $ax + 7 = 12$
   - $\frac{3+x}{7} = \frac{x-9}{4}$
   - Solve for $x$: $\frac{2}{3}x + 9 < 18$

2. Match each inequality in items 1 – 3 with the number line in items A – F that represent the solution to the inequality.

**1** $\boxed{-4x < -12}$

**2** $\boxed{2(x + 2) < 8}$

**3** $\boxed{5 - 2x < 2 - x}$




*Solutions:* **1.** F **2**. B **3.** F

**Instructional Strategies:**

There are two major reasons for discussing the topic of inequalities along with equations: First, there are analogies between solving equations and inequalities that help students understand them both**.** Second, the applications that lead to equations almost always lead in the same way to inequalities.

In grades 6-8, students solve and graph linear equations and inequalities. Graphing experience with inequalities is limited to graphing on a number line diagram. Despite this work, some students will still need more practice to be proficient. It may be beneficial to remind students of the most common solving techniques, such as converting fractions from one form to another, removing parentheses in the sentences, or multiplying both sides of an equation or inequality by the common denominator of the fractions. Students must be aware of what it means to check an inequality's solution. The substitution of the end points of the solution set in the original inequality should give equality regardless of the presence or the absence of an equal sign in the original sentence**.** The substitution of any value from the rest of the solution set should give a correct inequality.

Careful selection of examples and exercises is needed to provide students with meaningful review and to introduce other important concepts, such as the use of properties and applications of solving linear equations and inequalities. Stress the idea that the application of properties is also appropriate when working with equations or inequalities that include more than one variable, fractions and decimals. Regardless of the type of numbers or variables in the equation or inequality, students have to examine the validity of each step in the solution process.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(B + b)$ when solving for *b* is similar to solving an equation with one variable. Provide students with an opportunity to abstract from particular numbers and apply the same kind of manipulations to formulas as they did to equations. One of the purposes of doing abstraction is to learn how to evaluate the formulas in easier ways and use the techniques across mathematics and science.

Draw students' attention to equations containing variables with subscripts. The same variables with different subscripts (e.g., $x_1 and\ x_2$) should be viewed as different variables that cannot be combined as like terms. A variable with a variable subscript, such as $a_n$, must be treated as a single variable – the $n^{th}$ term, where variables *a* and *n* have different meaning.

### Common Misconceptions:

Some students may believe that for equations containing fractions only on one side, it requires "clearing fractions" (the use of multiplication) only on that side of the equation. To address this misconception, start by demonstrating the solution methods for equations similar to $\frac{1}{4}x + \frac{1}{5}x + \frac{1}{6}x + 46 = x$ and stress that the Multiplication Property of Equality is applied to both sides, which are multiplied by 60.

Students may confuse the rule of changing a sign of an inequality when multiplying or dividing by a negative number with changing the sign of an inequality when one or two sides of the inequality become negative (for ex., $3x > -15\ or\ x < -5$).

Some students may believe that subscripts can be combined as $b_1 + b_2 = b_3$ and the sum of different variables *d* and D is $2D(d + D = 2D)$.

### Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.B
  - Integer Solutions to Inequality
- A-REI.B.3
  - Reasoning with linear inequalities

# Domain: Reasoning with Equations and Inequalities (A.REI)

▶ **Cluster:** *Solve equations and inequalities in one variable.*

**Standard**: **A.REI.3**

Solve equations in one variable and give examples showing how extraneous solutions may arise.

A.REI.3a. **(9/10/11)** Solve rational, absolute value and square root equations. **(A.REI.2)**

**(9/10)** Limited to simple equations such as, $2\sqrt{x-3}+8=16, \frac{x+3}{2x-1}=5, x \neq \frac{1}{2}$.

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.

**Connections**: A.SSE.2, A.CED, A.REI.1, A.REI.2

**Explanations and Examples:**

Extending multiple standards about solving equations (A.CED, A.REI.1 & 2), this standard specifies that students in 9th or 10th grade will solve equations that extend beyond linear and quadratic. Students will solve simple rational, absolute value, and square root function during the first two years of high school and during 11th grade, these types of equations will become more complicated.

Extraneous solutions arise when students reverse a step that isn't reversible, such as taking the square root, or if the solution is not viable in the context. To avoid extraneous solutions, students must check the solutions when the problem is a type where this is a risk.

**Examples:**

1. Solve for x:
   - $\sqrt{x+2}=5$
   - $\frac{7}{8}\sqrt{2x-5}=21$
   - $\frac{x+2}{x+3}=2$
   - $\sqrt{3x-7}=-4$
2. Solve the following two equations by isolating the radical on one side and squaring both sides:

   i. $\sqrt{2x+1}-5=-2$

   ii. $\sqrt{2x+1}+5=2$
3. If we raise both sides of an equation a power, we sometimes obtain an equation which has more solutions than the original one. (Sometimes the extra solutions are called extraneous solutions.)

4. Which of the following equations result in extraneous solutions when you raise both sides to the indicated power? Explain.
    i. $\sqrt{x} = 5$, square both sides
    ii. $\sqrt{x} = -5$, square both sides
    iii. $\sqrt[3]{x} = 5$, cube both sides
    iv. $\sqrt[3]{x} = -5$, cube both sides

5. Create a square root equation similar to the one in problem 1 that has an extraneous solution.

   Show the algebraic steps you would follow to look for a solution, and indicate where the extraneous solution arises.

***Selected Solutions:***

2. i.
$$\begin{aligned}\sqrt{2x+1}-5&=-2\\ \left(\sqrt{2x+1}\right)^2&=(3)^2\\ 2x+1&=9\\ 2x&=8\\ x&=4\end{aligned}$$

Checking:
$$\begin{aligned}\sqrt{2\cdot 4+1}-5&=-2\\ \sqrt{9}-5&=-2\\ 3-5&=-2\\ -2&=-2\end{aligned}$$

So, *x* = 4 is the solution to the equation.

ii.
$$\begin{aligned}\sqrt{2x+1}+5&=2\\ \left(\sqrt{2x+1}\right)^2&=(-3)^2\\ 2x+1&=9\\ 2x&=8\\ x&=4\end{aligned}$$

Checking:
$$\begin{aligned}\sqrt{2\cdot 4+1}+5&=2\\ \sqrt{9}+5&=2\\ 3+5&=2\\ 8&\neq 2\end{aligned}$$

So this equation has no solution.

4. The only one of the equations that produces an extraneous solution is $\sqrt{x} = -5$

The square root symbol (like all even roots) is defined to be the positive square root, so a positive root can never be equal to a negative number.  Squaring both sides of the equation will make that discrepancy disappear;  the square of a positive number is positive but so is the square of a negative number, so we'll end up with a solution to the new equation even though there was no solution to the original equation.

This isn't the case with odd roots - a cube root of a positive number is positive, and a cube root of a negative number is negative. When we cube both sides of the last equation, the negative remains, and we end up with a true solution to the equation.

**Instructional Strategies:** See A.REI.1

**Resources/Tools:**

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI-A.2
    - Radical Equations
    - Basketball

# Domain: Reasoning with Equations and Inequalities (A.REI)

▶ **Cluster:** *Solve equations and inequalities in one variable.*

**Standard: A.REI.5**

Solve quadratic equations and inequalities

A.REI.5a. **(9/10)** Solve quadratic equations by inspection *(e.g. for $x^2 = 49$)*, taking square roots, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives no real solutions. **(A.REI.4b)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: A.REI, A.APR.4

**Explanations and Examples:**

Students are expected to solve quadratic equations using 4 different methods:

- Inspection
- Square roots
- Quadratic Formula
- Factoring

Notice completing the square (A.SSE.3b) has become an 11th grade expectation. Students are also expected to know when each strategy is appropriate, based on the initial form of the equation. In 9/10 grade, students are only expected to recognize no real solutions, they do not use complex numbers. Students do not have to use the discriminant to identify the number of real solutions.

While this is not a modeling standard, several ALL standards should be connected with this standard and those standards require modeling. Solving quadratic equations requires students to see the parts of the expression as individual pieces (A.SSE.1a) and as larger composite pieces (A.SSE.1b). Also, recognizing which solution strategy will be used requires students to identify the structure of an expression (A.SSE.2) because solving by inspection and square roots requires the student to isolate the $x^2$ term while solving with the quadratic formula and factoring requires setting the whole equation equal to zero. Finally, A.CED.1 applies to solving all equations in one variable.

**Examples:**

1. Solve using each of the four methods at least once.
   a. $x^2 - 100 = 0$
   b. $x^2 = 180$

   c. $x^2 + x - 12 = 0$
   d. $(x-2)^2 - 16 = 0$
   e. $x^2 - 8x = 1$
   f. $4x^2 - 19x - 5 = 0$
   g. $3x^2 + 5x = -1$

2. The height, 190 in feet, of an object above the ground is given by $h = -16t^2 + 64t + 190$, where $t$ is the time in seconds. Find the time it takes the object to strike the ground.

Solution:

Let's first find the time it takes for the object to hit the ground. Since $h$ represents the height above the ground, we would like to know at what time $h = 0$. So in the equation $h = -16t^2 + 64t + 190$ we will set $h = 0$ and solve for the time, $t$.
We have

$$h = -16t^2 + 64t + 190$$
$$0 = -16t^2 + 64t + 190$$

So we simply want to solve this quadratic equation. It is easiest to use the quadratic formula in this situation. So we get

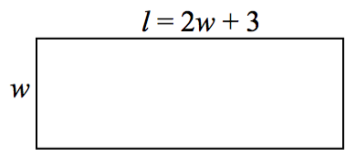
$$\begin{aligned} t &= \frac{-64 \pm \sqrt{64^2 - 4(-16)(190)}}{2(-16)} \\ &= \frac{-64 \pm \sqrt{16256}}{-32} \\ &= \frac{-64 \pm 8\sqrt{254}}{-32} \\ &= \frac{8 \pm \sqrt{254}}{4} \\ &\approx 5.98, -1.98 \end{aligned}$$

However, since $t$ represents time, we must throw out –1.98. Therefore, it takes 5.98 seconds for the object to strike the ground.

3. The length of a rectangle is three more than twice the width. Determine the dimensions that will give a total area of 27 $m^2$.

Solution:

First we need to draw a picture to visualize the problem. Since the length is 3 more that twice the width, we will have $l = 2w + 3$. So we have the following picture




Now, since both parts of this question deal with the area of this rectangle, lets begin by generating a function for the area. Since $A = l \cdot w$ we have

$$A = (2w+3)w$$
$$= 2w^2 + 3w$$

For the first part, we want to know what dimensions make an area of 27 m$^2$. Thus, we can insert 27 for *A* into our function and solve for *w*. We have

$$27 = 2w^2 + 3w$$
$$2w^2 + 3w - 27 = 0$$
$$(w-3)(2w+9) = 0$$
$$w = 3, -\tfrac{9}{2}$$

So, since *w* represents the width of a rectangle we must omit the negative value. Therefore, we have $w = 3$. Plugging that value into $l = 2w + 3$ we get $l = 2 \cdot 3 + 3 = 9$.

Therefore, the dimensions that give the rectangle an area of 27 m$^2$ are 9 m by 3 m.

**Instructional Strategies:**

Start by inspecting equations, such as $x^2 = 9$, that has two solutions, x = 3 and x = -3. Next, progress to equations such as $(x-7)^2 = 9$ by replacing the expression *x* – 7 for a single variable such, *x,* and solving them either by "inspection" or by taking the square root on each side:

$$(x-7)^2 = 9$$
$$x - 7 = \pm 3$$
$$x = 7 \pm 3$$

Graph both pairs of solutions (-3 and 3, 4 and 10) on the number line and notice that 4 and 10 are 7 units to the right of – 3 and 3. So, the substitution of *x* – 7 for *x* moved the solutions 7 units to the right. Next, graph the function $y = (x-7)^2 - 9$, pointing out that the *x*-intercepts are 4 and 10, and emphasizing that the graph is the translation of 7 units to the right and 9 units down from the position of the graph of the parent function $y = x^2$ that passes through the origin (0, 0). Generate more equations of the form $y = a(x-h)^2 + k$ and compare their graphs using a graphing technology.

Highlight and compare different approaches to solving the same problem. Use technology to recognize that two different expressions or equations may represent the same relationship. For example, since $x^2 - 10x + 25 = 0$ can be rewritten as $(x-5)(x-5) = 0$ *or* $(x-5)^2 = 0$ *or* $x^2 = 25$, these are all representations of the same equation that has a solution x = 5. Support it by putting all expressions into graphing calculator. Compare their graphs and generate their tables displaying the same output values for each expression.

Discourage students from giving a preference to a particular method of solving quadratic equations. Students need experience in analyzing a given problem to choose an appropriate solution method before their computations become burdensome. Point out that the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is a universal tool that can solve any quadratic equation; however, it is not efficient to use the Quadratic Formula when the quadratic equation is missing either a middle term, *bx*, or a constant term, *c*. When it is missing a constant term, (e.g., $3x^2 - 10x5 = 0$) a factoring method becomes more efficient. If a middle term is missing (e.g., $2x^2 - 15 = 0$), a square root method is the most appropriate.

**Common Misconceptions:**

Some students may think that rewriting equations into various forms (taking square roots, completing the square, using quadratic formula and factoring) are isolated techniques within a unit of quadratic equations. Teachers should help students see the value of these skills in the context of solving higher degree equations and examining different families of functions.

**Resources/Tools:**

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.B.4.b
  - Braking Distance
  - Quadratic Sequence 1
  - Quadratic Sequence 2
  - Quadratic Sequence 3
  - Springboard Dive
  - Zero Product Property 4

# Domain: Reasoning with Equations and Inequalities (A.REI)

- **Cluster:** *Solve systems of equations.*

**Standard**: **A.REI.6**

**(9/10)** Analyze and solve pairs of simultaneous linear equations.

A.REI.6a. **(9/10)** Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. **(8.EE.8a)**

A.REI.6b. **(9/10)** Solve systems of two linear equations in two variables algebraically, and estimate solutions by graphing the equations. Solve simple cases by inspection. *For example,* $3x + 2y = 5$ *and* $3x + 2y = 6$ *have no solution because* $3x + 2y$ *cannot simultaneously be 5 and 6.* **(8.EE.8b)**

A.REI.6c. **(9/10)** Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.* **(8.EE.8c)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

**Connections**: A.REI, F.IF

**Explanations and Examples:**

Systems of linear equations can also have one solution, infinitely many solutions or no solutions. Students will discover these cases as they graph systems of linear equations and solve them algebraically.

Systems of linear equations can also have one solution, infinitely many solutions or no solutions. Students will discover these cases as they graph systems of linear equations and solve them algebraically.

By making connections between algebraic and graphical solutions and the context of the system of linear equations, students are able to make sense of their solutions. Students need opportunities to work with equations and context.

**Examples:**

Find *x* and *y* using elimination and then using substitution.

$$3x + 4y = 7$$
$$\underline{-2x + 8y = 10}$$

Plant A and Plant B are on different watering schedules. This affects their rate of growth. Compare the growth of the two plants to determine when their heights will be the same.

Let *W* = number of weeks

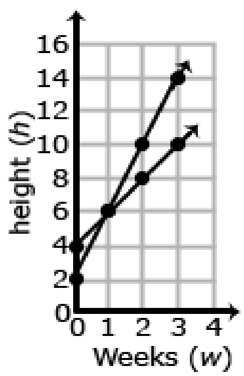
Let *H* = height of the plant after *W* weeks

| Plant A | | |
|---|---|---|
| **W** | **H** | |
| 0 | 4 | (0,4) |
| 1 | 6 | (1,6) |
| 2 | 8 | (2,8) |
| 3 | 10 | (3,10) |

| Plant B | | |
|---|---|---|
| **W** | **H** | |
| 0 | 2 | (0,2) |
| 1 | 6 | (1,6) |
| 2 | 10 | (2,10) |
| 3 | 14 | (3,14) |

Given each set of coordinates, graph their corresponding lines.

*Solution*:




Write an equation that represent the growth rate of Plant A and Plant B.

*Solution*:

Plant A: $H = 2W + 4$

Plant B: $H = 4W + 2$

At which week will the plants have the same height?

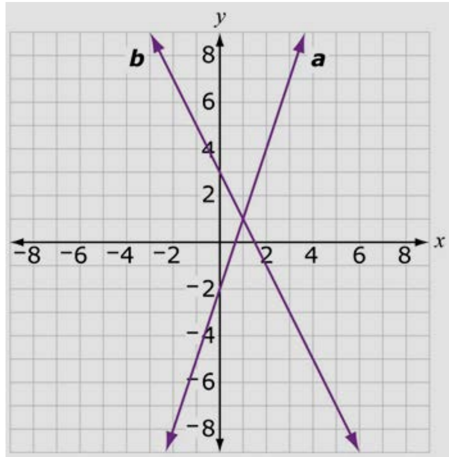
*Solution*:

The plants have the same height after one week.

| Plant A | Plant B |
|---|---|
| $H = 2W + 4$ | $H = 4W + 2$ |
| $H = 2(1) + 4$ | $H = 4(1) + 4$ |
| $H = 6$ | $H = 6$ |

After one week, the height of Plant A and Plant B are both 6 inches.

The graphs of line *a* and line *b* are shown on this coordinate grid.




Match each line with its equation. Click on an equation and then drag it to the corresponding box for each line.

The equation of line *a* is

.

The equation of line *b* is

| | | |
|---|---|---|
| $y = -2x + 3$ | $y = 2x + 3$ | $y = 3x - 2$ |
| $y = -\frac{1}{2}x + 3$ | $y = -\frac{1}{3}x - 2$ | |

*Solution*: The equation of line *a* is $y = -2x + 3$.
The equation of line *b* is $y = 3x - 2$.

Line *a* is shown on the coordinate grid. Construct line *b* on the coordinate grid so that

- line *a* and line *b* represent a system of linear equations with a solution of (7, –2)
- the slope of line *b* is greater than –1 and less than 0
- the *y*-intercept of line *b* is positive

**Instructional Strategies:**

This cluster builds on the informal understanding of slope from graphing unit rates and graphing proportional relationships with a stronger, more formal understanding of slope. It extends solving equations to understanding solving systems of equations, or a set of two or more linear equations that contain one or both of the same two variables. Once again the focus is on a solution to the system. Most student experiences should be with numerical and graphical representations of solutions. Beginning work should involve systems of equations with solutions that are ordered pairs of integers, making it easier to locate the point of intersection, simplify the computation and hone in on finding a solution. More complex systems can be investigated and solve by using graphing technology.

Contextual situations will add meaning to the solution to a system of equations. Students should explore many problems for which they must write and graph pairs of equations leading to the generalization that finding one point of intersection is the single solution to the system of equations. Provide opportunities for students to connect the solutions to an equation of a line, or solution to a system of equations, by graphing, using a table and writing an equation.

Students should receive opportunities to compare equations and systems of equations, investigate using graphing calculators or graphing utilities, explain differences verbally and in writing, and use models such as equation balances.

Problems such as, "Determine the number of movies downloaded in a month that would make the costs for two sites the same, when Site A charges $6 per month and $1.25 for each movie and Site B charges $2 for each movie and no monthly fee."

Students write the equations letting *y* = the total charge and *x* = the number of movies.

Site A: $y = 1.25x + 6$ Site B: $y = 2x$

Students graph the solutions for each of the equations by finding ordered pairs that are solutions and representing them in a t-chart. Discussion should encompass the realization that the intersection is an ordered pair that satisfies both equations. And finally students should relate the solution to the context of the problem, commenting on the practicality of their solution.

Problems should be structured so that students also experience equations that represent parallel lines and equations that are equivalent. This will help them to begin to understand the relationships between different pairs of equations: When the slope of the two lines is the same, the equations are either different equations representing the same line (thus resulting in many solutions), or the equations are different equations representing two not intersecting, parallel, lines that do not have common solutions.

Solving systems should include estimating solutions graphically, solving using substitution, and solving using elimination. Students again should gain experience by developing conceptual skills using models that develop into abstract skills of formal solving of equations.

Provide opportunities for students to change forms of equations (from a given form to slope- intercept form) in order to compare equations.

**Resources/Tools:**

Illustrative Mathematics **Grade 8** tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.C.8
  - How Many Solutions?
  - Fixing the Furnace
  - Cell Phone Plans
  - Kimi and Jordan
  - Folding a Square into Thirds
- 8.EE.C.8.a
  - The Intersection of Two Lines
- 8.EE.C.8.c
  - Quinoa Pasta 1
  - Summer Swimming

"Cara's Candles and DVD's", Georgia Department of Education.

Students are given two tasks; both require writing two equations and solving the resulting system of equations.

Also see engageNY Modules: https://www.engageny.org/resource/grade-8-mathematics

## Domain: Reasoning with Equations and Inequalities (A.REI)

**Cluster:** *Understand solving equations as a process of reasoning and explaining the reasoning.*

**Standard**: **A.REI.8 (all)**

Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). **(A.REI.10)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

**Connections**: Functions Domain

**Explanations and Examples:**

Quoting from Teaching Mathematics in Grades 6-12:Developing Research Based Instructional Practices:

> "Research suggests that multiple representations of functions are not used to their fullest extent in traditional mathematics instruction. Knuth (2000) noted that traditional algebra instruction emphasizes **producing graphs from symbolic representations** of functions (e.g., the task, "Produce a graph of $y = x^2$"), but generally does not ask students to **reason from graphs back to symbolic representations.** To illustrate the detrimental effects of this practice, Knuth gave high school students tasks in which they were to determine the equation of a given linear graph. In one task, students were asked to determine the value of "?" in $?x + 3y = -6$. They were given a graph of the equation to use. Many of the students who gave a **correct solution** to the task used the **inefficient process** of calculating the slope of the graph provided, determining the y-intercept from the graph, writing the equation in slope-intercept form, and then converting it back to standard form. **Students did not seem to recognize that every point on the graph represented a solution** to $?x + 3y = -6$. If they understood this idea, called the **Cartesian connection,** they likely would have chosen any point (x, y) from the graph to substitute into the equation to determine the value of the "?" symbol. Although students in traditional algebra classes produce tables, graphs, and equations for functions, **the act of producing these representations becomes a rote process devoid of meaning** if problems that prompt them to recognize ideas such as the Cartesian connection are not included.
>
> Students who do not fully grasp the Cartesian connection may also **lack skill in choosing the most efficient representations for solving problems.** Slavit (1998) examined the algebraic problem solving strategies of students in a precalculus course where the instructor emphasized graphical representations. Despite this emphasis, some **students persisted in using equations and symbol manipulation even when it was inefficient to do so.** For example, when given a task requiring a solution to $-0.1x^2 + 3x + 80 = x$, one of the students interviewed first tried to factor. When factoring became difficult, she used the quadratic formula. Although she was prompted by the interviewer to discuss other solution strategies, **approaching the problem graphically never occurred to her.** A graphical approach might involve locating the roots of the parabola $y = -0.1x^2 + 2x + 80$ or determining the intersection point of $y = -0.1x^2 + 3x + 80$ and *y = x*. Slavit partially attributed

the lack of use of graphical representations to past instruction focusing heavily on symbol manipulation. **When instruction emphasizes only one representational system, some students come to see graphical and symbolic representations as two separate systems of procedures to follow rather than as representations that complement one another.**

When students view **function representations as complementary** to one another, they develop **better understanding of the attributes of functions**. Schwarz and Hershkowitz (1999) found that students who explored multiple representations with technology understood functions more deeply than those who did not. Students using the technology were able to generate a broader range of examples of functions and also understood multiple representations as different descriptions of the same function. The zooming and scaling features of the technology helped students develop better part-whole reasoning about function representations. **The results of the study strongly suggest that technology capable of generating multiple representations of functions should be a foundational part of algebra instruction."**

Groth, Randall E.. Teaching Mathematics in Grades 6 - 12: Developing Research-Based Instructional Practices (p. 225). SAGE Publications. Kindle Edition.

**Examples:**

1. Which of the following points is on the circle with equation $(x-1)^2 + (y+2)^2 = 5$?

   (a) (1, -2) (b) (2, 2) (c) (3, -1) (d) (3, 4)

2. Graph the equation and determine which of the following points are on the graph of $y = 3^x + 1$.

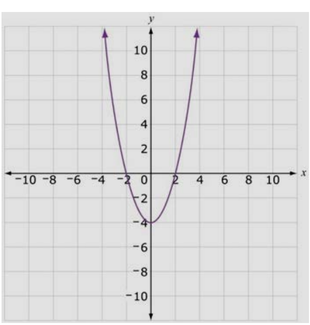
   (a) (2, 7) (b) $\left(-1, \frac{4}{3}\right)$ (c) (2, 10) (d) (0, 1)

$\sqrt{x-4}$

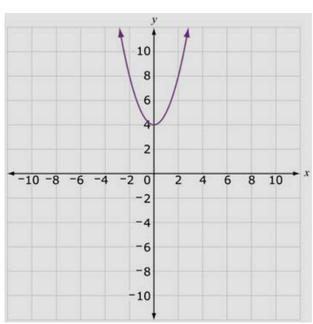
3. Which graph could represent the solution set of $y = \sqrt{x-4}$?

   *Solution*: B

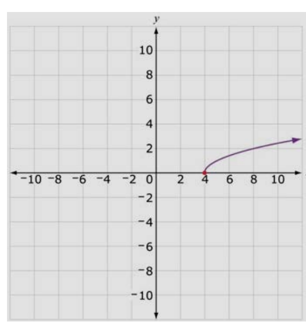
A.



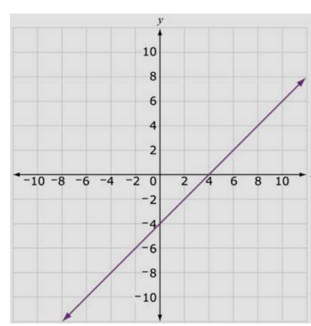
B.



C.



D.



## Instructional Strategies:

Beginning with simple, real-world examples, help students to recognize a graph as a set of solutions to an equation. For example, if the equation $y = 6x + 5$ represents the amount of money paid to a babysitter (i.e., $5 for gas to drive to the job and $6/hour to do the work), then every point on the line represents an amount of money paid, given the amount of time worked.

Explore visual ways to solve an equation such as $2x + 3 = x - 7$ by graphing the functions $y = 2x + 3$ and $y = x - 7$. Students should recognize that the intersection point of the lines is at (-10, -17). They should be able to verbalize that the intersection point means that when *x* = -10 is substituted into both sides of the equation, each side simplifies to a value of -17. Therefore, -10 is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear or both.

Using technology, have students graph a function and use the trace function to move the cursor along the curve. Discuss the meaning of the ordered pairs that appear at the bottom of the calculator, emphasizing that every point on the curve represents a solution to the equation.

Begin with simple linear equations and how to solve them using the graphs and tables on a graphing calculator. Then, advance students to nonlinear situations so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions.

Use the table function on a graphing calculator to solve equations. For example, to solve the equation $x^2 = x + 12$, students can examine the equations $y = x^2$ *and* $y = x + 12$ and determine that they intersect when $x = 4$ and when $x = -3$ by examining the table to find where the *y*-values are the same.

Investigate real-world examples of two-dimensional inequalities. For example, students might explore what the graph would look like for money earned when a person earns *at least* $6 per hour. (The graph for a person earning *exactly* $6/hour would be a linear function, while the graph for a person earning at least $6/hour would be a half-plane including the line and all points above it.)

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered “best practice” to ask these questions in isolation frequently.)

“Optimization Problems: Boomerangs” – Mathematics Assessment Project

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.D.10
  - Collinear Points

## Domain: Reasoning with Equations and Inequalities (A.REI)

▶ **Cluster:** *Represent and solve equations and inequalities graphically.*

**Standard: A.REI.9 ★**

**(9/10/11)** Solve an equation $f(x) = g(x)$ by graphing $y = f(x)$ and $y = g(x)$ and finding the $x$-value of the intersection point. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. For **(9/10)** focus on linear, quadratic, and absolute value. **(A.REI.11)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

**Connections**: A.REI, F.IF. Modeling, A.REI.8

**Explanations and Examples:**

The ALL Flipbook shares research on the Cartesian Connection in A.REI.8. In summary, research says that students do not fully connect all the different representations of a function. Students who can connect across representations are able to flexibly solve problems. One solution strategy that capitalizes on these connections is described in this standard. Let's work backwards, starting with two function rules.

$$f(x) = x - 3$$

| Input | Rule | Output |
|---|---|---|
| -2 | -2-3= | -5 |
| -1 | -1-3= | -1 |
| 0 | 0-3 | -3 |
| 1 | 1-3 | -2 |
| 2 | 2-3 | -1 |
| 3 | 3-3 | 0 |
| 4 | 4-3 | 1 |

**Function Rule**

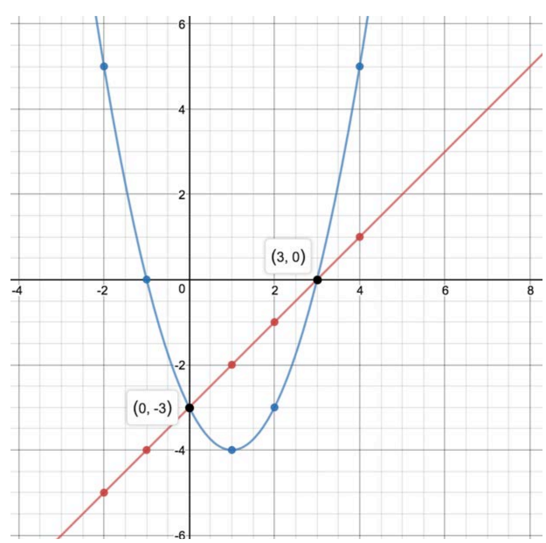
A function takes an input and assigns it to exactly one output.

Solving a system is the process of finding an input for both functions and results in the same output.

$$g(x) = x^2 - 2x - 3$$

| Input | Rule | Output |
|---|---|---|
| -2 | $(-2)^2 - 2(-2) - 3 =$ | 5 |
| -1 | $(-1)^2 - 2(-1) - 3 =$ | 0 |
| 0 | $(0)^2 - 2(0) - 3 =$ | -3 |
| 1 | $(1)^2 - 2(1) - 3 =$ | -4 |
| 2 | $(2)^2 - 2(2) - 3 =$ | -3 |
| 3 | $(3)^2 - 2(3) - 3 =$ | 0 |
| 4 | $(3)^2 - 2(3) - 3 =$ | 5 |

The ALL standard A.REI.10 expects that students realize the graph represents all possible solutions to an equation. So another view of the system is graphing the equations and identifying the points of intersection.



A.REI.9 uses this type of thinking to solve equations. For example, consider the equation

$$x - 3 = x^2 - 2x - 3$$

This equation can be viewed as a question asking for what input are the two functions equal.

$$f(x) = g(x)$$

When provided an equation to solve, the equation can be separated into two functions to identify when the two functions have equal outputs. See the examples below along with some notes about how this thinking can be used to support student learning.

---

$$\mathbf{4x - 2 = 10}$$

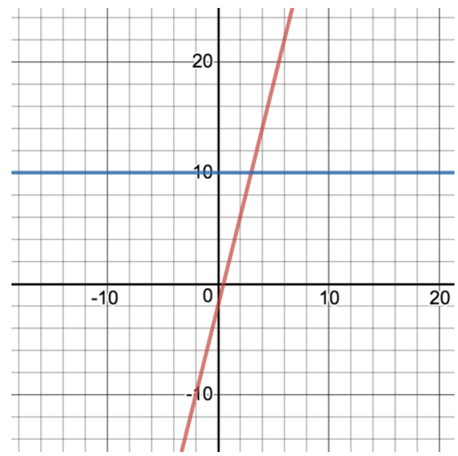
$f(x) = 4x - 2$

This equation can be solved by finding the point on the graph where the function output is 10 *or* by finding the point of intersection.

$g(x) = 10$

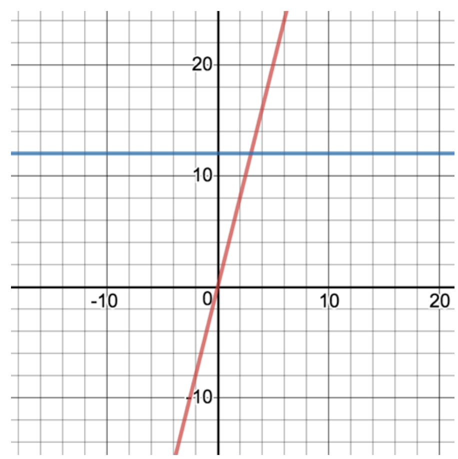
Not only does solving by graphing provide an entry point for all students, it also provides them independence when checking their solutions found through algebraic manipulation. They will begin the problem knowing the answer is 2.

With each step taken toward the solution, the student can graph the equation to ensure the result is still equivalent to the original problem.



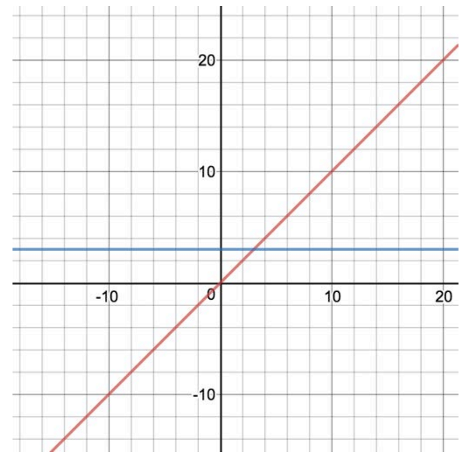

$$f(x) = 4x - 2$$
$$g(x) = 10$$
$$Intersection\ (3, 10)$$

The solution is x=3




$$f(x) = 4x$$
$$g(x) = 12$$
$$Intersection\ (3, 12)$$

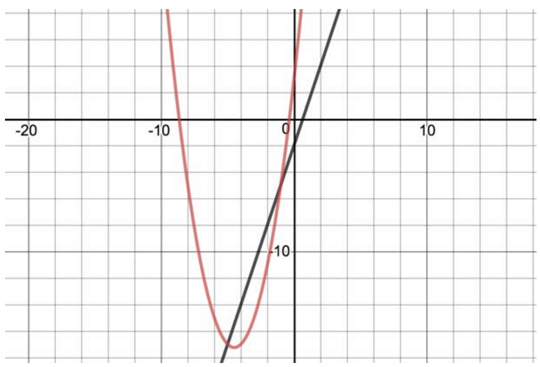
$$f(x) = x$$
$$g(x) = 3$$
$$Intersection\ (3, 3)$$




Let's look at another concept that students struggle with: setting a quadratic equal to zero before solving and then referring to the corresponding answer as both solutions, zeros, and x-intercept.

$$3x - 2 = x^2 + 9x + 3$$

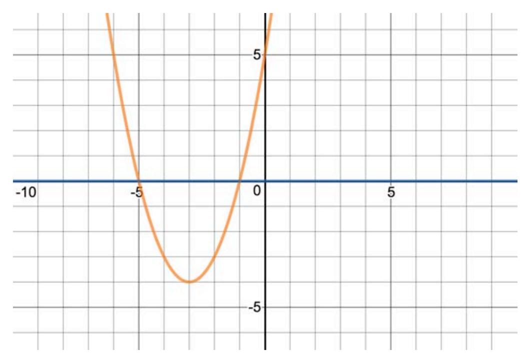
First, graph the pair of functions to identify the solution goal.




The points of intersection are: (-1, -5) and (-5, -17) so the solutions are x=-1 and -5

Setting the equation equal to zero, the new equation is
$x^2 + 6x + 5 = 0$

- The intersection is (-5, 0) and (-1, 0).
- The solutions are x= -1 and -5.
- They are the zeros of the function when the equation is equal to zero.
- They are the x-intercepts when the equation is equal to zero because graphing f(x)=0 creates a line that coincides with the x-axis.



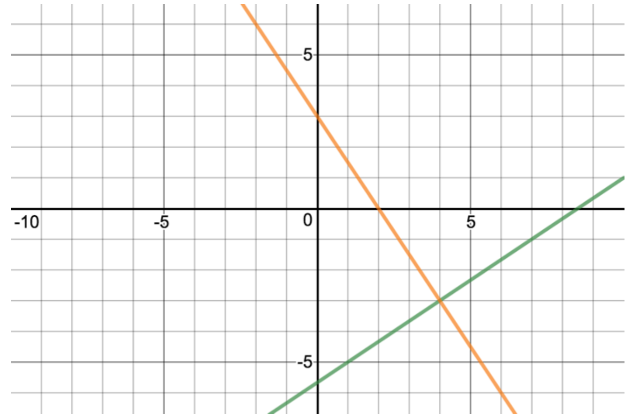

These examples *do not imply* that the only purpose for this standard is to support the teaching and learning of algebraic manipulation. There is value in learning to solve equations by graphing, in addition to solving with tables, reasoning about numbers, *and* algebraic manipulation. Solving equations by graphing is another tool and is not intended to *replace* algebraic manipulation.

BUT, you might be thinking, if a student learns to solve equations by graphing *why* would they want to learn algebraic manipulation? Excellent point! Excellent question! Teachers need to ask better questions than "find the solution" to encourage students to use all their resources (such as asking students to rewrite the function into a different form, combining functions using arithmetic, finding the exact solution when the solution is irrational, etc.). We want to ask questions with multiple entry points and then have a rich discussion about why the student chose a particular solution strategy. Not only is solving an equation by graphing a valuable tool but it also helps visualize the relationship between the functions.

**Examples:**

1. Solve and justify, using substitution, that the solution is correct.

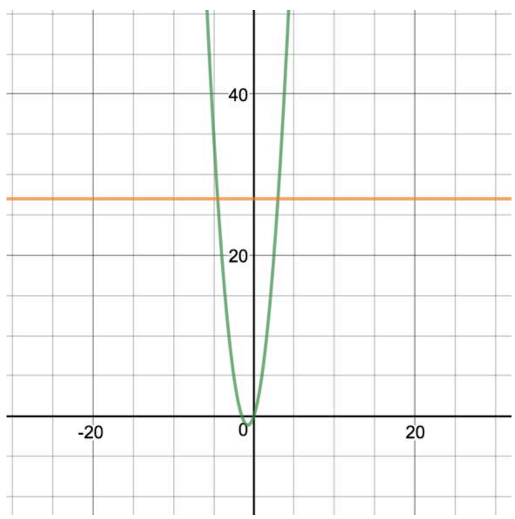
$$\frac{2x-17}{3} = -3(\frac{1}{2}x - 1)$$




The intersection is (4, -3). The solution is *x=4*.

2. The length of a rectangle is three more than twice the width. Determine the dimensions that will give a total area of 27 $m^2$.

<u>Solution:</u>
$w(2w+3) = 27$




Intersections are (-4.5, 27) and (3, 27). When constrained by the situation, the width is 2 and the length is 9.

3. What do the points on the graph of the function $A(w) = w(2w+3)$ represent?
<u>Solution:</u>
Along the x-axis, the *w* variable represents the width of the rectangle. Along the *y*-axis is the output of the function, Area. Each point along the graph represents the width of the rectangle and its associated Area.

4. In the situation above, one student used the function $g(w) = w(2w + 3)$ to find the answer and another used the function $f(w) = 2w^2 + 3w - 27 = 0$. The points for each graph mean different things. Describe the meaning of the points on each graph and how to use the graph to find the width with an Area of 27 $m^2$.

   Solution:

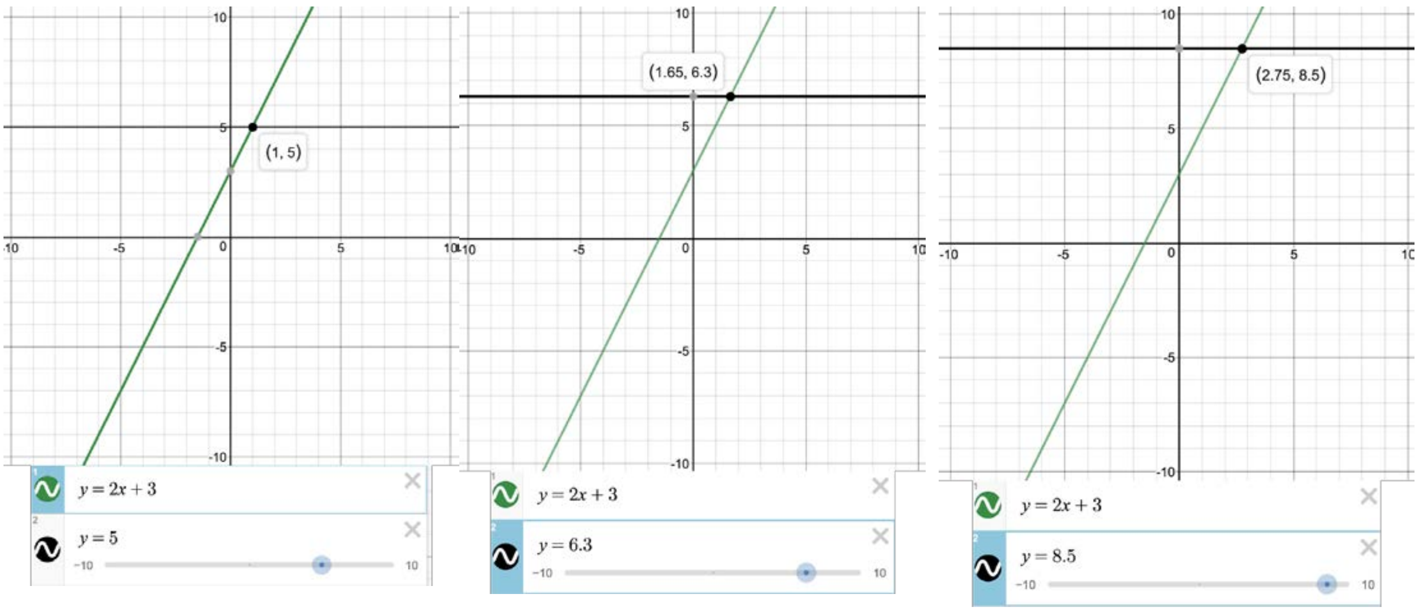
   For the *g* function, each point on the graph represents the width of the rectangle and its associated Area. To find the width, find the point where g(w)=27. The *f* function represents the same rectangle translated down 27 units so that the x-intercept represents an Area of *exactly* 27 $m^2$. Each point on the graph of *f* represents the width of the rectangle and the increase or decrease of the area above or below 27 $m^2$.

**Instructional Strategies:**

Using technology, students can easily find the points of intersection and explore how the solutions are related to the functions. In particular, desmos easily replaces numbers in the equation with parameters that can become sliders to explore how the solutions changes as the numbers in the equation change. This again expands the classroom discussion beyond "find the answer" into developing number sense and estimation skills. For example, when reviewing solving two-step equations such as $2x + 3 = 4$, you could explore some ideas such as:

1. **As the value that the expression $2x + 4$ is equal increases, how does the solution to the equation change?**

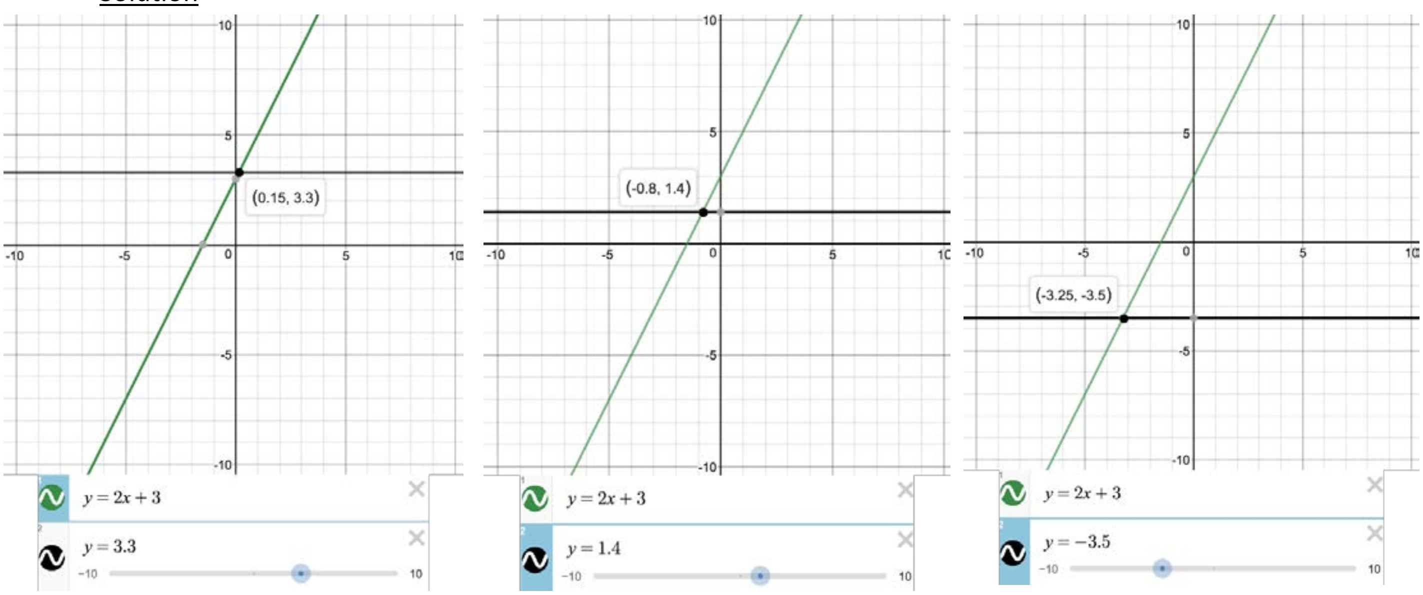
   Solution




   As the value of that the expression is equal to increases, the solution increases. This will happen along the entire function because, as the line the expression intersects with rises, the function also moves further to the right.

2. **As the value that the expression is equal to decreases, how does the solution to the equation change?**

   Solution

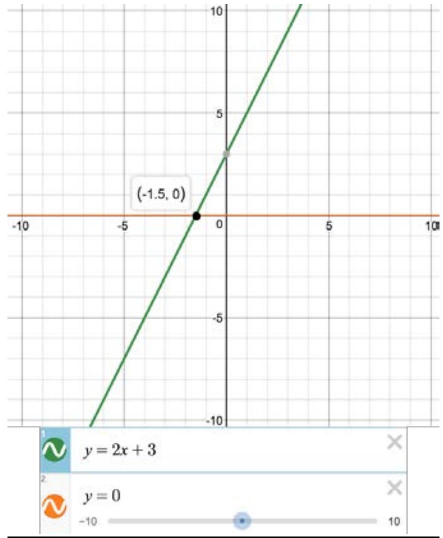



   As the value the expression is equal to decreases, the solution to the equation will decrease because the slope is positive.

3. **When the value of the expression equals zero, what key feature for the function $f(x) = 2x + 3$ is identified? Will that be true or *any* equation?**

   Solution

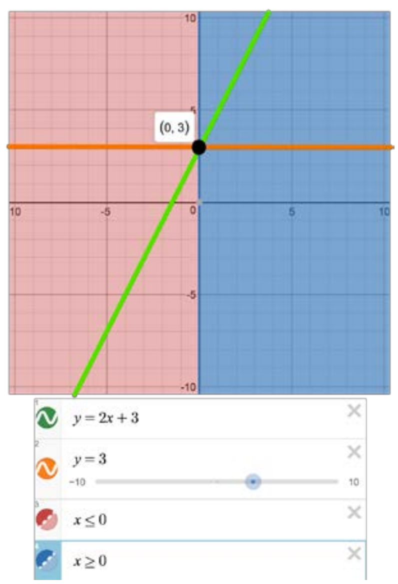
   When the expression is equal to zero, the function graphed lies on the *x-axis* so the solution is the same as the *x*-intercept. This would be true for any equation.




4. **When does the solution become negative?**

   Solution

   When the *x* value (the solution) is negative, the graph will be on the left side of the *y*-axis. When the *x* value (the solution), is positive, the graph will be on the right side of the *y*-axis. Therefore, the solution changes from positive to negative when the graph intersects with the *y*-axis.




5. **For *any* equation, is that location where the solution will *always* change its sign?**

   Solution

   Yes because the *y*-axis will always be the place where the *x* value changes from positive to negative and the *x* value is the solution.

**Resources/Tools:**

- Desmos graphing calculator

Desmos activity: Picture Perfect

# Domain: Reasoning with Equations and Inequalities (A.REI)

▶ **Cluster:** *Represent and solve equations and inequalities graphically.*

**Standard**: **A.REI.10**

**(9/10)** Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes. **(A.REI.12)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

**Connections**: 7.EE.4, A.REI.8

**Explanations and Examples:**

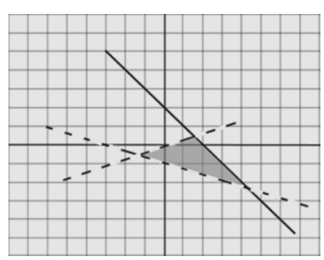
Building on A.REI.8, which reinforces that the graph of an equation is the set of all solutions, students learn to view an inequality as the boundary line for the region of all possible solutions. Students may use graphing calculators, programs or applets to model and find solutions for inequalities or systems of inequalities.

**Examples:**

- Graph the solution: $y \leq 2x + 3$.
- A publishing company publishes a total of no more than 100 magazines every year. At least 30 of these are women's magazines, but the company always publishes at least as many women's magazines as men's magazines. Find a system of inequalities that describes the possible number of men's and women's magazines that the company can produce each year consistent with these policies. Graph the solution set.
- Graph the system of linear inequalities below and determine if (3, 2) is a solution to the system.

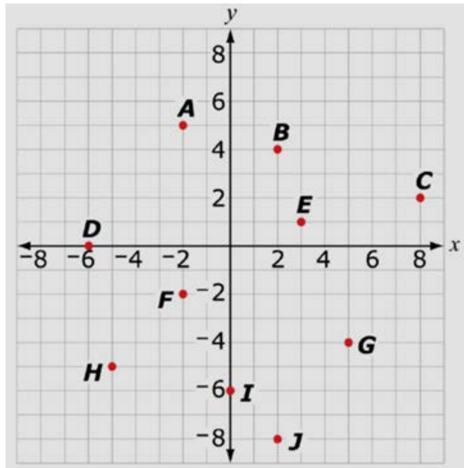
$$\begin{cases} x - 3y > 0 \\ x + y \leq 2 \\ x + 3y > -3 \end{cases}$$

*Solution:*



(3, 2) is not an element of the solution set (graphically or by substitution).

4. The coordinate grid shows points *A* through *J*.



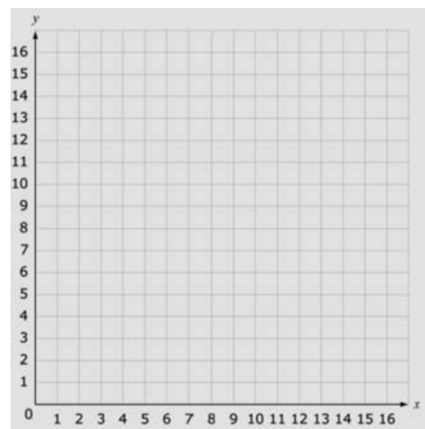

- Given the system of inequalities shown below, name all the points that are solutions to this system of inequalities.

$$\begin{cases} x + y < 3 \\ 2x - y > 6 \end{cases}$$

*Solution:* points *G* and *J*

5. Graph this system of inequalities on the given coordinate grid.

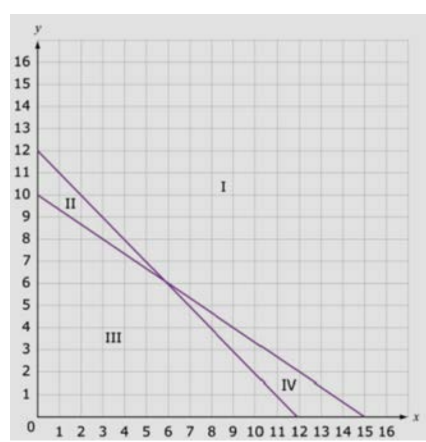
$$\begin{cases} x + y \ge 12 \\ 20x + 30y \le 300 \end{cases}$$




To create a line, click in the grid to create the first point on the line. To create the second point on the line, move the pointer and click. The line will be automatically drawn between the two points. Use the same process to create additional lines.

When both inequalities are graphed, select the region in your graph that represents the solution to this system of inequalities. To select a region, click anywhere in the region. To clear a selected region, click anywhere in the selected region.

*Solution:* One line contains points (12, 0) and (0, 12), the other line contains the points (15,0) and (0, 10). Region IV represents the solution to the system of inequalities.



**Instructional Strategies:** See A.REI.8

**Resources/Tools:**

Mathematics Assessment Project:

- "Defining Regions Using Inequalities" – This lesson unit is intended to help you assess how well students are able to use linear inequalities to create a set of solutions. In particular, the lesson will help you identify and assist students who have difficulties in:
  - Representing a constraint by shading the correct side of the inequality line.
  - Understanding how combining inequalities affects a solution space.

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.D.12
  - Fishing Adventures 3
  - Solution Sets

# High School – Functions

## Domain: Interpreting Functions (F.IF)

**Cluster:** *Understand the concept of a function and use function notation.*

**Standard**: **F.IF.1 (all)**

Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then $f(x)$ denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation $y = f(x)$. **(F.IF.1)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

**Connections**: Functions and Algebra Domain

**Explanations and Examples:**

The goal with the functions domain is teach students a *process* for analyzing functions, rather than teaching each function family in isolation, causing students to miss the larger structure holding them all together. For that reason, several of the Function standards were chosen as ALL standards and should be present throughout all mathematics instruction. In fact, the five ALL functions standards so clearly define that process and are so intertwined that it is difficult to separate them into individual standards to discuss here. As a result, this section of the flipbook will discuss the common process here and address any individual nuances in each additional standard.

1. Functions should be **analyzed qualitatively** (from a global perspective) for key features (F.IF.4)
   a) Direction of change (increasing/decreasing)
   b) Type of change (constant linear or non-constant nonlinear)
   c) Minimum and maximum values.
   d) Symmetry
   e) End behavior
   f) Periodicity
2. Functions should be **analyzed quantitatively** (from a local perspective) by
   a) Identifying the domain and range and relating it back to the relationship
   b) Representing the function across all representations and identifying key features.
   c) Understanding function notation in the context of the situation.
   d) Understanding how the graph of the function is related to the equation.

The first concepts for students to wrestle with are:

- Defining a function as the rule that *assigns* an *every* element from the input set to *exactly one* element of the output set
- Naming the input set domain and the output set range
- Introducing notation naming the input *x* and the output *f(x)*
- Defining the graph of *f* as the graph of the equation *y=f(x)*

**Define the function relationship**

- The standard uses an "assignment" definition for a function using a rule to assign an input to an output.
- Inputs and outputs are named domain and range and are a fundamental part of the definition for a function.
- The definition does **not** require a function to be an equation, graph, or even numerical.
- Assignment requires two elements**:** (1) an element from the domain is assigned (2) to exactly one element of the range.

**Identifying functions from a table**

- It is **not correct** to describe a procedure for determining if a table (or set of ordered pairs) is a function by saying **"the x value cannot repeat."** This neglects an essential requirement for a function: a correspondence between two values.
- In a data table the input value might repeat but, if it does, it must ALWAYS be assigned to the same output value.

**Identifying a function from a graph**

- Focus on the definition: that every element of the domain is assigned to exactly one element of the range.
- The vertical line test provides an easy procedure to instruct students but there are numerous reasons to **avoid this strategy.**
- Once students learn the vertical line test, they tend to blindly apply it to all graphs rather than having a procedure that can grow with them through geometric transformations, inverse function, polar functions, etc.
- Focus on clearly identifying the input (independent variable or domain) and matching it to a single output (dependent variable or range). This also provides essential practice the students need with domain and range.

**Function notation**

- Asking students to **interpret function notation** in non-quantitative situations can help reinforce what it means for *x* to be the input value in a function rule and how *f(x)*, the output value, is the associated output.
- x is a number and a specific input, *f(x)* is a number and the corresponding output. (See N.Q.1)
- The name of the function is *f*, for example, not *f(x).*
- Dr. Bill McCallum wrote about how to interpret the statement *y=f(x).*

  As for the $y = f(x)$ notation, when we say something like "the function $y = x^2$" we are using abbreviated language for "the function defined by the equation $y = x^2$, where *x* is the independent variable and *y* is the dependent variable." You can't say that every time, so we have a shortened form, which depends on certain conventions: the dependent variable occurs on the left and an expression in the independent variable occurs on the right."

**Examples:**

1. Determine which of the following tables represent a function and explain why.

| A | |
|---|---|
| ***x*** | ***f(x)*** |
| 0 | 1 |
| 1 | 2 |
| 2 | 2 |
| 3 | 4 |

| B | |
|---|---|
| ***x*** | ***f(x)*** |
| 0 | 0 |
| 1 | 2 |
| 1 | 3 |
| 4 | 5 |

*Solution:* Table A represents a function because for each element in the domain there is exactly one element in the range. Table B does not represent a function because when x = 1, there are two values for $f(x)$: 2 and 3.

2. For the functions a. through f. below:

- List the algebraic operations in order of evaluation. What restrictions does each operation place on the domain of the function?
- Give the function's domain.

a. $y = \dfrac{2}{x-3}$

b. $y = \sqrt{x-5} + 1$

c. $y = 4 - (x-3)^2$

d. $y = \dfrac{7}{4-(x-3)^2}$

e. $y = 4 - (x-3)^{\frac{1}{2}}$

f. $y = \dfrac{7}{4-(x-3)^{\frac{1}{2}}}$

3. Is a geometric transformation an example of a function? If not, why? If so, how does that support its use in formal proofs?

Viewing transformations as functions is essential to proving congruence through rigid transformations. We have to know that if I do a translation, for example, that there is exactly one guaranteed output. Once we have a guaranteed output, we can apply geometric reasoning to the sequence of transformations and know, beyond all doubt, that the image is congruent.

**Instructional Strategies:**

Provide applied contexts in which to explore functions. For example, examine the amount of money earned when given the number of hours worked on a job, and contrast this with a situation in which a single fee is paid by the "carload" of people, regardless of whether 1, 2, or more people are in the car.

Use diagrams to help students visualize the idea of a function machine. Students can examine several pairs of input and output values and try to determine a simple rule for the function.

Rewrite sequences of numbers in tabular form, where the input represents the term number (the position or index) in the sequence, and the output represents the number in the sequence.

Help students to understand that the word "domain" implies the set of all possible input values and that the integers are a set of numbers made up of {…-2, -1, 0, 1, 2, …}.

Distinguish between relationships that are not functions and those that are functions (e.g., present a table in which one of the input values results in multiple outputs to contrast with a functional relationship). Examine graphs of functions and non-functions, recognizing that if a vertical line passes through at least two points in the graph, then *y* (or the quantity on the vertical axis) is not a function of x (or the quantity on the horizontal axis).

**Common Misconceptions:**

Students may believe that all relationships having an input and an output are functions, and therefore, misuse the function terminology.

Students may also believe that the notation *f(x)* means to multiply some value *f* times another value *x*. The notation alone can be confusing and needs careful development. For example, *f*(2) means the output value of the function *f* when the input value is 2.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.A
  - Interpreting the graph
- F-IF.A.1
  - The Parking Lot
  - Your Father
  - Parabolas and Inverse Functions
  - Using Function Notation I
  - The Customers
  - Points on a graph
  - Domains

# Domain: Interpreting Functions (F.IF)

**Cluster:** *Understand the concept of a function and use function notation.*

**Standard**: **F.IF.2 (all)**

Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. **(F.IF.2)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

**Connections**: F.IF.1

**Explanations and Examples:**

In addition to explanation about function notation discussed in N.Q.1 and F.IF.1, it is also important to analyze function notation across multiple contexts and representations.

**Examples:**

An Illustrated Math Task:

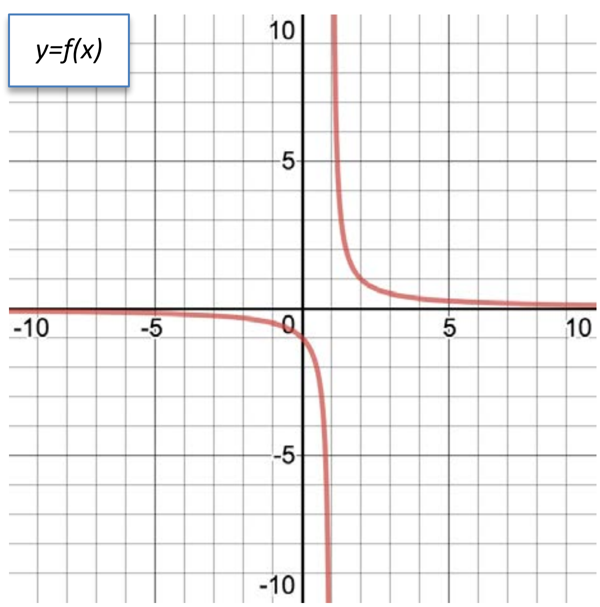
1. You put a yam in the oven. After 45 minutes, you take it out. Let $f$ be the function that assigns to each minute after you placed the yam in the oven, its temperature in degrees Fahrenheit.
   a. Write a sentence explaining what $f(0) = 65$ means in everyday language
   b. Write a sentence explaining what $f(5) < f(10)$ means in everyday language.
   c. Write a sentence explaining what $f(40) = f(45)$ means in everyday language
   d. Write a sentence explaining what $f(45) > f(60)$ means in everyday language.

Use the table and graph to answer the questions below.

2. Find $f(4)$

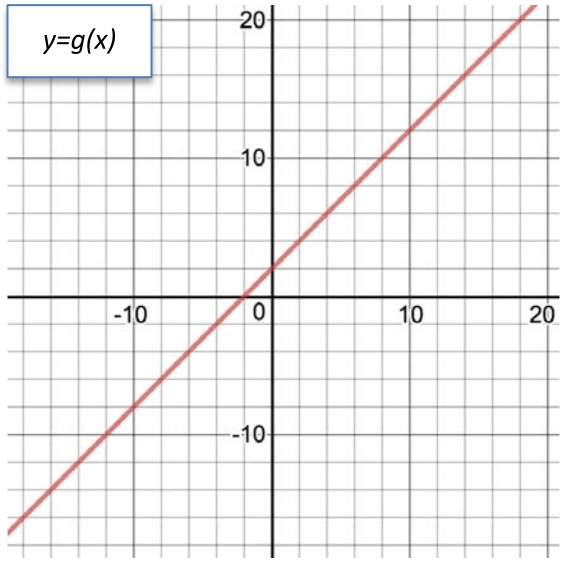
3. If $f(x) = 2$, find x.

| x | f(x) |
|---|---|
| **-2** | -4 |
| **-1** | -1 |
| **0** | 2 |
| **1** | 5 |
| **2** | 8 |
| **3** | 11 |
| **4** | 14 |



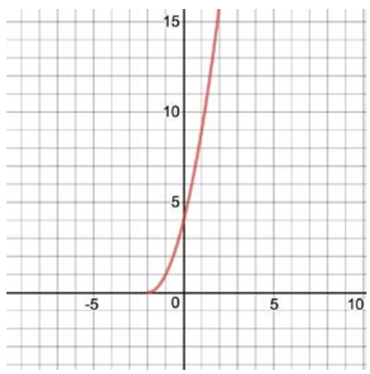
4. Use the table and/or the equation to perform the given function operation. Graph the result.

| x | f(x) |
|---|---|
| **-2** | 4 |
| **-1** | 1 |
| **0** | 0 |
| **1** | 1 |
| **2** | 4 |
| **3** | 9 |
| **4** | 16 |

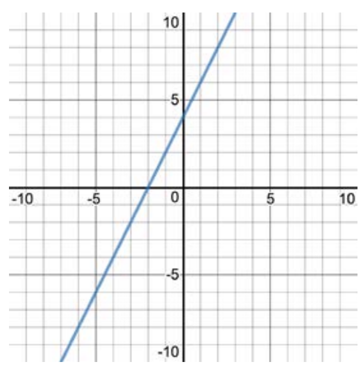



*Solution:*

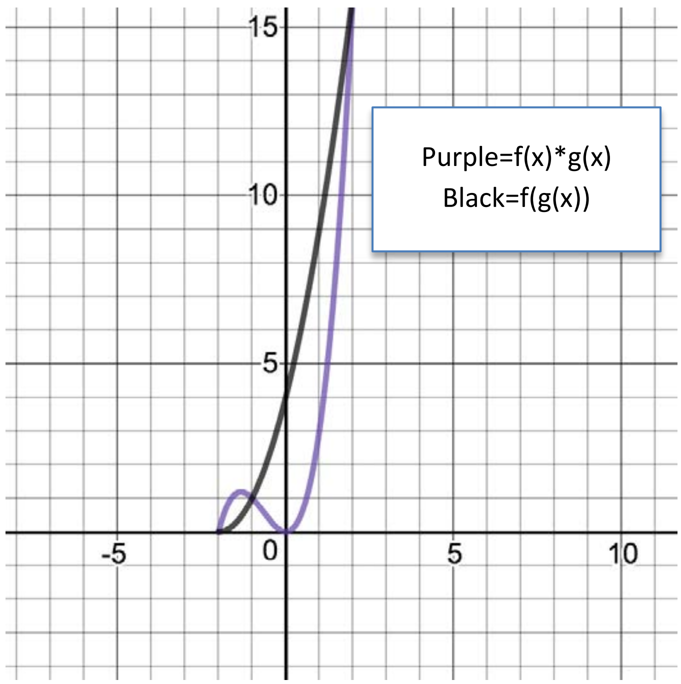
| x | *f(x)* | *f(x+2)* |
|---|---|---|
| -2 | 4 | *f(-2+2)=0* |
| -1 | 1 | *f(-1+2)=1* |
| 0 | 0 | *f(0+2)=4* |
| 1 | 1 | *f(1+2)=9* |
| 2 | 4 | *f(2+2)=16* |
| 3 | 9 | |
| 4 | 16 | |



| x | *g(x)* | *2g(x)* |
|---|---|---|
| -2 | 0 | 2g(-2)=0 |
| -1 | 1 | 2g(-1)=2 |
| 0 | 2 | 2g(0)=4 |
| 1 | 3 | 2g(1)=6 |
| 2 | 4 | 2g(2)=8 |
| 3 | 5 | 2g(3)=10 |
| 4 | 6 | 2g(4)=12 |



| x | *f(x)* | *g(x)* | *f(x)*g(x)* | *f(g(x))* |
|---|---|---|---|---|
| -2 | 4 | 0 | f(-2)*g(-2)=0 | f(g(-2))=f(0)=0 |
| -1 | 1 | 1 | f(-1)*g(-1)=1 | f(g(-1))=f(1)=1 |
| 0 | 0 | 2 | f(0)*g(0)=0 | f(g(0))=f(2)=4 |
| 1 | 1 | 3 | f(1)*g(1)=3 | f(g(1))=f(3)=9 |
| 2 | 4 | 4 | f(2)*g(2)=16 | f(g(2))=f(16)=2 |
| 3 | 9 | 5 | | |
| 4 | 16 | 6 | | |




**Resources/Tools:**
(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.A.2
  - Using Function Notation II
  - Yarn in the Oven
  - The Random Walk
  - Cell phones
  - Random Walk II

## Domain: Interpreting Functions (F.IF)

● **Cluster:** *Understand the concept of a function and use function notation.*

**Standard**: **F.IF.3**

**(9/10/11)** Recognize patterns in order to write functions whose domain is a subset of the integers. **(9/10)** Limited to linear and quadratic. *For example, find the function given* $\{(-1,4),(0,7),(1,10),(2,13)\}$. **(F.IF.3)**

### Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: F.IF.1-2, F.BF.1-2

### Explanations and Examples:

This standard is not about sequences but is, more generally, about writing the function rule from a pattern. Previous pattern work asked students to describe the rule that generated the pattern recursively (i.e. "it's a plus 2 pattern" for 3, 5, 7, 9, 11, …). Now students are asked to number the terms in the pattern, with the number of the term the input and the term in the pattern the output, and create a function rule that generates the pattern. Students are not expected to write the recursive rule or identify the connection between the recursive description and the function rule but these concepts might be scaffolding for students struggling with writing the rule.

### Examples:

Write the function rule that generates the following patterns:

1. 5, 8 11, 14, 17, …
2. -7, -7.5, -8, -8.5, …
3. 2, 8, 18, 32, 50, …
4. 0, 1, 4, 9, 16, 25, ….

### Instructional Strategies:

Instruction for linear patterns will be very similar to instruction for arithmetic sequences but instruction for quadratic patterns might be new to many students. Students should be able to recognize the output values for the parent function of quadratic family. Comparing the output from the pattern to the parent function and analyzing the differences as a transformation can help students write a function to define the pattern.

### Resources/Tools:

Generalizing Patterns: Table Tiles- This lesson unit is intended to help you assess how well students are able to identify linear and quadratic relationships in a realistic context: the number of tiles of different types needed for a range of square tabletops.

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.B.4.b
  - Quadratic Sequence 1
  - Quadratic Sequence 2
  - Quadratic Sequence 3

## Domain: Interpreting Functions (F.IF)

**Cluster:** *Understand the concept of a function and use function notation.*

**Standard**: **F.IF.4 (all) ★**

For a function that models a relationship between two quantities, interpret key features of expressions, graphs and tables in terms of the quantities, and sketch graphs showing key features given a description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. **(F.IF.4)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: F.IF.7

**Explanations and Examples:**

This standard is often paired with F.IF.7 but F.IF.7 focuses on the key feature from the graph, while this standard focuses on identifying key features across all representations. Another key difference between the two standards is that F.IF.7 is not a modeling standard while F.IF.4 is a modeling standard. As a modeling standard, the focus should be on *interpreting the quantities* in context. This standard is an ALL standard because all the functions studied in high school have this standard applied and practice.

**Examples:**

- A rocket is launched from 180 feet above the ground at time *t* = 0. The function that models this situation is given by $h = -16t^2 + 96t + 180$, where *t* is measured in seconds and *h* is height above the ground measured in feet.
    - What is a reasonable domain restriction for *t* in this context?
    - Determine the height of the rocket two seconds after it was launched.
    - Determine the maximum height obtained by the rocket.
    - Determine the time when the rocket is 100 feet above the ground.
    - Determine the time at which the rocket hits the ground.
    - How would you refine your answer to the first question based on your response to the second and fifth questions?

- Compare the graphs of $y = 3x^2$ and $y = 3x^3$.
- Let $R(x) = \frac{2}{\sqrt{x-2}}$. Find the domain of *R*(*x*). Also find the range, zeros, and asymptotes of *R*(*x*).
- Let $f(x) = x^2 - 5x + 1$. Graph the function and identify end behavior and any intervals of constancy, increase, and decrease.
- It started raining lightly at 5am, then the rainfall became heavier at 7am. By 10am the storm was over, with a total rainfall of 3 inches. It didn't rain for the rest of the day. Sketch a possible graph for the number of inches of rain as a function of time, from midnight to midday.

**Instructional Strategies:**

Flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

Examine a table of related quantities and identify features in the table, such as intervals on which the function increases, decreases, or exhibits periodic behavior.

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5. However, if a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Given a table of values, such as the height of a plant over time, students can estimate the rate of plant growth. Also, if the relationship between time and height is expressed as a linear equation, students should explain the meaning of the slope of the line. Finally, if the relationship is illustrated as a linear or non-linear graph, the student should select points on the graph and use them to estimate the growth rate over a given interval.

Begin with simple, linear functions to describe features and representations, and then move to more advanced functions, including non-linear situations.

Provide students with many examples of functional relationships, both linear and non-linear. Use real-world examples, such as the growth of an investment fund over time, so that students can not only describe what they see in a table, equation, or graph, but also can relate the features to the real-life meanings.

Allow students to collect their own data sets, such as the falling temperature of a glass of hot water when removed from a flame versus the amount of time, to generate tables and graphs for discussion and interpretation.

**Common Misconceptions:**

Students may believe that it is reasonable to input any *x*-value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.

Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B
  - Pizza Place Promotion
- F-IF.B.4
  - Influenza epidemic
  - Warming and Cooling
  - How is the weather?
  - Telling a Story with Graphs
  - Logistic Growth Model , Abstract Version
  - Logistic Growth Model , Explicit Version

## Domain: Interpreting Functions (F.IF)

**Cluster:** *Understand the concept of a function and use function notation*..

### Standard: F.IF.5 (all) ★

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★ **(F.IF.5)**

### Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

**Connections**: F.IF.1

### Explanations and Examples:

Given the graph if a function, determine the practical domain of the function as it relates to the numerical relationship it describes.

Students may explain orally or in written format, the existing relationships.

**Examples:**

- If the function $h(n)$ gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function.
- A hotel has 10 stories above ground and 2 levels in its parking garage below ground. What is an appropriate domain for a function. $T(n)$ that gives the average number of times an elevator in the hotel stops at the $n^{th}$ floor each day.
- Oakland Coliseum, home of the Oakland Raiders, is capable of seating 63,026 fans. For each game, the amount of money that the Raiders' organization brings in as revenue is a function of the number of people, $n$, in attendance. If each ticket costs $30, find the domain and range of this function.

*Sample Response*:

Let *r* represent the revenue that the Raider's organization makes, so that $r = f(n)$. Since *n* represents a number of people, it must be a nonnegative whole number. Therefore, since 63,026 is the maximum number of people who can attend a game, we can describe the domain of *f* as follows: $\{Domain = n: 0 \leq n \leq 63{,}026 \ and \ n \ is \ an \ integer\}$.

The range of the function consists of all possible amounts of revenue that could be earned. To explore this question, note that $r = 0$ if nobody comes to the game, $r = 30$ if one person comes to the game, $r = 60$ if two people come to the game, etc. Therefore, *r* must be a multiple of 30 and cannot exceed $(30 \cdot 63{,}026) = 1{,}890{,}780$, so we see that $\{Range = r: 0 \leq r \leq 1{,}890{,}780 \ and \ r \ is \ an \ integer \ multiple \ of \ 30\}$.

**Instructional Strategies:**

The deceptively simple task above asks students to find the domain and range of a function from a given context. The function is linear and if simply looked at from a formulaic point of view, students might find the formula for the line and say that the domain and range are all real numbers. However, in the context of this problem, this answer does not make sense, as the context requires that all input and output values are non-negative integers, and imposes additional restrictions. This problem could serve different purposes. Its primary purpose is to illustrate that the domain of a function is a property of the function in a specific context and not a property of the formula that represents the function. Similarly, the range of a function arises from the domain by applying the function rule to the input values in the domain. A second purpose would be to illicit and clarify a common misconception, that the domain and range are properties of the formula that represent a function. Finally, the context of the task as written could be used to transition into a more involved modeling problem, finding the Raiders' profit after one takes into account overhead costs, costs per attendee, etc.

**Resources/Tools:**

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F.IF.B.5
  - Oakland Coliseum
  - Average Cost

# Domain: Interpreting Functions (F.IF)

◆ **Cluster:** *Interpret functions that arise in applications in terms of the context*

**Standard**: **F.IF.6** ★

**(9/10/11)** Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph. ★ **(9/10)** Limited to linear functions.★**(F.IF.6)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

**Connections**: F.IF.4, F.IF.7, S.ID.6, 8.EE

**Explanations and Examples:**

Consider: Is there a difference between "slope" and "rate of change"?

Slope is the description of a physical feature of the graph. It describes the *incline* of the line. Rate of change describes how the function changes and is not limited to any one representation. This standard focuses on calculating the rate of change across multiple representations. In grades 9/10, problems should be limited to linear functions.

Additionally, this standard is a modeling standard so students should be working problems from modeling situations. Collecting data, creating a scatterplot, and modeling the line of best fit is an opportunity for students to calculate and interpret the rate of change.

The wording "over a specified interval" can provide an opportunity for discussions about comparisons or how to rewrite the rate of change so that the description is more meaningful. For example, consider the following situation:

**Example 1:**

While planning for a vacation to Orlando, Florida during the month of June, you compare the cost of a taxi, an uber, and renting a car. Describe the conditions that would make each vehicle the best choice. Be sure to include any assumptions you made to create your predictive model and the rate of change for each transportation type. (My estimates are in red as an example of what a student might do.)

| Estimate 2 trips every 5 miles. Cost shows maximum miles for that range. | Rental Vehicle | Taxi $4.50 base fee, 2.40/mile | Uber $9.11 base fee, $2.70 booking fee, $0.79/mile |
|---|---|---|---|
| 1 mile | $255 | $6.90 | $12.60 |
| 10 miles | $255 | $33.00 | $31.52 |
| 20 miles | $255 | $66.00 | $63.04 |
| 30 miles | $255 | $99.00 | $94.56 |
| 60 miles | $255 | $198.00 | $189.12 |
| 100 miles | $255 | $330.00 | $315.20 |

- Rate of change for the rental vehicle: $\frac{255-255}{20-10} = \$0/mile$
- Rate of change for taxi: $\frac{66-33}{20-10} = \$3.30/mile$
- Rate of change for uber: $\frac{63.04-31.52}{20-10} = \$3.15/mile$

Even though the rental vehicle has the smallest rate of change, $0/mile, it is not the cheapest option unless I plan to travel more than 100 miles while I am in Orlando, the rental vehicle would not be the best option. Uber has the smallest rate of change at $3.15 and would be the best option for trips longer than 5 miles because the single booking fee and smaller mileage fee makes that the best option. If I will only take short trips, less than 5 miles an hour, than the taxi is the cheaper option because the lower base fee will offset the increased mileage cost.

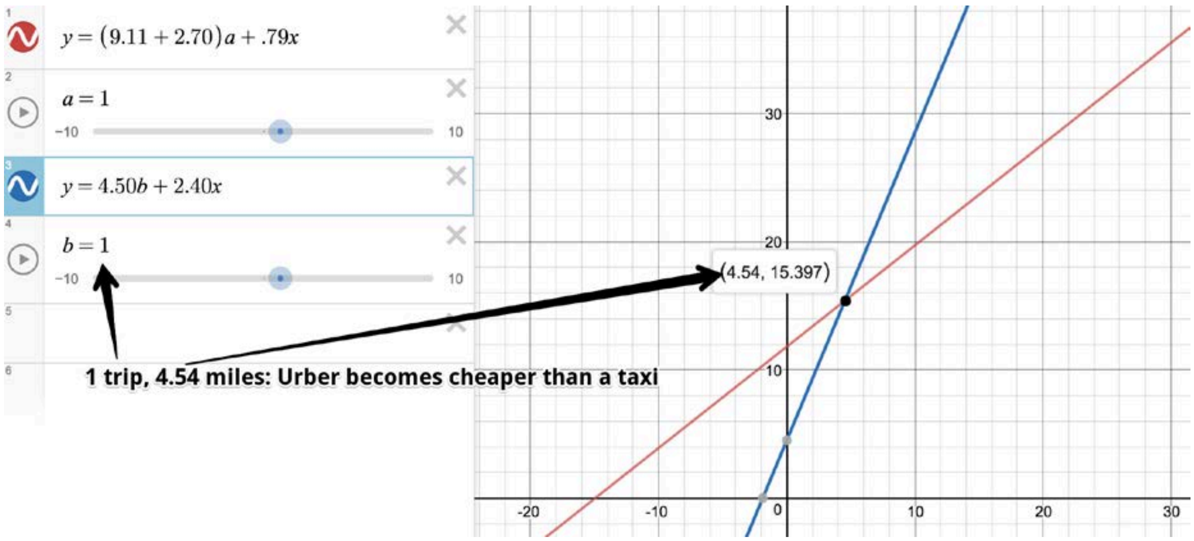
Note: Desmos can help you explore the multiple variables in this situation. This creates a more realistic model and would be a great extension, depending on student interest and ability.

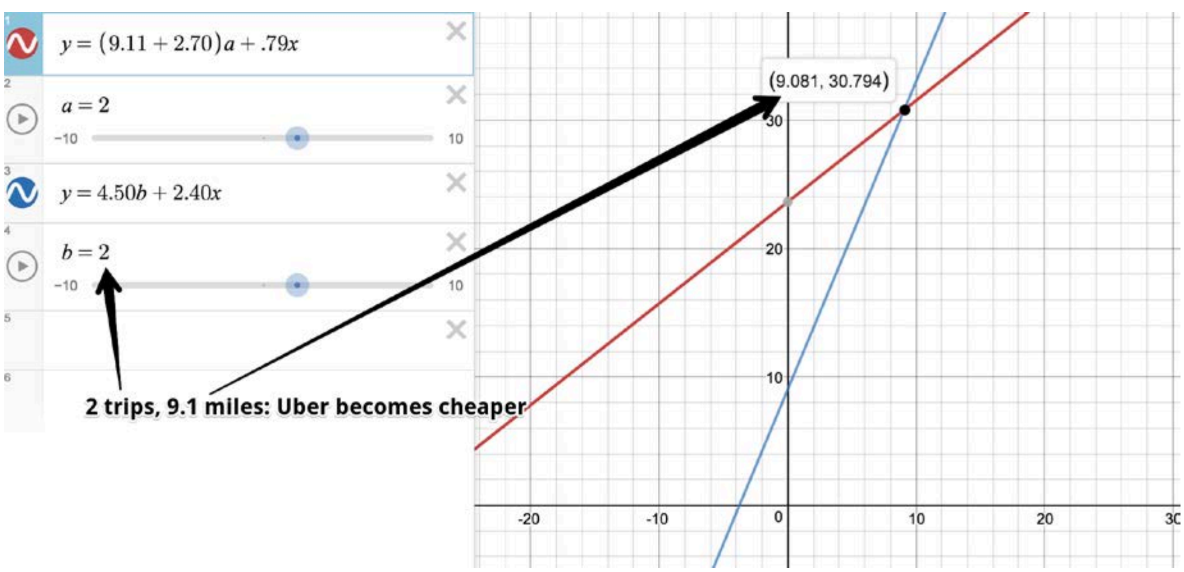
Function model for uber $f(x) = (9.11 + 2.70) * number\ of\ trips + 0.79 * number\ of\ miles$

$$f(x) = (9.11 + 2.70) * a + 0.79 * x$$

Function model for taxi $f(x) = 4.50 * number\ of\ trips + 2.40 * number\ of\ miles$

$$f(x) = 4.50 * b + 2.40x$$



**Example 2:**

Rewrite the interpretation for the given rate of change so that the interval of change is more easily understandable.

a. $0.003/mile
b. 0.2 bolts/lbs.
c. 0.5 people/seat

Possible solutions:

a. Change the interval from every 1 mile to every 1000 miles. $3 for every 1000 miles
b. Change the interval from every 1 lb. to every 10 lbs. 2 bolts for every 10 lbs.
c. Change the interval from every 1 seat to every 2 seats. 1 person for every 2 seats

### Instructional Strategies:

The instructional strategies for this standard are not necessarily different than those used with slope, which was taught in 8th grade. The key difference is to connect the rate of change across representations and to connect the meaning to the context.

### Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B.6
    - The High School Gym
    - Mathemafish Population
    - Temperature Change

## Domain: Interpreting Functions (F.IF)

▶ **Cluster:** *Analyze functions using different representations.*

**Standard: F.IF.7 ★**

Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

F.IF.7a. **(9/10)** Graph linear, quadratic and absolute value functions and show intercepts, maxima, minima and end behavior.★ **(F.IF.7a)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

**Connections**: A.REI.8-10, F.IF.4, F.BF.3

**Explanations and Examples:**

Three function families: linear, quadratic, and absolute value, are studied in this modeling/function standard. In all cases, the function is presented symbolically. There are similarities between this standard and F.IF.4, which requires students to interpret the key features. This standard requires students to connect the key features of the function to the graph. The key features focused on for these functions families are intercepts, maxima/minima, and end behavior (in addition to rate of change from F.IF.6).

Graphing by hand merges F.BF.3 with F.IF.7 so that students can translate the function from the original parent function. Graphing with technology could be an online application such as desmos or a graphing calculator.

**Examples:**

1. The linear model that shows the total income for a car salesman is $f(x) = 300x + 40{,}000$, where x is the number of cars sold in one month. Graph the function, identify the key features in the graph and describe their meaning in the context of this situation.
    a. The salesman was offered a job selling motorcycles with the following offer for pay, $f(x) = 100x + 45{,}000$. Should he take the job? Write a convincing argument to justify your recommendation. You can use the following information:
        i. The top car salesman sells 50 cars a month.
        ii. The top motorcycle salesman sells 75 motorcycles a month
        iii. On average, the salesman offered this job sells 25 cars a month.
2. Amtrak s annual passenger revenue for the years 1980-2000 is modeled approximately by the formula $f(t) = -40|x - 11| + 990$ where *f(t)* is the annual revenue in millions of dollars and *t* is the number of years since January 1, 1980.
    a. Describe the meaning of the key features in the context of the situation.
    b. In what years was the passenger revenue $790 million?
    c. When the model was created in 2000, should the company have continued its current business practices for changed? Justify your conclusion with information from the graph.

3. A distributor of apple juice has 5000 bottles in the store that it wishes to distribute in a month. From experience, it is known that demand $D$ is given by $D = -2000p^2 + 2000p + 17000$. Find the price per bottle that will result in zero inventory. What key feature of the graph corresponds with this value?

4. Make up an equation for a quadratic function whose graph satisfies the given condition. Use whatever form is most convenient.
   a. Has a vertex at (2, 5).
   b. Has a y-intercept at (0, −6)
   c. Has x-intercepts at (3,0) and (5,0)
   d. Has x-intercepts at the origin and (4,0)
   e. Goes through the points (4,2) and (1,2)

**Instructional Strategies:** See F.BF.3

**Resources/Tools:**
Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.C.7.a
  - Graphs of Quadratic Functions

Mathematics Assessment Project:

- Classifying Parallel and Perpendicular Lines
- Representing Quadratic Functions Graphically
- Representing Functions of Everyday Situations (with the number of problems restricted to linear, quadratic, and absolute value)
- Comparing Lines and Linear Equations

Desmos:

- Linear Bundle
- Quadratic Bundlle
- Polygraph: Absolute Value

## Domain: Interpreting Functions (F.IF)

▶ **Cluster:** *Analyze functions using different representations.*

**Standard: F.IF.8**

Write a function in different but equivalent forms to reveal and explain different properties of the function.

F.IF.8a. **(9/10)** Use different forms of linear functions, such as slope-intercept, standard, and point-slope form to show rate of change and intercepts. **(2017)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: F.IF, A.CED.4, A.SSE.1-3

**Explanations and Examples:**

While A.SSE.3 is similar, and can be used in conjunction with F.IF.8, the primary difference is that A.SSE.3 focuses on the quantity represented by the expression while F.IF.8 is related to the properties of the function. The other standards in F.IF identify the properties of the function which should be investigated here i.e. intercepts, rate of change (including increasing/decreasing and positive/negative).

**Examples:**

1. The Granda Theater has a special rate for groups of 10 or more people: $40 for the first 10 people and $3 for each additional person. Which of the following expressions tells the amount that a group of 10 or more will have to pay if n represents the number of people in the group, where $n \geq 10$:
   a. $40 + 3n$
   b. $(40 + 3)n$
   c. $40 + 3(n + 10)$
   d. $40 + 3(n - 10)$
2. Rewrite the equation into slope-intercept form. How are the slope and y-intercept related to the original situation?
3. Write a line parallel to $3x - 4y = 20$ that meets the following conditions:
   a. The line through the point (1,2) and in standard form
   b. Any parallel line in slope intercept form

**Instructional Strategies:**

One authentic problem type for this standard is to write the function based on the pattern seen or information provided and then rewrite the equation into one of the three main forms for a linear function and identify how the new equation is related to the original situation. (See example 1). Students might also benefit from classifying equation types similar to language used to investigate addition situations in elementary school. The addition situation "start/change/result" is similar to slope intercept form and a "part/part/whole" addition situation is related to standard form.

**Resources/Tools:**

Mathematics Assessment Project:

- Comparing Lines and Linear Equations

Desmos:

- Linear Bundle

## Domain: Interpreting Functions (F.IF)

**Cluster:** *Understand the concept of a function and use function notation*

**Standard:** **F.IF.9 (all)**

Compare properties of two functions using a variety of representations (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, a quantity increasing exponentially eventually exceeds a quantity increasing linearly.* **(F.IF.9)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

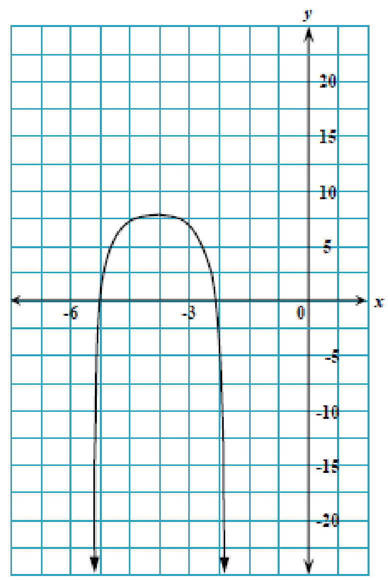
**Connections**: F.IF.4, F.IF.7

**Explanations and Examples:** See F.IF.1 and F.IF.4

**Examples:**

1. Examine the functions below. Which function has the larger maximum? How do you know?

$$f(x) = 2x^2 - 8x + 20$$




**Resources/Tools:**

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.C.9
  - o Throwing Baseballs

# Domain: Building Functions (F.BF)

◆ **Cluster:** *Build a function that models a relationship between two quantities.*

**Standard**: **F.BF.1**

Use functions to model real-world relationships.

F.BF.1a. **(9/10)** Combine multiple functions to model complex relationships. *For example,* $p(x) = r(x) - c(x); (profit = revenue - cost)$. **(F.BF.1b)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.7 Look for and make use of structure.

**Connections**: F.IF.8, A.SSE.1-3, A.APR.1

**Explanations and Examples:**

This standard provides an excellent perspective on the "all" standard A.SSE.1b "Interpret complicated expressions by viewing one or more of their parts as a single entity." In the case of this standard, the single entity is a function that is combined with another function to create a new model. In fact, this standard could provide scaffolding for that standard by providing explicit instruction on how the complicated function might be built.

**Examples:**

1) A shoe store has three different locations. Location A's daily inventory can be modeled by the function $f(x) = 250 - 27.9x$. The inventory for Location B is modeled by the function $h(x) = -\frac{1}{2}x^2 + 617$. Finally, Location C has a daily inventory modeled by $g(x) = -42.8x + 1253$. To calculate the inventory for the entire company, the chief first needs a function to model the total inventory. Write a function to model the company's entire daily inventory.
2) The length of a rectangle is a function of its width and can be modeled by the function $f(w) = 3w - 5$.
   a) Write a function to model the perimeter of the rectangle.
   b) Write a function to model the area.
3) An excel spreadsheet uses the formulas seen below. Fill in the remainder of the table by substituting the values in rows 2 and 3

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | A1 | B1 | A1-B1 | A1+B1 | C1*D1 |
| 2 | 15 | 3 | | | |
| 3 | X | Y | | | |

**Instructional Strategies:**

Color coding each function can help student follow the function through the combination.

**Resources/Tools:**

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-BF.A.1.b
    - A Sum of Functions

## Domain: Building Functions (F.BF)

● **Cluster:** *Build new functions from existing functions*

**Standard**: **F.BF.3**

**(9/10/11)** Transform parent functions $(f(x))$ by replacing $f(x)$ with $f(x)+k,\ kf(x),\ f(kx),$ and $f(x+k)$ for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. For **(9/10)** focus on linear, quadratic, and absolute value functions. **(F.BF.3)**

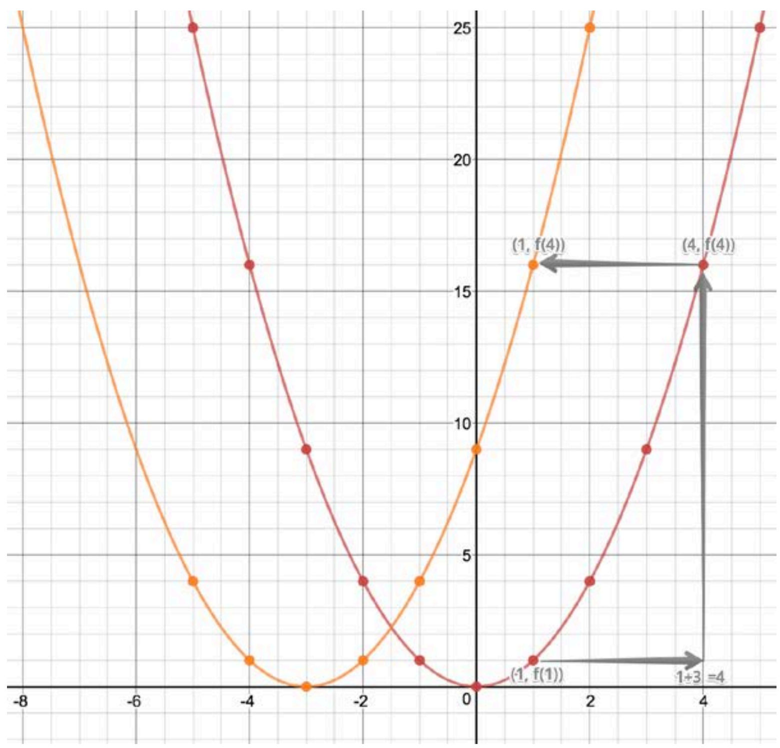
**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: F.IF.1, F.IF.7

**Explanations and Examples:**

This standard defines transformations using reasoning function notation so students should investigate transformations by reasoning about functions using function notation. Students could use tables of values, equations, or graphs but are only asked to find the value of *k* when given the graph. Students should also be able to investigate and illustrate the effect on the graph using technology. The focus in 9/10 should be on linear, quadratic and absolute value functions. Because of this focus, identifying even and odd functions is reserved for 11th grade.

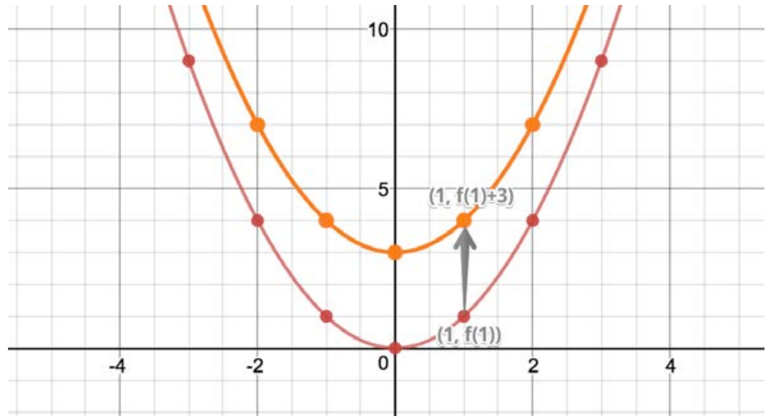



To investigate what it means to view transformations through the perspective of function notation, compare two transformations $f(x+k)$ and $f(x)+k$. Restating these transformations, the first adds the value *k* to the input while the second adds the value to the output. Students also need to know the parent function, which might be presented as an equation, table, or graph.

If *k=3* then the first transformation is $f(1+3) = f(4) = 16$. So if the input is *x=1* then the output is the output associated with $f(4)$=16.

If *x=-3*, then the output is $f(-3+3) = f(0) = 0$

After several examples, students should notice that the output shifts three units to the left.




The other function transformation, $f(x) + k$, adds the value *k* to the output instead of the input.
If *k=3* and x=1, then the transformation $f(1) = 1 + 3 = 4$.

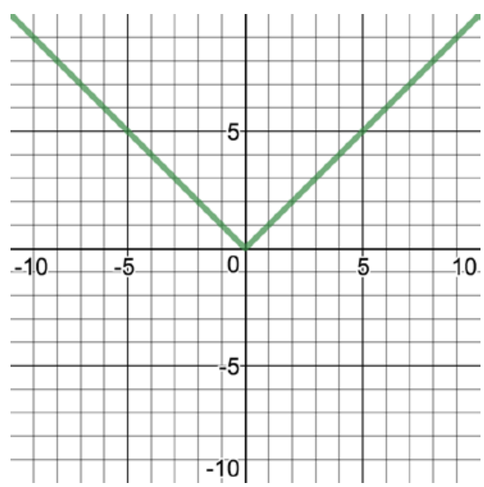
If *x=-3*, then the output is
$f(-3) = 9 + 3 = 12$

After a few examples, students can predict that changes to the input will affect the graph horizontally while changes to the output will affect the graph vertically. Students could then verify their conclusions using technology to make a prediction and then verify that their prediction was accurate.

Exploring transformations from an algebraic perspective provides another view of transformations and, in the case of linear and absolute value functions, explores different but equivalent forms of the same function. For example, if the parent function is $f(x) = 2x + 3$, justify that $f(x - 1) = f(x) - 2$ using at least two different representations. An example that might be even more confusing for students is when the parent function is $f(x) = x$. In this situation $f(x + 3) = f(x) + 3$ and $(-2x) = -2f(x)$ . If these transformations are studied separate from quadratic functions, students might overgeneralize and conclude that it does not make a difference if the change happens to the input or output.

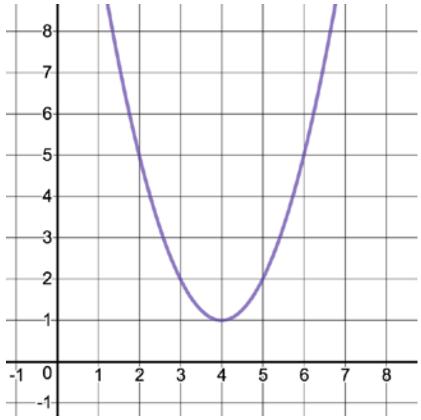
**Examples:**

1. Describe how to draw the parent function $y = |x|$ by hand, without technology.
2. Draw the following functions on the provided graph:
   a. $f(x - 1)$
   b. $f(x) + 3$
   c. $2f(x)$
   d. $f(-\frac{1}{3}x)$




3. Explain, algebraically why $f(x) = x^2$ and $f(x) = (-x)^2$ are equivalent by $f(x) = -x^2$ is not equivalent.

4. Write the function transformation, using function notation, that produces the following graph:




**Instructional Strategies:**

This standard provides a great opportunity to reinforce some ALL standards. Specifically, reinforcing the meaning of function notation can strengthen students understanding of both functions and function notation. Helping students focus on the key features for these three function families can help them find more efficient strategies for transforming the function. Using multiple representations will continue to help students think flexibly about functions. So while this standard can support F.IF.7, which focuses on graphing, students should experience transformations with a table of values and algebraically.

Desmos polygraph activities and marble slide activities are engaging opportunities to explore and practice transformation.

**Resources/Tools:**

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-BF.B.3
  - Medieval Archer
  - Building a Quadratic Function
  - Transforming the Graph of a Function

Desmos:

- Function Transformations: Practice with Symbols
- Function Transformations Bundle

# High School – Statistics & Probability

## Domain: Interpreting Categorical and Quantitative Data (S.ID)

▶ **Cluster:** *Summarize, represent, and interpret data on a single count or measurement variable.*

**Standard: S.ID.1**

**(9/10)** Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (**interquartile range**, standard deviation) of two or more different data sets. **(S.ID.2)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

**Connections**: 7.SP.1-4,

**Explanations and Examples:**

Given two sets of data or two graphs, identify the similarities and differences in shape, center and spread.

Compare data sets and be able to summarize the similarities and difference between the shape, and measures of center and spreads of the data sets.

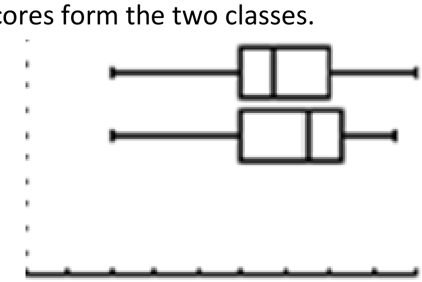
Use the correct measure of center and spread to describe a distribution that is symmetric or skewed.
Identify outliers and their effects on data sets.

Students may use spreadsheets, graphing calculators and statistical software for calculations, summaries, and comparisons of data sets.

**Examples:**

Example1:

The box plots show the distribution of scores on a district writing test of two classes at a school.
Compare the range and medians of the scores form the two classes.



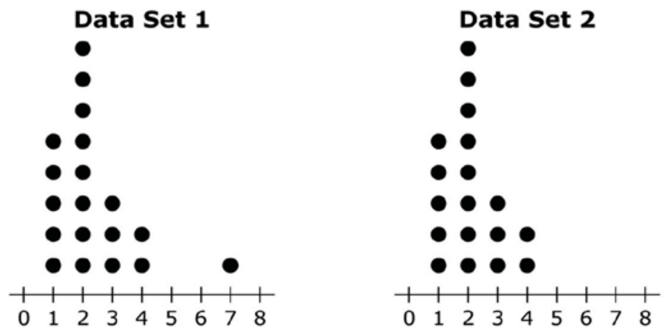
Example 2:

Given a set of test scores: 99, 96, 94, 93, 90, 88, 86, 77, 70, 68, find the mean, median, and standard deviation.

Explain how the values vary about the mean and median. What information does this give the teacher?

Example3:

The frequency distributions of two data sets are shown in the dot plots below.




For each of the following statistics, determine whether the value of the statistic is greater for Data Set 1, equal for both data sets, or greater for Data Set 2.

| | **Greater for Data Set 1** | **Equal for Both Data Sets** | **Greater for Data Set 2** |
|---|---|---|---|
| Mean | | | |
| Median | | | |
| Standard Deviation | | | |

*Solution:*

- Row 1: Greater for Data Set 1
- Row 2: Equal for both data sets
- Row 3: Greater for Data Set 1

### Instructional Strategies:

Measures of center and spread for data sets without outliers are the mean and standard deviation, whereas median and interquartile range are better measures for data sets with outliers.

Introduce the formula of standard deviation by reviewing the previously learned MAD (mean absolute deviation). The MAD is very intuitive and gives a solid foundation for developing the more complicated standard deviation measure.

Informally observing the extent to which two boxplots or two dot plots overlap begins the discussion of drawing inferential conclusions. Don’t shortcut this observation in comparing two data sets.

As histograms for various data sets are drawn, common shapes appear. To characterize the shapes, curves are sketched through the midpoints of the tops of the histogram’s rectangles. Of particular importance is a symmetric unimodal curve that has specific areas within one, two, and three standard deviations of its mean. It is called the Normal distribution and students need to be able to find areas (probabilities) for various events using tables or a graphing calculator.

Students may believe:

- That a bar graph and a histogram are the same. A bar graph is appropriate when the horizontal axis has categories and the vertical axis is labeled by either frequency (e.g., book titles on the horizontal and number of students who like the respective books on the vertical) or measurement of some numerical variable (e.g., days of the week on the horizontal and median length of root growth of radish seeds on the vertical). A histogram has units of measurement of a numerical variable on the horizontal (e.g., ages with intervals of equal length).
- That the lengths of the intervals of a boxplot (min,Q1), (Q1,Q2), (Q2,Q3), (Q3,max) are related to the number of subjects in each interval. Students should understand that each interval theoretically contains one-fourth of the total number of subjects. Sketching an accompanying histogram and constructing a live boxplot may help in alleviating this misconception.
- That all bell-shaped curves are normal distributions. For a bell-shaped curve to be Normal, there needs to be 68% of the distribution within one standard deviation of the mean, 95% within two, and 99.7% within three standard deviations.

**Resources/Tools:**

EngageNY Algebra I Module 2:

- "Descriptive Statistics" - (This Module includes lessons for standards S.ID. 1-3,5-9) In this module, students reconnect with and deepen their understanding of statistics and probability concepts first introduced in Grades 6, 7, and 8. Students develop a set of tools for understanding and interpreting variability in data, and begin to make more informed decisions from data. They work with data distributions of various shapes, centers, and spreads. Students build on their experience with bivariate quantitative data from Grade 8. This module sets the stage for more extensive work with sampling and inference in later grades.

**S.ID.1-4**

Mathematics Assessment Project:

- "Representing Data 1: Using Frequency Graphs" – This lesson unit is intended to help you assess how well students:
  - Are able to use frequency graphs to identify a range of measures and make sense of this data in a real-world context.
  - Understand that a large number of data points allow a frequency graph to be approximated by a continuous distribution.
- "Representing Data 2: Using Box Plots" – This lesson unit is intended to help you assess how well students are able to interpret data using frequency graphs and box plots. In particular this unit aims to identify and help students who have difficulty figuring out the data points and spread of data from frequency graphs and box plots. It is advisable to use the lesson: Representing Data 1: Frequency Graphs, before this one.

Illustrative Mathematics High School Statistics & Probability tasks: **Scroll to the appropriate section to find named tasks.**

- S-ID.A
  - Accuracy of Carbon 14 Dating I
- S-ID.A.1
  - Haircut Costs
  - Speed Trap

## Domain: Interpreting Categorical and Quantitative Data (S.ID)★

- **Cluster:** *Summarize, represent, and interpret data on a single count or measurement variable.*

**Standard**: **S.ID.2**

**(9/10)** Interpret differences in shape, center, and spread in the context of the data sets using dot plots, histograms, and box plots, accounting for possible effects of extreme data points (**outliers**). **(S.ID.3)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics

**Connections**: S.ID.1

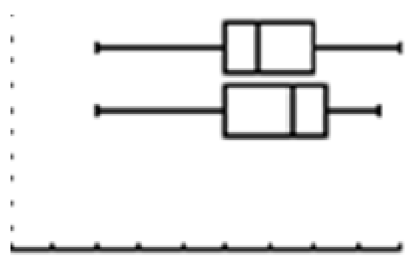
**Explanations and Examples:**

Use data from multiple sources to interpret differences in shape, center and spread.
Discuss the effect of outliers on measures of center and spread and the effect on the shape.
Predict the effect an outlier will have on the shape, center, and spread of a data set.
Decide whether to include the outliers as part of the data set or to remove them.

Students may use spreadsheets, graphing calculators and statistical software to statistically identify outliers and analyze data sets with and without outliers as appropriate.

**Examples:**

Example1:
The box plots show the distribution of scores on a district writing test of two classes at a school.
Which class performed better? Justify your conclusion.



Example 2:
Find two similar data sets A and B. Choose and create a plot or graph to represent the data.
What changes would need to be made to data set A to make it look like data set B?

Example 3:

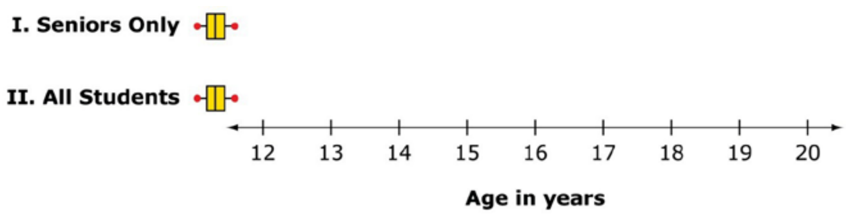
The ages of the students in a certain high school are to be graphed on a set of parallel box plots according to the following:

- Set I: All seniors in the school (grade 12)
- Set II: All students in the school (grades 9 through 12)

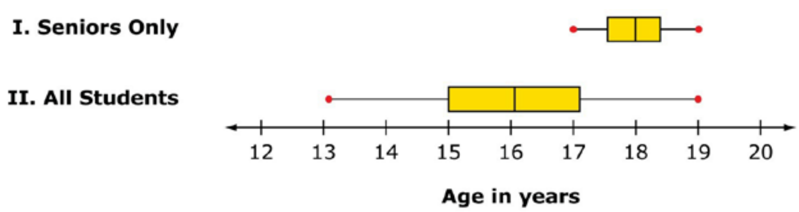
In the figure below, drag each of the two box plots into position above the number line to approximate the ages of the two sets of students. To do this:

- o First move each box plot at an appropriate location according to its center.
- o Then drag each endpoint to stretch the box plot to represent the spread.

*Note: There are no outliers in either set.*



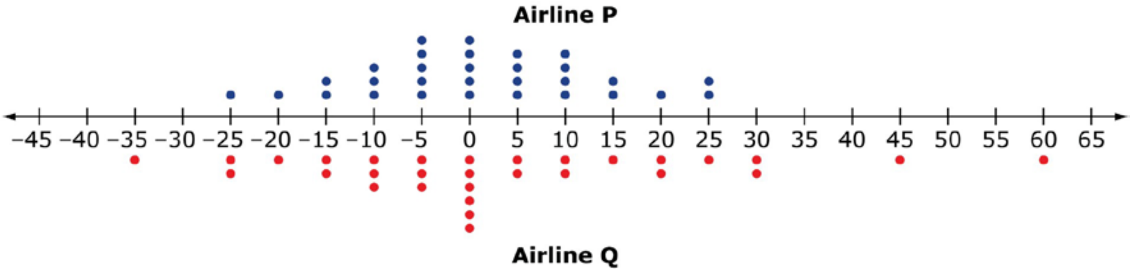

*Solution:*




*Graphs should show: Median of I > Median of II Range of I < Range of II Max of I ≤ Max of II*

Example4:

The dot plots below compare the number of minutes 30 flights made by two airlines arrived before or after their scheduled arrival times.




- Negative numbers represent the minutes the flight arrived **before** its scheduled time.
- Positive numbers represent the minutes the flight arrived **after** its scheduled time.
- Zero indicates the flight arrived **at** its scheduled time.

Based on these data, from which airline will you choose to buy your ticket?
Use the ideas of center and spread to justify your choice.

*Sample Response*:

I would choose to buy the ticket from Airline P. Both airlines are likely to have an on-time arrival since they both have median values at 0. However, Airline Q has a much greater range in arrival times. Airline Q could arrive anywhere from 35 minutes early to 60 minutes late. For Airline P, this flight arrived within 10 minutes on either side of the scheduled arrival time about $\frac{2}{3}$ of the time, and for Airline Q, that number was only about $\frac{1}{2}$. For these reasons, I think Airline P is the better choice.

**Instructional Strategies:** See S.ID.1

**Resources/Tools:**

Illustrative Mathematics High School Statistics & Probability tasks: **Scroll to the appropriate section to find named tasks.**

- S-ID.A.1
  - Haircut Costs
  - Speed Trap

## Domain: Interpreting Categorical and Quantitative Data (S.ID)★

◆ **Cluster:** *Summarize, represent, and interpret data on two categorical and quantitative variables.*

**Standard**: **S.ID.4**

**(9/10)** Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies). Recognize possible associations and trends in the data. **(S.ID.5)**

### Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

**Connections**: 6.RP, 7.RP

### Explanations and Examples:

Create a two-way frequency table from two categorical variables; read and interpret data displayed in a two way table.

Write clear summaries of data displayed in a two-way frequency table.

Calculate joint, marginal, and conditional relative frequencies. Make appropriate displays of joint, marginal, and conditional distributions.

Describe patterns observed in the data. Recognize the association between two variables by comparing conditional and marginal percentages.

Students may use spreadsheets, graphing calculators, and statistical software to create frequency tables and determine associations or trends in the data.

### Examples:

**Two-way Frequency Table**

A two-way frequency table is shown below displaying the relationship between age and baldness. We took a sample of 100 male subjects, and determined who is or is not bald. We also recorded the age of the male subjects by categories.

| Two-Way Frequency Table | | | |
|---|---|---|---|
| Bald | Age | | Total |
| | Younger than 45 | 45 or older | |
| No | 35 | 11 | 46 |
| Yes | 24 | 30 | 54 |
| Total | 59 | 41 | 100 |

The *total row* and *total column* entries in the table above report the marginal frequencies, while entries in the body of the table are the joint frequencies.

**Two-way Relative Frequency Table**

Example 1:

The relative frequencies in the body of the table are called conditional relative frequencies.

| Two-Way Relative Frequency Table | | | |
|---|---|---|---|
| Bald | Age | | Total |
| | Younger than 45 | 45 or older | |
| No | 0.35 | 0.11 | 0.46 |
| Yes | 0.24 | 0.30 | 0.54 |
| Total | 0.59 | 0.41 | 1.00 |

Example 2:

Given the data in the table below, what is the joint frequency of students who have chores and a curfew? Which marginal frequency is the largest?

| | **Curfew: Yes** | **Curfew: No** | **Total** |
|---|---|---|---|
| **Chores: Yes** | 13 | 5 | 18 |
| **Chores: No** | 12 | 3 | 15 |
| **Total** | 25 | 8 | |

Example 3:

The 54 students in one of several middle school classrooms were asked two questions about musical preferences: "Do you like rock?" "Do you like rap?" The responses are summarized in the table below.

| | **Like Rap** | | |
|---|---|---|---|
| **Like Rock** | **Yes** | **No** | **Row Totals** |
| **Yes** | **27** | **6** | **33** |
| **No** | **4** | **17** | **21** |
| **Column Totals** | **31** | **23** | **54** |

1. Is this a random sample, one that fairly represents the opinions of all students in the middle school?
2. What percentage of the students in the classroom like rock?
3. Is there evidence in this sample of a positive association in this class between liking rock and liking rap? Justify your answer by pointing out a feature of the table that supports it.
4. Explain why the results for this classroom might not generalize to the entire middle school.

*Solution:*

1. This is not a randomly selected sample that fairly represents the students in the school. See part (d) for more details.
2. $\frac{33}{54} = 61.1\%$
3. Yes, there is evidence of a positive association. Of those who like Rap, $\frac{27}{31} = 87.1\%$like Rock, too.
   This means that the percentage of those who like Rock is higher among those who like Rap than among the entire sample.

The sample is not necessarily a random sample. While it might be true that the association holds in other classes, we have no evidence of this. It is possible, for instance, that this was an unusual class at this school; maybe this class consisted entirely of music students, and their preferences would be different than in other classes or than in the entire school.

**Instructional Strategies:**

In the categorical case, begin with two categories for each variable and represent them in a two-way table with the two values of one variable defining the rows and the two values of the other variable defining the columns. (Extending the number of rows and columns is easily done once students become comfortable with the 2X2 case.) The table entries are the joint frequencies of how many subjects displayed the respective cross-classified values. Row totals and column totals constitute the marginal frequencies. Dividing joint or marginal frequencies by the total number of subjects define relative frequencies (and percentages), respectively. Conditional relative frequencies are determined by focusing on a specific row or column of the table. They are particularly useful in determining any associations between the two variables.

In the numerical or quantitative case, display the paired data in a scatterplot. Note that although the two variables in general will not have the same scale, e.g., total SAT versus grade-point average, it is best to begin with variables with the same scale such as SAT Verbal and SAT Math. Fitting functions to such data will avoid difficulties such as interpretation of slope in the linear case in which scales differ. Once students

**Resources/Tools:**

Illustrative Mathematics High School Statistics & Probability tasks: **Scroll to the appropriate section to find named tasks.**

- S-ID.B.5
  - Musical Preferences

# Domain: Interpreting Categorical and Quantitative Data (S.ID)★

◆ **Cluster:** *Summarize, represent, and interpret data on two categorical and quantitative variables.*

**Standard**: **S.ID.5**

Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.

S.ID.5a. **(9/10)** Use a given linear function to solve problems in the context of data. **(S.ID.6a)**

S.ID.5b. **(9/10)** Fit a linear function to data and use it to solve problems in the context of the data. **(S.ID.6a)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: 8.SP.1-3

**Explanations and Examples:**

Create a scatter plot from two quantitative variables; identify the independent and dependent variables; and describe the relationship of the variables. Describe the form, strength and direction of the relationship.

Students may use spreadsheets, graphing calculators, and statistical software to represent data, describe how the variables are related, fit functions to data, perform regressions, and calculate residuals.

**Examples:**

- Measure the wrist and neck size of each person in your class and make a scatter plot. Find the least squares regression line. Calculate and interpret the correlation coefficient for this linear regression model. Graph and evaluate the fit of the linear equations.
- The following data shows the age and average daily energy requirements for make children and teens.

| **Age** | 1 | 2 | 5 | 11 | 14 | 17 |
|---|---|---|---|---|---|---|
| **Daily Energy** | 1110 | 1300 | 1800 | 2500 | 2800 | 3000 |

  Create a graph and find a linear function to fit the data. Using your function, what is the daily energy requirement for a male 15 years old? Would your model apply to an adult male? Explain your reasoning.

- Collect data on forearm length and height in a class. Plot the data and estimate a linear function for the data. Compare and discuss different student representations of the data and equations students discover.

  Could the equations(s) be used to estimate the height for any person with a known forearm length?
  Why or why not?

**Instructional Strategies:** See S.ID.4

**Resources/Tools:**

EngageNY Algebra I Module 2:

- "Descriptive Statistics" - (This Module includes lessons for standards S.ID. 1-3,5-9) In this module, students reconnect with and deepen their understanding of statistics and probability concepts first introduced in Grades 6, 7, and 8. Students develop a set of tools for understanding and interpreting variability in data, and begin to make more informed decisions from data. They work with data distributions of various shapes, centers, and spreads. Students build on their experience with bivariate quantitative data from Grade 8. This module sets the stage for more extensive work with sampling and inference in later grades.

Illustrative Mathematics High School Statistics & Probability tasks: **Scroll to the appropriate section to find named tasks.**

- S-ID.B.6
  - Used Subaru Foresters I
  - Battery Charge 2

## Domain: Interpreting Categorical and Quantitative Data (S.ID)★

- **Cluster:** *Interpret linear models*

**Standard**: **S.ID.6**

**(9/10)** Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data. **(S.ID.6)**

**Suggested Standards for Mathematical Practice (MP):**

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

**Connections**: S.ID.5, 8.SP.1-3

**Explanations and Examples:**

See F.IF.4-7 to review interpreting rate of change and intercept in the terms of context.

**Examples:** See S.ID.5

**Instructional Strategies:** See F.IF.4-7

**Resources/Tools:** See S.ID.5

# APPENDIX: TABLE 1 *The Properties of Operations*

| Name of Property | Representation of Property | Example of Property, Using Real Numbers |
|---|---|---|
| **Properties of Addition** | | |
| **Associative** | $(a+b)+c=a+(b+c)$ | $(78+25)+75=78+(25+75)$ |
| **Commutative** | $a+b=b+a$ | $2+98=98+2$ |
| **Additive Identity** | $a+0=a$ *and* $0+a=a$ | $9875+0=9875$ |
| **Additive Inverse** | For every real number $a$, there is a real number $-a$ such that $a+-a=-a+a=0$ | $-47+47=0$ |
| **Properties of Multiplication** | | |
| **Associative** | $(a\times b)\times c=a\times(b\times c)$ | $(32\times 5)\times 2=32\times(5\times 2)$ |
| **Commutative** | $a\times b=b\times a$ | $10\times 38=38\times 10$ |
| **Multiplicative Identity** | $a\times 1=a$ *and* $1\times a=a$ | $387\times 1=387$ |
| **Multiplicative Inverse** | *For every real number a,* $a\neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$ | $\frac{8}{3}\times\frac{3}{8}=1$ |
| **Distributive Property of Multiplication over Addition** | | |
| **Distributive** | $a\times(b+c)=a\times b+a\times c$ | $7\times(50+2)=7\times 50+7\times 2$ |

*(Variables a, b, and c represent real numbers.)*

*Excerpt from NCTM's* ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

## *TABLE 2. The Properties of Equality*

| Name of Property | Representation of Property | Example of property |
|---|---|---|
| **Reflexive Property of Equality** | $a = a$ | $3{,}245 = 3{,}245$ |
| **Symmetric Property of Equality** | $If\ a = b, then\ b = a$ | $2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$ |
| **Transitive Property of Equality** | $If\ a = b\ and\ b = c, then\ a = c$ | $If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$ |
| **Addition Property of Equality** | $If\ a = b, then\ a + c = b + c$ | $If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$ |
| **Subtraction Property of Equality** | $If\ a = b, then\ a - c = b - c$ | $If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$ |
| **Multiplication Property of Equality** | $If\ a = b, then\ a \times c = b \times c$ | $If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$ |
| **Division Property of Equality** | $If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$ | $If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$ |
| **Substitution Property of Equality** | If $a = b$, then $b$ may be substituted for $a$ in any expression containing $a$. | $If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$ |

*(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)*

# *TABLE 3. The Properties of Inequality*

Exactly one of the following is true: $a < b, a = b, a > b$.

$If\ a > b\ and\ b > c\ then\ a > c.$

$If\ a > b, then\ b < a.$

$If\ a > b, then - a < -b.$

$If\ a > b, then\ a \pm c > b \pm c.$

$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$

$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$

$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$

$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

# *Table 4. Cognitive Rigor Matrix/Depth of Knowledge (DOK)*

Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

| Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom) | DOK Level 1<br>Recall & Reproduction | DOK Level 2<br>Basic Skills & Concepts | DOK Level 3<br>Strategic Thinking & Reasoning | DOK Level 4<br>Extended Thinking |
|---|---|---|---|---|
| **Remember** | • Recall conversions, terms, facts | | | |
| **Understand** | • Evaluate an expression<br>• Locate points on a grid or number on number line<br>• Solve a one-step problem<br>• Represent math relationships in words, pictures, or symbols | • Specify, explain relationships<br>• Make basic inferences or logical predictions from data/observations<br>• Use models/diagrams to explain concepts<br>• Make and explain estimates | • Use concepts to solve non-routine problems<br>• Use supporting evidence to justify conjectures, generalize, or connect ideas<br>• Explain reasoning when more than one response is possible<br>• Explain phenomena in terms of concepts | • Relate mathematical concepts to other content areas, other domains<br>• Develop generalizations of the results obtained and the strategies used and apply them to new problem situations |
| **Apply** | • Follow simple procedures<br>• Calculate, measure, apply a rule (e.g., rounding)<br>• Apply algorithm or formula<br>• Solve linear equations<br>• Make conversions | • Select a procedure and perform it<br>• Solve routine problem applying multiple concepts or decision points<br>• Retrieve information to solve a problem<br>• Translate between representations | • Design investigation for a specific purpose or research question<br>• Use reasoning, planning, and supporting evidence<br>• Translate between problem & symbolic notation when not a direct translation | • Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results |
| **Analyze** | • Retrieve information from a table or graph to answer a question<br>• Identify a pattern/trend | • Categorize data, figures<br>• Organize, order data<br>• Select appropriate graph and organize & display data<br>• Interpret data from a simple graph<br>• Extend a pattern | • Compare information within or across data sets or texts<br>• Analyze and draw conclusions from data, citing evidence<br>• Generalize a pattern<br>• Interpret data from complex graph | • Analyze multiple sources of evidence or data sets |
| **Evaluate** | | | • Cite evidence and develop a logical argument<br>• Compare/contrast solution methods<br>• Verify reasonableness | • Apply understanding in a novel way, provide argument or justification for the new application |
| **Create** | • Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept | • Generate conjectures or hypotheses based on observations or prior knowledge and experience | • Develop an alternative solution<br>• Synthesize information within one data set | • Synthesize information across multiple sources or data sets<br>• Design a model to inform and solve a practical or abstract situation |

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