4/4/2019

2017 Kansas Mathematics Standards

Flip Book

High School- Geometry

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope"; 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path; if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size." In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important, but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster-Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move "toward improved instructional practice" and support "one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students" (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in High School

The State Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students Grades K-12. Below are a few examples of how these Practices may be integrated into tasks that High School students complete.

Practice	Explanation and Example
1) Make sense of problems and persevere in solving them.	Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. High School students make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. They might transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. They can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They constantly check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They understand the approaches of others to solving complex problems and identify correspondence between different approaches.
2) Reason abstractly and quantitatively.	Mathematically proficient students make sense of quantities and their relationships in problem situations. They represent a wide variety of real world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*—to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.
3) Construct viable arguments and critique the reasoning of others.	Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. High School students reason inductively about data, making plausible arguments that take into account the context from which the data arose. They compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High School students determine domains to which an argument applied, they listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4) Model with mathematics.	Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. High School students might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximation to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.
5) Use appropriate tools strategically.	Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil/paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software. They are sufficiently familiar with tools appropriate for their grade to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.
6) Attend to precision.	Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussions with others and in their *own* reasoning. They state the meaning of the symbols they choose. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. High school students have learned to examine claims and make explicit use of definitions.
7) Look for and make use of structure.	Mathematically proficient students look closely to discern a pattern or structure. For example, high school students recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5-3(x-y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.
8) Look for and express regularity in repeated reasoning.	Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1,2) with slope 3, they might abstract the equation $\frac{(y-2)}{(x-1)}=3$. Noticing the regularity in the way terms cancel when expanding $(x-1)\,(x\ +1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2\ +x+1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Implementing Standards for Mathematical Practice

This guide was created to help educators implement these standards into their classroom instruction. These are the practices for the **students,** and the teacher can assist students in using them efficiently and effectively.

#1 – Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others' solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line…, chart..., table…?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools. (Tools may include; concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
  - a task when there is no need to have an exact answer
  - a task when there is not enough information to get an exact answer
  - a task to check if the answer from a calculation is reasonable

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value; working with properties; etc.)
- Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm. When "unit size" cannot be equally distributed, it is necessary to break down into a smaller "unit size". (example below)

$4\overline{)351}$ $\underline{-32}$	3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.
31	There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.
$\underline{-28}$ 3	This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation. i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for Mathematics in Geometry

Introduction

The fundamental purpose of the Geometry course is to formalize and extend students' geometric experiences from the middle grades. In this high school Geometry course, students explore more complex geometric situations and deepen their explanations of geometric relationships by presenting and hearing formal mathematical arguments. Important differences exist between this course and the historical approach taken in geometry classes. For example, transformations are emphasized in this course. Close attention should be paid to the introductory content for the Geometry conceptual category.

For the high school Geometry course, instructional time should focus on five critical areas:

1. Students develop experience with drawing triangles based on given measurements and performing rigid motions including translations, reflections, and rotations. They use these to developnotions about what it means for two objects to be congruent. Students establish triangle congruence criteria, based on analyses of rigid motions and formal constructions. They use triangle congruence as a familiar foundation for the development of formal proof. Students prove theorems— using a variety of formats including deductive and inductive reasoning and proof by contradiction—and solve problems about triangles, quadrilaterals, and other polygons. They apply reasoning to complete geometric constructions and explain why they work.

2. Students apply their earlier experience with proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use similarity to solve problems, and apply similarity in right triangles to understand right triangle trigonometry, with particular attention to special right triangles and the Pythagorean Theorem. They construct arguments about triangles using theorems. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity, and AA.

3. Students construct arguments about basic theorems related to circles, with particular attention to perpendicularity, in order to see symmetry in circles and as an application of triangle congruence criteria. They identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle as an application of similarity. They construct arguments using properties of polygons inscribed and circumscribed about circles.

4. In the Cartesian coordinate system, students use the distance formula to write the equation of a circle when given the radius and the coordinates of its center. Given an equation of a circle, they draw the graph in the coordinate plane. They use coordinates to prove simple geometric theorems algebraically, including the use of slope, distance, and midpoint formulas. Furthermore, they prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems. Building on their work with the Pythagorean Theorem in eighth grade to find distances, students use the rectangular coordinate system to verify geometric relationships, including properties of special triangles and quadrilaterals, and slopes of parallel and perpendicular lines, which relates back to work done in the Algebra I course.

5. Apply geometric concepts in modeling situations. Apply concepts of density and displacement based on area and volume in modeling situations

The Standards for Mathematical Practice complement the content standards so that students increasingly engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle, and high school years.

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

Growth Mindset

The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.

You can find a variety of resources related to **Growth Mindset** at: http://community.ksde.org/Default.aspx?tabid=6383.

youcubed at Stanford University

Building a Mathematical Mindset Community

MATHEMATICAL MINDSETS

Teachers and students believe everyone can learn maths at HIGH LEVELS.
- Students are not tracked or grouped by achievement
- All students are offered high level work
- "I know you can do this" "I believe in you"
- Praise effort and ideas, not the person
- Students vocalize self-belief and confidence

Communication and connections are valued.
- Students work in groups sharing ideas and visuals.
- Students relate ideas to previous lessons or topics
- Students connect their ideas to their peers' ideas, visuals, and representations.
- Teachers create opportunities for students to see connections.
- Students relate ideas to events in their lives and the world.

The maths is VISUAL.
- Teachers ask students to draw their ideas
- Tasks are posed with a visual component
- Students draw for each other when they explain
- Students gesture to illustrate their thinking

The maths is OPEN.
- Students are invited to see maths differently
- Students are encouraged to use and share different ideas, methods, and perspectives
- Creativity is valued and modeled.
- Students' work looks different from each other
- Students use ownership words - "my method", "my idea"

The environment is filled with WONDER and CURIOSITY.
- Students extend their work and investigate
- Teacher invites curiosity when posing tasks
- Students see maths as an unexplored puzzle
- Students freely ask and pose questions
- Students seek important information
- "I've never thought of it like that before."

The classroom is a risk-taking, MISTAKE VALUING environment
- Students share ideas even when they are wrong
- Peers seek to understand rather than correct
- Students feel comfortable when they are stuck or wrong
- Teachers and students work together when stuck
- Tasks are low floor/high ceiling
- Students disagree with each other and the teacher

Developed by Jo Boaler/Youcubed.org and Tulare County Office of Education

High School Notation

(★) **Modeling Standards:** Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol. The star symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

Grade Level Classifications

To assist with the organization of high school mathematics courses, the standards have grade level classifications to identify the appropriate grade at which they should be taught. The classifications were designed with the following framework in mind:

Year of School	Traditional Course Sequence	Integrated Course Sequence
9th Grade	Algebra I	Mathematics 1
10th Grade	Geometry	Mathematics 2
11th Grade	Algebra II	Mathematics 3

There will be variation with student placement in the courses listed above. At the present time, the "gateway" math class in Kansas for postsecondary schooling is College Algebra. The standards committee used this as a guide when identifying grade level classifications.

The grade level classifications are as follows

(9/10)	These standards are required for all students by the end of their first two years of high school math courses.
(11)	These standards are required for all students by the end of their third year math course.
(9/10/11)	These standards are required for all students in their first three years of high school math courses. These standards are often further divided to (9/10) and (11) to identify specific concepts and their appropriate grade level. *(9/10) should primarily accomplish the standards described as linear, quadratic and absolute value while (11) should primarily accomplish the standards described as logarithmic, square root, cube root, and exponential .*
(all)	These standards should be taught throughout every high school math course, and often represent over-arching themes or key features of the mathematical concept. These standards should be taught in conjunction with the appropriate grade level standards.
(+)	These standards should be taught as **extensions** to grade level standards when possible, or in a 4th year math course. These standards prepare students to take advanced courses in high school such as college algebra, calculus, advanced statistics, or discrete mathematics.

High School- Geometry Overview

High School—Modeling

Modeling

High School – Number and Quantity

Quantities (★) (N.Q)

A. Reason quantitatively and use units to solve problems.
N.Q.1 (★) N.Q.2 (★) N.Q.3 (★)

High School – Algebra

Seeing Structure in Expressions (A.SSE)

A. Interpret the structure of expressions.
A.SSE.1 (★) A.SSE.2

Creating Equations (★) (A.CED)

A. Create equations that describe numbers or relationships.
A.CED.1 (★) A.CED.2 (★) A.CED.3 (★) A.CED.4 (★)

Reasoning with Equations and Inequalities (A.REI)

A. Understand solving equations as a process of reasoning and explain the reasoning.
A.REI.1
B. Solve equations and inequalities in one variable.
A.REI.2
D. Represent and solve equations and inequalities graphically.
A.REI.8

High School – Functions

Interpreting Functions (F.IF)

A. Understand the concept of a function and use function notation.
F.IF.1 F.IF.2
B. Interpret functions that arise in applications in terms of the context.
F.IF.4 (★) F.IF.5 (★)
C. Analyze functions using different representations.
F.IF.9

High School –Geometry

Congruence (G.CO)

A. Experiment with transformations in the plane.
G.CO.1 G.CO.2
B. Understand congruence in terms of rigid motions.
G.CO.3 G.CO.4
C. Construct arguments about geometric theorems using rigid transformations and/or logic.
G.CO.7 G.CO.8 G.CO.9 G.CO.10
D. Make geometric constructions.
G.CO.11

Similarity, Right Triangles, and Trigonometry (G.SRT)

A. Understand similarity in terms of similarity transformations.
G.SRT.1 G.SRT.2 G.SRT.3 G.SRT.4
B. Construct arguments about theorems involving similarity.
G.SRT.5 G.SRT.6
C. Define trigonometric ratios and solve problems involving right triangles.
G.SRT.7 G.SRT.8 (★) G.SRT.9

Circles (G.C)

A. Understand and apply theorems about circles.
G.C.1 G.C.2 G.C.3

Expressing Geometric Properties with Equations (G.GPE)

A. Translate between the geometric description and the equation for a conic section.
G.GPE.1
B. Use coordinates to prove simple geometric theorems algebraically.
G.GPE.6 G.GPE.7 G.GPE.8 (★)

Modeling with Geometry (G.MG) (★)

A. Apply geometric concepts in modeling situations.
G.MG.1 (★) G.MG.2 (★) G.MG.3 (★)

High School – Modeling

Domain: Modeling (★)

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See ★ standards on the overview page.

Explanations and Examples:

The goal for this section is to expand on the information in the Modeling section of the standards by adding information from research using an article summarizing our current knowledge base "Quality Teaching of Mathematical Modeling: What Do We Know, What Can we Do?" from Werner Blum.

The word "modeling" is a word that is difficult to define because it is used to describe both a process and a product. The process of *modeling* creates a product called a *model*. The standards expect students can successfully use the process to create a model and that, given a model; they can successfully interpret and understand how the math model is related to the real world situation. But what *exactly* is a model? Niss (2007) defines a model as "a deliberately simplified and formalized image of some part of the real world, formally speaking: a triple (D, M, f) consisting of a domain D of the real world, a subset M of the mathematical world and a mapping from D to M (Niss et al. 2007)."

The standards describe a six step modeling cycle:

1. Identify the variables in the situation and select those that represent essential features.
2. Formulate a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables.
3. Analyze and perform operations on these relationships to draw conclusions.
4. Interpret the results of the mathematics in terms of the original situation.
5. Validate the conclusions by comparing them with the situation and then either improve the model or, if is acceptable, move to step 6.
6. Report the conclusions and the reasoning behind them.

Throughout the cycle, students will make choices, assumptions, and approximations.

Blum, in his research summary, identifies an important first step that is not explicitly described in the modeling process- to construct a mental model of the situation. This requires understanding the situation, being able to mentally imagine all the parts of the situation. Research has found that many students get stuck here, in this "pre-step." The reason some students don't gain entry into the process is because they have been *taught* to ignore the context, find the numbers,

and apply a familiar operation. This has been labeled by researchers as the "suspension of sense-making" and occurs whenever students are processing any word problem. Robert Kaplinksy created a video illustrating this phenomenon. He asked 32 8th grade students the following question:

"There are 125 sheep and 5 dogs in a flock. How old is the shepherd?"

Sadly 75% of students performed math operations and provided a numerical answer. This question has been replicated across a variety of settings since 1993 with the same consistent results.

After students have created a mental model of the situation, they are ready to begin the modeling process. The first step is to simplify the mental model down to the critical elements. This requires making assumptions and estimating any missing information. This is another source of difficulties for students- they are afraid to make assumptions. For example, one PISA task that asks students to make assumptions to solve the problem had low success rates across multiple countries.

Fig. 6 PISA task "Rock Concert"

For this PISA task, given to 15 year olds, the success rates were:

- Finland- 37%
- Korea- 21%
- USA- 26%

ROCK CONCERT

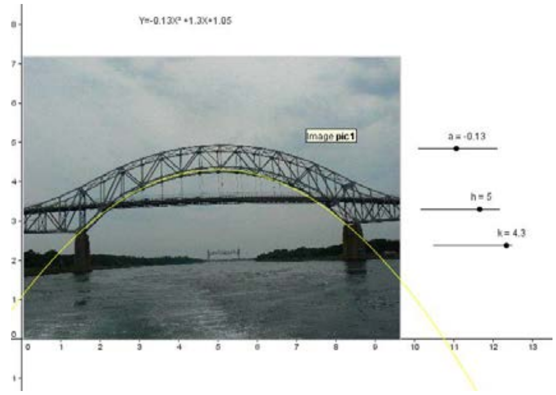
For a rock concert a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

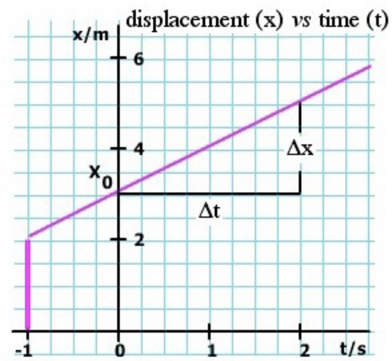
A 2 000
B 5 000
C 20 000
D 50 000
E 100 000

This question required students to estimate about how many people could fit in a square meter and an assumption that each square meter had the same density. The students should have realized that A and B did not match the scenario of a full stadium because it would be one or fewer persons per square meter. Choices D and E were also poor estimates because those choices require 10 or 20 people per square meter. Leaving the only reasonable estimate choice C. Even countries more familiar with metric measurement than the USA struggled with this type of estimation.

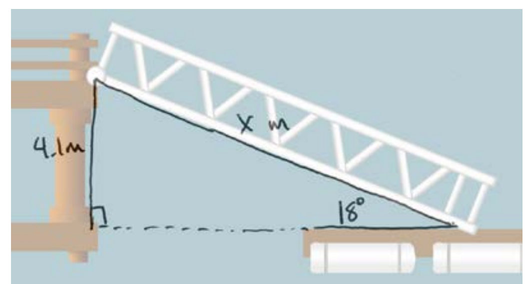
The real world is messy, filled with irrelevant data and partial information. If students are only presented problems that have been simplified and all the assumptions are made, then they do not get practice with this critical step. Developmentally, once given the freedom to find holes or irrelevant information in a problem, adolescents are often excited to explore a problem in this way.



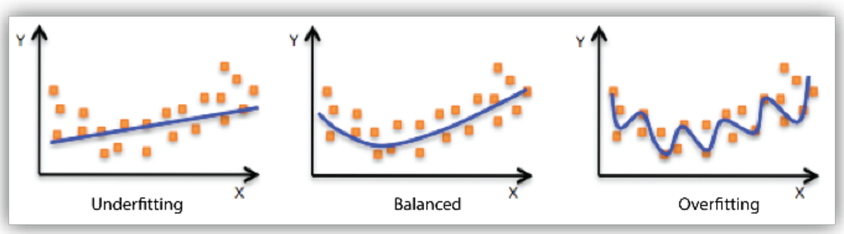
Step 2 is to mathematize the problem by creating geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables. This requires students recognize the general structure of a problem, to understand how the rate of change identifies the function family, the parameters which describe the situation, and possibly the representation that best demonstrates the relationship and can be used to find the needed information.



Step 3 and 4 are the steps we ask students to complete in a typical word problem. Unfortunately these are often the only two steps the students are regularly asked to work. The problem has been sufficiently mathematized and structured so that there are few questions about the correct structure for the problem. Students must "do the math" and interpret the results.



Step 5- validating the conclusions- involves more than interpreting the results and is another step often skipped by students. This step involves determining if the model is suitable in the real world. For example, F.LQE asks students to compare linear, quadratic, and exponential models and use the model to solve problems. If a student selects a model inappropriate to the situation, they will still be able to complete steps 3 and 4. It is once they have reached step 5 to validate the conclusions that they are given the opportunity to re-evaluate the model. Or the validation might decide the function family is correct but that the parameters chosen could be modified to better represent the situation. This step is when, in statistics, the researcher must try to fit the model to the data but also be careful to not over fit the data so that it can't be generalized to other similar situations.



The final step is to have students write up the process and conclusions. Asking students to convince a friend that they are correct can help students structure their persuasive and descriptive argument. Another reason that writing up the process, the assumptions, the simplified structure, etc. is difficult is because the problems we provide are not truly modeling problems- they are word problems. There is one solution path and it isn't messy. Providing problems with multiple viewpoints and different conclusions will help students have something to talk about. For example, when analyzing data do not clean the data for the students. Let them decide how to approach incorrect data and outliers. Using Dan Meyer's Three-Act Math Videos can provide common input data for the students with multiple paths to the solution.

These six steps are cognitively demanding and difficult because they require math knowledge, non-math knowledge, and a specific set of beliefs and attitudes about ones ability to do mathematics. So *why* do we take valuable class time to go through this entire process? Research has identified there are four different justifications and perspectives which drive the modeling process, depending on the type of problem presented to students. Understanding these justifications and perspectives can help the teacher present a wide variety of problem types and to be more intentional about highlighting the purpose for the chosen problem.

1. Applied Math: Applied mathematics justifies the modeling process because the mathematics will help the learner understand the real world. The other three justifications use the situation to support math understanding so applied mathematics is the only justification with the purpose of supporting a deeper theoretical understanding about the world. When working these modeling problems, students are practicing sensemaking through understanding the real-world.
2. Educational Modeling: Another reason to practice the modeling process is to formatively assess the thinking of students. For these problems, the examples are concrete and authentic. They are cognitively rich and include time for students to reflect on their process. When the purpose is educational modeling, students are making sense of the problem through the lens of their own growth.
3. Cultural Modeling: Modeling has the ability to connect the outside world to the math classroom, to allow students to see how math can *help* the world around them. The problems that students work will be authentic and will show how math shapes the world around them or will allow the student to see that mathematics is a science. Students will make sense of these problems by seeing the role of math in the real-world.
4. Pedagogical Modeling: Psychologically modeling problems have the ability to spark interest, motivate students, and increase retention in STEM fields. These problems are interesting, illustrating how math will benefit the student, or are rich enough to deepen students understanding of a mathematics concept (sometimes called conceptual modeling). Students will make sense by finding joy in mathematics or puzzling through a math concept.

It is clear that modeling is an important process in mathematics but also that modeling is demanding. There must be significant efforts to make this process accessible for all learners. There are many resources available by performing an internet search for STEM problems. Below are four examples to start you on your journey.

Resources/Tools:

Quantamagazine:

- https://www.quantamagazine.org/the-math-of-causation-puzzle-20180530/
- https://www.quantamagazine.org/puzzle-solution-bongard-problems-20170628/

Quantamagazine

- https://www.nytimes.com/spotlight/learning-stem

NRICH Math:

- https://nrich.maths.org/6458

NRICH

Dan Meyer's Three-Act Math Tasks:

- https://docs.google.com/spreadsheets/d/1jXSt_CoDzyDFeJimZxnhgwOVsWkTQEsfqouLWNNC6Z4/edit#gid=0

High School – Number and Quantity

Domain: Quantities ★ (N.Q)

Cluster: *Reason quantitatively and use units to solve problems.*

Standard: **N.Q.1 ★ (all)**

Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. **(N.Q.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: Algebra and Functions

Explanations and Examples:

Across a wide variety of problems and applications, units can and should be used a way to understand a problem and make an effective problem solving tool, guiding the student to the relevant measurements and operations. Interpreting units consistently could be as simple as interpreting the meaning of the y-intercept to as complicated as using the units to support the selection of the appropriate regression model.

Particular attention should be paid to creating graphs that follow standard mathematical and scientific conventions for graphing or discussing decisions made when graphing in cases of no consensus.

- Graphs must be partitioned into equal intervals.
- Intervals should be chosen so that the area interest is easily visible (for example, small enough to see an intersection or large enough to view the vertex of a parabola).
- Intervals should allow global analysis of direction of change, maximum/minimum, end behavior, etc. For example, it is possible to zoom in or out so much that a nonlinear graph appears linear.

Things to consider:

- Is it more important for the graph to take up the majority of the graphing space or should the intervals on the domain and range be the same? Taking up more space might make it easier to see the key features of interest but can distort the appearance of rate of change. Keeping the intervals the same helps create a visual of the rate of change but might not make sense if the domain is $0 \leq x \leq 2$ while the range is $150 \leq y \leq 500$.
- Should the graph include the origin or use a "compressed scale" to begin the scale at a higher number? Compressing the y-axis has the benefit of using more of the graphable space but might create a false y-intercept.

This is an "all" standard because there is no one right answer to most of these questions. Fluency and skill with making these decisions and interpreting the decision of others comes only after consistent and explicit discussion during learning.

Geometry Examples:

1. Why are the units of measure not necessary when proving $\overline{AB} \cong \overline{XY}$?

[How many cells are in the human body?]

2. About how many cells are in the human body? You can assume that a cell is a sphere with radius 10^{-3} cm and that the density of a cell is approximately the density of water which is 1g/cm3.

Instructional Strategies:

As you think about this the standard, the first few words of the standard should guide you "Use units *as a way to understand problems…"* This standard should be highlighted when it will enhance student understanding and not as part of a procedural checklist or as an addition to question that confuses more than it supports. Remember that this is an ALL standard because mastery is developed *over time*. Initially the conversations will be difficult but students should progress in sophistication throughout their time in high school mathematics. Think instructionally about how you can monitor, assess, and provide feedback to students on growth in this area, as well as all areas in the "ALL" category.
This standard also provides an opportunity to ensure alignment with other departments. The science department likely has some criteria for the graphs that they draw, which might be discussed during class. Sharing the outcome for this standard with other departments might give them some ideas for supporting mathematics within their class.

Some contextual problems may require an understanding of derived measurements and capability in unit analysis. Keeping track of derived units during computations and making reasonable estimates and rational conclusions about accuracy and the precision of the answers help in the problem-solving process.

Common Misconceptions:

Students may not realize the importance of the units' conversions in conjunction with the computation when solving problems involving measurements. Students often have difficulty understanding how ratios expressed in different units can be equal to one**.** For example, $\frac{5280\,ft}{1\,mile}$ is simply one, and it is permissible to multiply by that ratio.

Students need to make sure to put the quantities in the numerator or denominator so that the terms can cancel appropriately. Since today's calculating devices often display 8 to 10 decimal places, students frequently express answers to a much greater degree of precision than the required.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

N.Q.1-3

"Relationships Between Quantities & Reasoning with Equations & Their Graphs" - EngageNY Algebra 1 Module 1:
In this module students analyze and explain precisely the process of solving an equation. Through repeated reasoning, students develop fluency in writing, interpreting, and translating between various forms of linear equations and inequalities and make conjectures about the form that a linear equation might take in a solution to a problem. They reason abstractly and quantitatively by choosing and interpreting units in the context of creating equations in two variables to represent relationships between quantities. They master the solution of linear equations and apply related solution techniques and the properties of exponents to the creation and solution of simple exponential equations.

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A
    - Traffic Jam
- N-A.A.1
    - Weed Killer
    - Ice Cream Van
    - How Much is a Penny Worth?
    - Selling Fuel Oil at a Loss
    - Fuel Efficiency
    - Runners' World

Domain: Number and Quantities ★ (N.Q)

Cluster: *Reason quantitatively and use units to solve problems.*

Standard: **N.Q.2 ★ (all)**

Define appropriate quantities for the purpose of descriptive modeling. **(N.Q.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

Connections:

The Modeling domain provides a list of standards connected to this standard.

Explanations and Examples:

This standard focuses on critical aspect of modeling and these three words are each essential for students to use consistently: define, appropriate, and quantities.

First, students must clearly define the meaning for the variable. This requires them to attend to precision. For example, $t = time$. Time on the clock, time since the event started, time till the event ends?

It is critical to clearly define the variables to ensure that everyone understands what kind of input is expected and acceptable.

Second, the variable assignment must be appropriate. This means that students should be able to define the independent and dependent variables correctly. It also means they should be able to sift through extra information to identify the required information to answer the question. A research study from Dr. Marilyn Carlson and her team found that students struggle with identifying the appropriate variables. Students were asked to draw a graph showing the height of the fluid given the amount of fluid in the bottle. They found most misidentified the independent variable as the height and the dependent variable as the volume. Even more surprising, the students felt like *time* was an additional variable (i.e. if the water was poured in faster, the rate of change would be greater, if the water was poured and then stopped and then poured again, the graph would reflect those changes). This study illustrates the value in not only clearly defining variables but ensuring that students have made appropriate identifications and are not distracted by incorrect ideas about rate.

Finally, students *must* define the variable as quantities and use the variables as quantities. On Bill McCallum's forum about the standards, a question was asked about function notation which illustrates the importance of this concept:

> …"it can't literally be true that *f(x)* is a function, because it's a number, and a number is not a function. The letter *x* refers to a specific but unspecified number in the domain of *f*, and *f(x)* refers to the corresponding output. That's the way function notation works. I would worry that not being precise in this usage leads to confusion and misconceptions later on. I think your desire to use *f(x)* to refer to the function comes from a sense that *x* in some way represents all the input values at once. But this itself is dangerous, I think: **a lot of the trouble students have with algebra comes from a feeling that *x*** (or

whatever letter you are using) **isn't really a number** but is some vague mystical thing they have to perform mysterious rites on. So the more we can **keep students anchored** in the idea that the letters in algebraic expressions and equations are **just numbers,** and that the things you do to expressions and equations are just the things you can do to numerical expressions, the better.

Instructional Strategies: See N.Q.1

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.1
  - Harvesting the Fields

Domain: Quantities ★ (N.Q)

Cluster: ***Reason quantitatively and use units to solve problems.***

Standard: **N.Q.3 ★ (all)**

Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. **(N.Q.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See N.Q.1

Explanations and Examples:

Determine the accuracy of values based on their limitations in the context of the situation.

The margin of error and tolerance limit varies according to the measure, tool used, and context.

Examples:

- Determining price of gas by estimating to the nearest cent is appropriate because you will not pay in fractions of a cent but the cost of gas is $\frac{\$3.479}{gallon}$.
- A liquid weed-killer comes in four different bottles, all with the same active ingredient. The accompanying table gives information about the concentration of active ingredient in the bottles, the size of the bottles, and the price of the bottles. Each bottle's contents is made up of active ingredient and water.

	Concentration	Amount in Bottle	Price of Bottle
A	1.04%	64 fl oz.	$12.99
B	18.00%	32 fl oz.	$22.99
C	41.00%	32 fl oz.	$39.99
D	1.04%	24 fl oz.	$5.99

The margin of error and tolerance limit varies according to the measure, tool used, and context.

a) You need to apply a 1% solution of the weed killer to your lawn. Rank the four bottles in order of best to worst buy. How did you decide what made a bottle a better buy than another?
b) The size of your lawn requires a total of 14 fl. oz. of active ingredient. Approximately how much would you need to spend if you bought only the A bottles? Only the B bottles? Only the C bottles? Only the D bottles?

Supposing you can only buy one type of bottle, which type should you buy so that the total cost to you is the least for this particular application of weed killer?

The principal purpose of the task is to explore a real-world application problem with algebra, working with units and maintaining reasonable levels of accuracy throughout. Of particular interest is that the optimal solution for long-term purchasing of the active ingredient is achieved by purchasing bottle C, whereas minimizing total cost for a *particular* application comes from purchasing bottle B. Students might need the instructor's aid to see that this is just the observation that buying in bulk may not be a better deal if the extra bulk will go unused.

Solution:

a) All of the bottles have the same active ingredient, and all can be diluted down to a 1% solution, so all that matters in determining value is the cost per fl. oz. of active ingredient. We estimate this in the following table:

	Amount active in Bottle	**Price of bottle**	**Cost per ounce**
A	1.04% × 64 ≈ 0.64 fl oz	\$12.99 ≈ \$13	$\frac{13}{0.64} \approx \$20\ per\ fl\ oz$
B	18.00% × 32 ≈ 6 fl oz	\$22.99 ≈ \$23	$\frac{23}{6} \approx \$4\ per\ fl\ oz$
C	41.00% × 32 ≈ 13 fl oz	\$39.99 ≈ \$40	$\frac{40}{13} \approx \$3\ per\ fl\ oz$
D	1.04% × 24 ≈ 0.24 fl oz	\$5.99 ≈ \$6	$\frac{6}{0.24} \approx \$24\ per\ fl\ oz$

If we assume that receiving more active ingredient per dollar is a better buy than less active ingredient per dollar, the ranking in order of best-to-worst buy is C,B,A,D.

b) The A bottles have about 0.64 fl. oz. of active ingredient per bottle so to get 14 fl. oz. we need

c) $\frac{14\ fl.\ oz.}{0.64\ fl.\ oz./bottle} \approx 22\ bottles.$

Purchasing 22 A bottles at about \$13 each will cost about \$286.

The B bottles have a little less than 6 fl. oz. of active ingredient per bottle so to get 14 fl. oz. we need 3 bottles. Purchasing 3 B bottles at about \$23 each will cost about \$69.

The C bottles have a little more than 13 fl. oz. of active ingredient per bottle, so we need 2 bottles. Purchasing 2 C bottles at about \$40 each will cost about \$80.

The D bottles have only 0.24 fl. oz. of active ingredient per bottle, so to get 14 fl. oz. we need

$$\frac{14\ fl.\ oz.}{0.24\ fl.\ oz./bottle} \approx 58\ bottles.$$

Purchasing 58 D bottles at about \$6 each will cost about \$348.

Thus, although the C bottle is the cheapest when measured in dollars/fl. oz., the B bottles are the best deal for this job because there is too much unused when you buy C bottles.

Instructional Strategies: See N.Q.1

Resources/Tools:

Illustrative Mathematics High School Number & Quantity tasks: **Scroll to the appropriate section to find named tasks.**

- N-Q.A.3
    - Felicia's Drive
    - Calories in a sports drink
    - Dinosaur Bones
    - Bus and Car

High School – Algebra

Domain: Seeing Structure in Expressions ★ (A.SSE)

Cluster: *Interpret the structure of expressions.*

Standard: **A.SSE.1 ★ (all)**

Interpret expressions that represent a quantity in terms of its context.

A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients. **(A.SSE.1a)**

A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and* $(1+r)^n$. **(A.SSE.1b)**

Suggested Standards for Mathematical Practice (MP):

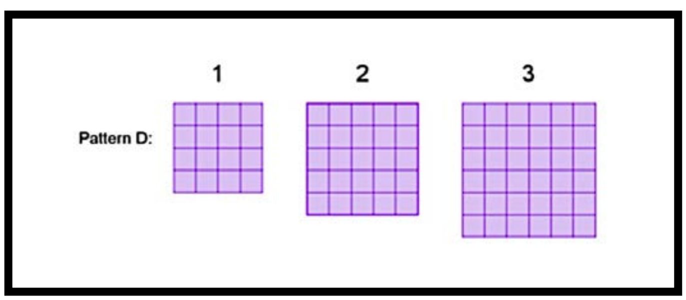
- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.

Connections: A.SSE.2

Explanations and Examples:

Viewing this standard as part of the Modeling Domain helps clarify that the purpose of this standard is to interpret in the context of the situation. Students are asked to reflect on the interaction between the situation and the equation. Students should be able to explain the individual parts and, as part of A.SSE.2, explain how the parts of an equivalent expression written in a different form continues to describe the same situation. A question asking students to describe a pattern algebraically provides a great opportunity for students to explain how they see the pattern algebraically.

For example, the following Illustrated Math task could be extended or adapted to demonstrate this standard. The number of tiles in step *n* of Pattern D is defined by $d(n) = (n+3)^2$.



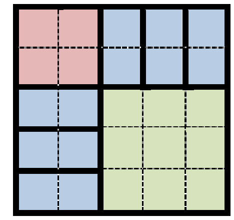
a) Explain "n+3" in the context of the situation?

The length of each side is three more than the number of the step.

b) Explain why the formula has an exponent of 2?

The pattern is describing the area of a square. The formula for the area of the square is $A = s^2$ with $s = n+3$. This is an example of viewing a one or more of their parts as a single entity. Being able to see the similarity between the area formula and this pattern helps students write their own equations to model a situation.

c) Expanding the pattern into standard form, explain how each component can still be seen the pattern. $d(n) = n^2 + 6n + 9$



It is unlikely that someone would notice the connection between this equation and the pattern and generate this model independently. But asking students to connect an equation they did not generate to a pattern is a good way to assess students' ability to connect an equation to its geometric representation. Students might see different parts "growing" in different ways. One possible interpretation is that the red squares in the corner represent n^2 and that square grows by one row and column each time. The green square is a constant 9 units each time. The blue represents the *6n*, meaning I have 6 groups of size 2. The next iteration will have 6 groups of size 3.

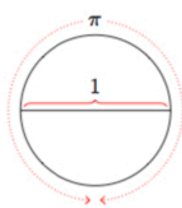
The analysis of this question illustrates how A.SSE 1 and 2 can work together to reveal new information about the problem. Explaining each part in context can create reach conversations with students, as well as reinforce the meaning behind the mathematics.

Geometry Example:

This [Illustrated Math Task] is an excellent example of interpreting parts of a formula in context. This task requires students to rethink how they interpret well known part of an equation. Additional examples can be found in explorations to generalize or derive geometric formulas.

Task

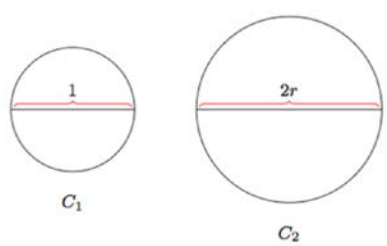
Suppose we define π to be the circumference of a circle whose diameter is 1:



Explain why the circumference of a circle with radius $r > 0$ is $2\pi r$.

The solution to the task above is copied directly from Illustrated Math:

Below is a picture of a circle of diameter 1, labeled C_1, and diameter $d = 2r$, labeled C_2:



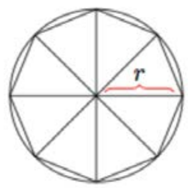
In the case pictured, d is larger than 1. All circles are similar and in this case the scale factor going from the circle of diameter 1 to the circle of diameter 2*r* is 2*r*. The circumference of a circle is a one dimensional measurement and so it scales in the same way as the diameters:

$$\frac{\text{Circumference}(C_2)}{\text{Circumference}(C_1)} = \frac{\text{diameter}(C_2)}{\text{diameter}(C_1)}$$
$$= \frac{2r}{1}$$

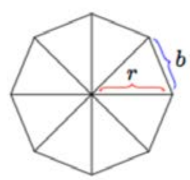
Since the circumference of C1 is π by definition, it follows from the above equation that the circumference of C2 is 2πr.

Solution: 2 Similarity of triangles

In this solution we approximate the circumference of a circle using polygons and then use similarity of triangles to explain the formula for the circumference of a circle. Below is a picture of a regular octagon inscribed inside a circle of radius r:



The circumference of the circle is a little bit more than the perimeter of the regular octagon which we can calculated using the picture below:



The perimeter of the octagon is 8b since it has been divided into eight congruent triangles each with a base of b. We can calculate the angles of these eight triangles using the fact that the eight inner angles combine to make a 360 degree circle so each measures 45 degrees. The triangles are all isosceles so this means that the base angles each measure $\frac{180-45}{2} = 67.5$ degrees. By AAA, two triangles with angles 67.5, 67.5, and 45 are similar. Therefore the ratio (b:r) does not depend on the size of the regular octagon. This means that the ratio (perimeter (octagon):r) also does not depend on the size of the regular octagon. As we add more and more sides, this ratio approaches the ratio of the circumference of the circle to its radius. We conclude that for a circle C of any radius r $(circumference\ (C){:}\,r) = \left(\pi{:}\,\frac{1}{2}\right)$.

Note the 12 comes from looking at the circle of diameter 1 and circumference π: the radius of this circle is 12. This is equivalent to the usual formula saying that the circumference of a circle with radius r is 2πr.

Instructional Strategies:

Using visuals to highlight the connection between the situation, the problem, and the equation is a great strategy to help students not get lost in the problem. Highlighting one piece of information, say time, the same color throughout the problem can help show where this piece of information goes throughout the problem. Another strategy is to use post-it notes to physically cover larger pieces of information with "a single entity" to help students see it as one large chunk. Flipping back and forth between the "big" piece written on one side and the "small" piece written on the other can help students view it as a chunk.

Without going into more detail than you might need here, let me briefly name what we asking students to do in part b of the standard so that you have the concept on hand for further research.

> "**The theory of reification** (Sfard, 1987, 1988, 1991, 1992, 1994; Sfard & Linchevski, 1994) describes how concepts come into existence from a cognitive perspective. The theory is based on the fact that many **mathematical concepts** are **conceived in two complementary ways, operationally and structurally**. Operational conceptions are "**about processes, algorithms, and actions rather than about objects**" (emphasis in original, Sfard, 1991, p. 4), in contrast to **structural conceptions** where mathematical entities are conceived as **objects, wholes, or as the result of a process instead of the process itself...** Reification is 'an ontological shift- a sudden ability to see something familiar in a totally new light" (Sfard, 1991, p. 19); what was previously only a process can now be seen as an object also."

Reification is difficult to achieve, thus, its placement as an ALL standard. It will require consistent practice.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

"Polynomial and Quadratic Expressions, Equations, and Functions" - EngageNY Algebra I Module 4:
In this module, students continue to interpret expressions, create equations, rewrite equations and functions in different but equivalent forms, and graph and interpret functions, but this time using polynomial functions, and more specifically quadratic functions, as well as square root and cube root functions.

"Interpreting Algebraic Expressions" – Mathematics Assessment Project:
This lesson unit is intended to help you assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It will help you to identify and support students who have difficulty in: Recognizing the order of algebraic operations. / Recognizing equivalent expressions. or Understanding the distributive laws of multiplication and division over addition (expansion of parentheses).

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.A.1
  - Mixing Fertilizer
  - Increasing or Decreasing? Variation 1
  - Throwing Horseshoes
  - Quadrupling Leads to Halving
  - Kitchen Floor Tiles
  - Mixing Candles
  - The Bank Account
  - Delivery Trucks
  - Radius of a Cylinder
  - The Physics Professor

Domain: Seeing Structure in Expressions ★ (A.SSE)

Cluster: *Interpret the structure of expressions.*

Standard: **A.SSE.2 ★ (all)**

Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$.* **(A.SSE.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.

Connections: A.SSE.1

Explanations and Examples:

This standard partners well with A.SSE.1 to support explorations when modeling (see A.SSE.1 for additional information) but the standard can also stand alone as an algebraic standard. There are many standards that focus on the specifics of re-writing an expression (i.e. factoring, completing the square, laws of exponents, trig identities, etc.) but this standard is not focused on typical algebraic manipulation. The goal is for students to take a step back and *see the structure* and connect the structure to the procedures. As with most ALL standards, there is not a specific procedure to teach students; rather this is a skill that develops over time and through intentional questions. For example, in Algebra 1 students learn the formula for slope ($m = \frac{y_2 - y_1}{x_2 - x_1}$). After developing an understanding of slope, they move into linear functions and look for the general form for a linear function. This standard supports the student development for the forms of a linear function because, through the structure, they can manipulate the equation into a more familiar form.

Point Slope Form: What equations could be written if we know the slope, *m,* and one point (x_1, y_1)?

$m = \frac{y - y_1}{x - x_1}$ ---We can multiply both sides by $x - x_1$

$m(x - x_1) = y - y_1$ **Point slope form**

$y = m(x - x_1) + y_1$ Not a typical form but it is structurally the same as transformations of functions. How can a linear function be

Slope Intercept Form: What equations could be written if we know the slope, *m,* and the y-intercept (0, b)?

$m = \frac{y - b}{x - 0}$ ---We can multiply both sides by $x - 0$, simply *x.*

$mx = y - b$

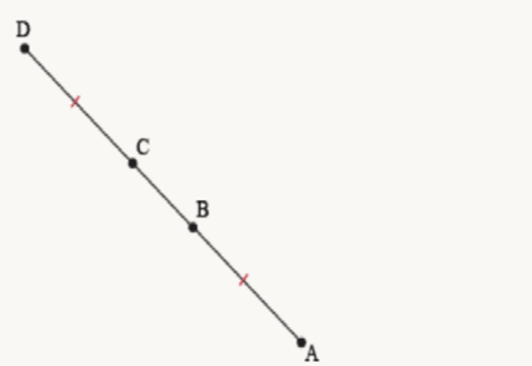
$y = mx + b$

There are many ways that these three examples rely on the structure of the equation.

- First, many students do not realize that understand that (x_1, y_1) is the convention for writing a specific point or value when generalizing an equation but that an equation must still have variables, such as *x* and *y.*
- They also forget that the equation must have an equal sign. This seems like basic understanding but, when working a new type of problem such as finding the general form of an equation, they tend to forget these basic structural requirements.
- Teachers tend to direct students toward the end result that we see (Point-Slope Form or Slope-Intercept Form) rather than letting students play with the equation to see what results they arrive at. As math teachers, if provided with an unknown equation to rewrite into a known form, we would naturally eliminate fractions. Students need to see this same structure and know that eliminating fractions is often valuable. Notice that neither result in the Point-Slope Form example distributed the *m* to $(x - x_1)$ but students would likely think this is a good next step. They need to learn that, structurally, there is often more information gained from the factored form than the distributed form.
- Another common missed opportunity is the ability to perform arithmetic to make an equation less complicated. Students need to recognize the structure of adding or subtracting 0 and multiplying or dividing by 1 and make use of these properties whenever possible.

Student's weakness with the structure of expressions is especially apparent in Geometry class. When students are asked to prove something by performing algebraic operations on geometric statements, they often fail to see that the structure of this equation is the same as the structure learned in Algebra. For example, see the two geometry problems below, which use the structure of the expression to identify ways to rewrite it.

Given the diagram below, show that $\overline{AC} \cong \overline{BD}$.



Statements	Reasons
1. $\overline{AB} \cong \overline{CD}$	Given (in figure)
2. $AB = CD$	Definition of congruence (1)
3. $AB + BC = BC + CD$	Addition of BC to (2)
4. $AC = BD$	Segment addition postulate (3)
5. $\overline{AC} \cong \overline{BD}$	Definition of congruence (4)

Many of the steps that rely on the structure of the equation are not written explicitly in the proof. How could additional steps be added to highlight the structure of the equation in relation to the geometric shape?

Show that if two angles are complementary to the same angle, then they're congruent.

Statements	Reasons
1. ∠1 and ∠3 are complementary	Given
2. ∠2 and ∠3 are complementary	Given
3. m∠1 + m∠3 = 90	Definition of complementary angles
4. m∠2 + m∠3 = 90	Definition of complementary angles
5. m∠1 = 90 − m∠3	Subtract m∠3 from (3)
6. m∠2 = 90 − m∠3	Subtract m∠3 from (4)
7. m∠1 = m∠2	Transitive property of equality (5 and 6)
8. ∠1 ≅ ∠2	Definition of congruence (7)

Similarly, students must recognize that that one solution path is to create the same expression in both equations in order to apply the transitive property. Another strategy could have been to solve for $m\angle 3$, set the resulting expressions equal and use the algebraic structure to solve. Both of these approaches rely on students' ability to recognize the structure of the expressions in order to create a strategy that will arrive at the needed proof statement. These types of problems are a struggle for students because there isn't a predictable algorithm and they find it difficult to see how the structure of the expression helps them identify a strategy. The ability to continually reinforce this standard, along with the need to develop this skill slowly over years of instruction is why this standard is an "ALL" standard.

Instructional Strategies:

Strategies from A.SSE.1 will also be useful here. In addition, teachers should use "think alouds" to focus on decision the decision they, as an expert, made as a result of the structure. Comparing and contrasting multiple correct solution strategies is another way to highlight how the structure of an expression can help the student rewrite it.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-SSE.A.2
  - Equivalent Expressions
  - Sum of Even and Odd
  - A Cubic Identity
  - Seeing Dots
  - Animal Populations

Domain: Creating Equations ★ (A.CED)

Cluster: *Create equations that describe numbers or relationships.*

Standard: **A.CED.1 ★**

Apply and extend previous understanding to create equations and inequalities in one variable and use them to solve problems. **(A.CED.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

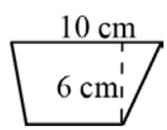
Connections: Modeling and Functions

Explanations and Examples:

Every year, in every course, students should be creating equations and inequalities. There isn't a time when we say, "The students have mastered it! They no longer need to develop this skill." For that reason, this standard is selected as an ALL standard. Algebra 1 focuses on creating equation and inequalities that are linear, quadratic, or exponential. Algebra 2 continues to increase in sophistication with linear, quadratic, and exponential but adds new function families such as rational, square root, logarithmic, and polynomial. Geometry reinforces algebraic skills while learning geometric properties by asking students to solve geometry problems using algebra skills.

Examples:

- Given that the following trapezoid has area $54\ cm^2$, set up an equation to find the length of the unknown base, and solve the equation.



- Lava coming from the eruption of a volcano follows a parabolic path. The height *h* in feet of a piece of lava *t* seconds after it is ejected from the volcano is given by $h(t) = -16t^2 + 64t + 936$. $h(t) = -t^2 + 16t + 936$. After how many seconds does the lava reach its maximum height of 1000 feet?

- The value of an investment over time is given by the equation $A(t) = 10{,}000(1.03)^t$. What does each part of the equation represent?

  Solution: The $10,000 represents the initial value of the investment. The 1.03 means that the investment will grow exponentially at a rate of 3% per year for *t* years.

- You bought a car at a cost of $20,000. Each year that you own the car the value of the car will decrease at a rate of 25%. Write an equation that can be used to find the value of the car after *t* years.

  Solution: $C(t) = \$20{,}000(0.75)^t$. The base is $1 - 0.25 = 0.75$ and is between 0 and 1, representing exponential decay. The value of $20,000 represents the initial cost of the car.

- Suppose a friend tells you she paid a total of $16,368 for a car, and you'd like to know the car's list price (the price before taxes) so that you can compare prices at various dealers. Find the list price of the car if your friend bought the car in:
  a) Arizona, where the sales tax is 5.6%.
  b) New York, where the sales tax is 8.25%.
  c) A state where the sales tax is *r*.

  Solution:
  a) If *p* is the list price in dollars then the tax on the purchase is 0.056*p*. The total amount paid is $p + 0.056p$, so

$$p + 0.056p(1 + 0.056) = 16{,}368$$
$$(1 + 0.056)p = 16{,}368$$
$$p = \frac{16{,}368}{1 + 0.056} = \$15{,}500.$$
$$p = \$15{,}500$$

  b) The total amount paid is $p + 0.0825p$ so

$$p + 0.0825p = 16{,}368$$
$$1 + 0.0825p = 16{,}368$$
$$p = \frac{16{,}368}{1 + 0.0825} =$$
$$p = \$15{,}120.55$$

  c) The total amount paid is $p + rp$ so

$$p + rp = 16{,}368$$
$$(1 + r)p = 16{,}368$$
$$\frac{16{,}368}{1 + r}$$
$$p = \frac{16{,}368}{1+r} \text{ dollars.}$$

Instructional Strategies:

Reading and comprehension strategies such as highlighting and annotating will help students make meaning from the problem. It is also important that the students understand the context and can visualize what is happening in the problem. These general strategies are well-known and can be effective literacy strategies that support writing equations.

A math specific strategy is to try and identify a general structure to math problems what can be applied to a variety of different situations. For example, there are many types of situations that fit into a part/part/whole or start/change/unknown situation in Algebra 1. Mixture problems is another general problem type with a fairly consistent problem structure. Identifying additional general problem types will help students see the larger structure present in

Algebra problems. The appendix at the back of the Standards and at the back of each flip book provides a general structure that is useful scaffolding for students. Many situations will fit within these computation situations and can help students see the pattern across a wide array of problems.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.1
    - Planes and wheat
    - Paying the rent
    - Buying a car
    - Sum of angles in a polygon
- A-CED.A.2
    - Throwing a Ball

Common Misconceptions:

Students may believe that equations of linear, quadratic and other functions are abstract and exist only "in a math book," without seeing the usefulness of these functions as modeling real-world phenomena.

Additionally, they believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.

Students may interchange slope and *y*-intercept when creating equations. For example, a taxi cab costs \$4 for a dropped flag and charges \$2 per mile. Students may fail to see that \$2 is a rate of change and is slope while the \$4 is the starting cost and incorrectly write the equation as $y = 4x + 2$ instead of $y = 2x + 4$.

Given a graph of a line, students use the x-intercept for b instead of the y-intercept.

Given a graph, students incorrectly compute slope as run over rise rather than rise over run. For example, they will compute slope with the change in x over the change in y.

Students do not know when to include the "or equal to" bar when translating the graph of an inequality.

Students do not correctly identify whether a situation should be represented by a linear, quadratic, or exponential function.

Students often do not understand what the variables represent. For example, if the height h in feet of a piece of lava t seconds after it is ejected from a volcano is given by $h(t) = -16t^2 + 64t + 936$ and the student is asked to find the time it takes for the piece of lava to hit the ground, the student will have difficulties understanding that $h = 0$ at the ground and that they need to solve for t.

Domain: Creating Equations ★ (A.CED)

Cluster: *Create equations that describe numbers or relationships.*

Standard: **A.CED.2 ★**

Apply and extend previous understanding to create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. **(A.CED.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

Connections: See A.CED.1

Explanations and Examples: See A.CED.1

Examples:

- The formula for the surface area of a cylinder is given by $A = 2\pi rh + 2\pi r^2$, where *r* represents the radius of the circular cross-section of the cylinder and *h* represents the height. Choose a fixed value for *h* and graph *V* vs. *r*. Then pick a fixed value for *r* and graph *V* vs. *h*. Compare the graphs.

  What is the appropriate domain for *r* and *h*? Be sure to label your graphs and use an appropriate scale.
- Gold is alloyed with different metals to make it hard enough to be used in jewelry. The amount of gold present in a gold ally is measured in 24ths called karats. 24-karat gold is 100% gold. Similarly, 18-karat gold is 75% gold.

  How many ounces of 18-karat gold should be added to an amount of 12-karat gold to make 4 ounces of 14-karat gold? Graph equations on coordinate axes with labels and scales.

- A metal alloy is 25% copper. How much of each alloy should be used to make 1000 grams of a metal alloy that is 45% copper?

- Find a formula for the volume of a single-scoop ice cream cone in terms of the radius and height of the cone. Rewrite your formula to express the height in terms of the radius and volume. Graph the height as a function of radius when the volume is held constant.

- David compares the sizes and costs of photo books offered at an online store. The table below shows the cost for each size photo book.

Book Size	Base Price	Cost for Each Additional Page
7-in. by 9-in.	$20	$1.00
8-in. by 11-in.	$25	$1.00
12-in. by 12-in.	$45	$1.50

1. Write an equation to represent the relationship between the cost, *y*, in dollars, and the number of pages, *x*, for each book size. Be sure to place each equation next to the appropriate book size. Assume that *x* is at least 20 pages.

Book Size	Equation
7-in. by 9-in.	
8-in. by 11-in.	
12-in. by 12-in.	

2. What is the cost of a 12-in. by 12-in. book with 28 pages?

3. How many pages are in an 8-in. by 11-in. book that costs $49?

Solution:

1. 7-in. by 9-in. $y = x$
   8-in. by 11-in. $y = x + 5$
   12-in. by 12-in. $y = 1.50x + 15$

2. $57
3. 44 pages

Instructional Strategies: See A.CED.1

Resources/Tools:
Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.2
  - Clea on an escalator
  - Silver rectangle

Domain: Creating Equations ★ (A.CED)

Cluster: *Create equations that describe numbers or relationships.*

Standard: **A.CED.3 ★**

Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.* ★ **(A.CED.3)**

Suggested Standards for Mathematical Practice (MP):

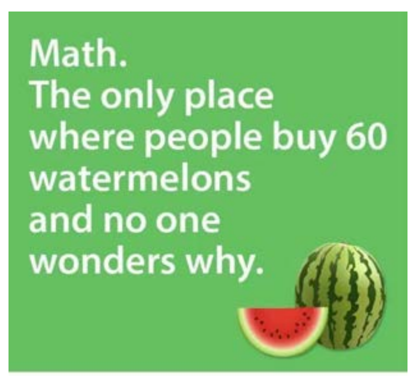
- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.

Connections: See A.CED.1

Explanations and Examples:

Part 1: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities.
A constraint is often thought about (by math teachers) as a linear programing problem. While that type of problem certainly fits within this standard, it would not be categorized as an ALL standard if that were the entire scope. The word "constrained" can apply to a wide variety of problem types. For example, if child tickets cost $3 and an adult's ticket cost $5, we can purchase tickets in any quantity- unconstrained by additional information. However if additional information were provided about how much money could be spent, then the solution set would be *constrained* to the values that add to no more than that amount. This situation also has a constrained domain; which is constrained to discrete values and that must be between zero and the maximum number of a single variable able to purchase tickets.

Part 2: and interpret solutions as viable or non-viable options in a modeling context.
This part of the standard has frequently been interpreted to address extraneous solutions but the phrase "in a modeling context" should redirect instruction back to thinking about the constraints on a problem, which could include constraints on the solution. For example, in the situation described above, it isn't possible to purchase half a ticket. Therefore, viable solutions would be whole numbers. Negative numbers are also not possible. The viable has broader meaning than simply possible or not possible. Is the solution "viable" or feasible? If we were working a rate problem, it might be possible for the speed of a motorcycle to be 350 mph according to all the constraints given in the situation but is it a viable solution? Thinking about the asymptote on an exponential function, as the value gets infinitesimally close to zero, does it remain viable?



desmos.com

As seen in many of the ALL standards, there is not a simple procedure or algorithm that will always produce the correct answer. A classroom discussion about the variety of factors that might constrain the equation or solution provides a rich opportunity for students to learn how to think about the world mathematically, in a truly real life situation. The statement above, posted frequently on social media, reflects the memes that result from a lack of attention to viability in a modeling context.

Examples:

A club is selling hats and jackets as a fundraiser. Their budget is \$1500 and they want to order at least 250 items. They must buy at least as many hats as they buy jackets. Each hat costs \$5 and each jacket costs \$8.

- Write a system of inequalities to represent the situation.
- Graph the inequalities.
- If the club buys 150 hats and 100 jackets, will the conditions be satisfied?
- What is the maximum number of jackets they can buy and still meet the conditions?

Represent inequalities describing nutritional and cost constraints o combinations of different foods.

The coffee variety *Arabica* yields about 750 kg of coffee beans per hectare, while *Robusta* yields about 1200 kg per hectare. Suppose that a plantation has *a* hectares of *Arabica* and *r* hectares of *Robusta*.

a) Write an equation relating *a* and *r* if the plantation yields 1,000,000 kg of coffee.
b) On August 14, 2003, the world market price of coffee was\$1.42 per kg of *Arabica* and \$0.73 per kg of *Robusta*. Write an equation relating *a* and *r* if the plantation produces coffee worth \$1,000,000.

This task is designed to make students think about the meaning of the quantities presented in the context and choose which ones are appropriate for the two different constraints presented. The purpose of the task is to have students *generate* the constraint equations for each part (though the problem and not to have students *solve* said equations. If desired, instructors could also use this task to touch on such solutions by finding and interpreting solutions to the system of equations created in parts (a) and (b).

Solution:

a) We see that *a* hectares of *Arabica* will yield 750*a* kg of beans, and that *r* hectares of *Robusta* will yield 1200r kg of beans. So the constraint equation is

$$750a + 1{,}200r = 1{,}000{,}000.$$

b) We know that a hectares of *Arabica* yield 750*a* kg of beans worth \$1.42/kg for a total dollar value of $1.42(750a) = 1065a$. Likewise, *r* hectares of *Robusta* yield 1200*r* kg of beans worth \$0.73/kg for a total dollar value of $0.73(1200r) = 876r$. So the equation governing the possible values of *a* and *r* coming from the total market value of the coffee is

$$1065a + 876r = 1{,}000{,}000$$

Instructional Strategies:

While this standard represents an exciting opportunity to open the math world up to big, wide, real world; that ambiguity can be intimidating and/or distracting. Students will enjoy thinking about ways the problem will be constrained by the context of the problem. Teaching students Talk Moves such as revoicing or adding on can provide some structure to guide the conversation. Another strategy is to restrict the types of constraints the students can discuss either by number or type. Finally, as with each of other ALL standards, the goal is to *enhance* student understanding of functions and how quantities vary together. Not every solution is possible. Infinite possibilities doesn't mean infinite viable solutions. Do not let the discussion drag you down rabbit hole or cause confusion.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.3
    - Dimes and Quarters
    - Writing Constraints
    - Growing Coffee
    - How Much Folate
    - Bernardo and Sylvia Play a Game

Domain: Creating Equations ★ (A.CED)

Cluster: *Create equations that describe numbers or relationships.*

Standard: A.CED.4 ★

Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law* $V = IR$ *to highlight resistance R.*★ **(A.CED.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: A.SSE.2, A.REI.2

Explanations and Examples:

Rearranging formulas is a critical skill in many applications such as computer programming or spreadsheet formulas. But, often, in the process of solving for the quantity of interest, student misunderstandings about the steps for solving equations will come to the forefront. Mistakes that they wouldn't make when solving other equations will begin to appear. The algebraic reasoning required to correct the flaw causes significant struggle, even when the same problem with numbers would have an immediate and correct answer. Usiskin provided some insight into this area of student difficulty- quoted below.

Quoting from Teaching Mathematics in Grades 6-12:Developing Research Based Instructional Practices:

Figure 8.2 Examples to illustrate the complexity of literal symbols (Usiskin, 1988, p. 40).

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot (\frac{1}{n})$
5. $y = kx$

"Usiskin (1988) provided a poignant example of the complexity involved in understanding the meanings of literal symbols by asking readers to consider the equations shown in Figure 8.2. Although all five symbol strings in Figure 8.2 are equations, each uses literal symbols in different ways. In equation 2, for example, x is often referred to as an unknown whose value can be determined by dividing each side by five. In equation 5, however, we usually think of x as being free to vary and take on numerous different values, making x feel more like a variable than a specific unknown. In equation 5, k is often thought of as a constant that specifies the slope of a line. Therefore, this set of five equations illustrates at least three different ways literal symbols can be used in algebra— to represent unknowns, variables, and constants. In addition, the equations themselves can be used for different purposes. To illustrate this, observe that equation 1 is commonly referred to as a formula, equation 3 as an identity, and equation 4 as a property."

Recognizing that students struggle with the meaning of the variable in a literal equation does not change the reasoning required. It does indicate that shifting back and forth between different meanings for a variable can be confusing for students and addressing this confusion directly might help some students.

So one goal for this standard is for students to become comfortable with different uses of the variable in the equation and to surface any flaws with their algebraic reasoning. Another goal is for students to develop the ability to think ahead to their goal and plan a path to get there. For example, if the variable of interest appears in several different terms, with different exponents, then factoring or completing the square might be required. If the variable of interest is in the exponent, then logarithms might be the strategy. As has been stated before, ALL standards require practice and the development of sophistication over time. With each new function family, an effort to should be made to solve a formula for a given variable.

Examples:

1. Solve for *h:* $A = \frac{1}{2}bh$

   An example like the problem above can highlight how students will move the $\frac{1}{2}$. Dividing by $\frac{1}{2}$ would work if the student remembers how to divide fractions. A more robust solution path would be to multiply by the multiplicative inverse of $\frac{1}{2}$ or 2. A student who writes the answer $h = \frac{A}{\frac{1}{2}b}$ is correctly reasoning about equations but does not see that the fraction divided by a fraction is unnecessary. Even more concerning is the student whose answer is $h = \frac{A}{2b}$ because they incorrectly divided by $\frac{1}{2}$.

2. Solve $A = P + Prt$ for *r.*

   When working this problem, some students arrive at the correct solution $r = \frac{A-P}{Pt}$ but will go one step too far and "cancel" the *P*. These students struggle with the reasoning necessary in A.SSE.2, seeing part of an expression as a single entity. The numerator cannot be separated in this way and should be viewed as a whole piece.

3. Solve $A = P + Prt$ for *P.*

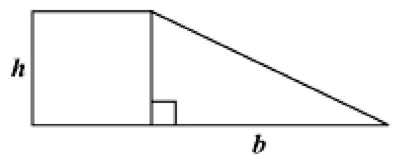
   This kind of problem is particularly challenging for students because they do not see how connection between factoring and distributing. With numbers, they understand that a problem like $2(x + y)$ will "give" the *x* and *y* a two but will not necessarily observe that *2x+2y* can be "undone" by factoring the two. A problem like the one above can highlight that students have missed this connection.

An Illustrated Math Task:

4. A bacteria population *P* is modeled by the equation $P = P_0 10^{kt}$ where time *t* is measured in hours, *k* is a positive constant, and P_0 is the bacteria population at the beginning of the experiment. Rewrite this equation to find *t* in terms of *P.*

   This equation will assess if students understand *when* to use logarithms.

   The figure below is made up of a square with height, *h* units, and a right triangle with height, *h* units, and base length, *b* units.



   The area of this figure is 80 square units.

   Write an equation that solves for the height, *h*, in terms of *b*.
   Show all work necessary to justify your answer.

   Sample Response:

$$h^2 + \frac{1}{2}bh = 80$$
$$h^2 + \frac{1}{2}bh + \frac{1}{16}b^2 = 80 + \frac{1}{16}b^2$$
$$\left(h + \frac{1}{4}b\right)^2 = 80 + \frac{1}{16}b^2$$
$$h + \frac{1}{4}b = \sqrt{80 + \frac{1}{16}b^2}$$
$$h = \sqrt{80 + \frac{1}{16}b^2} - \frac{1}{4}b$$

Instructional Strategies:

Substituting numbers in for the variables and solving it side by side with the literal equation can help scaffold the abstract thinking required.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered “best practice” to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-CED.A.4
  - Equations and Formulas

Domain: Reasoning with Equations and Inequalities (A.REI)

Cluster: *Understand solving equations as a process of reasoning and explain the reasoning.*

Standard: **A.REI.1**

Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. **(A.REI.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.

Connections: Algebra standards

Explanations and Examples:

In Algebra 1 students should focus on and master A.REI.1 for linear equations and be able to extend and apply their reasoning to other types of equations in future courses. In Algebra 2, extend to simple rational and radical equations.

Assuming an equation has a solution, construct a convincing argument that justifies each step in the solution process. Justifications may include the associative, commutative, and division properties, combining like terms, multiplication by 1, etc.

Properties of operations can be used to change expressions on either side of the equation to equivalent expressions. In addition, adding the same term to both sides of an equation or multiplying both sides by a non-zero constant produces an equation with the same solutions. Other operations, such as squaring both sides, may produce equations that have extraneous solutions.

Examples:

- Explain why the equation $\frac{x}{2}+\frac{7}{3}=5$ has the same solutions as the equation $3x+14=30$.
  Does this mean that $\frac{x}{2}+\frac{7}{3}$ is equal to $3x+14$?

- Show that $x=2$ and $x=-3$ are solutions to the equation $x^2+x=6$.
  Write the equation in a form that shows these are the only solutions, explaining each step in your reasoning.

- Transform $2x-5=7$ to $2x=12$ and tell what property of equality was used.
  Solution:

$$\begin{aligned} 2x-5&=7 \\ 2x-5+5&=7+5 \quad \text{Addition property of equality .} \\ 2x&=12 \end{aligned}$$

Instructional Strategies:

Challenge students to justify each step of solving an equation. Transforming 2*x* - 5 = 7 to 2*x* = 12 is possible because 5 = 5, so adding the same quantity to both sides of an equation makes the resulting equation true as well. Each step of solving an equation can be defended, much like providing evidence for steps of a geometric proof.

Provide examples for how the same equation might be solved in a variety of ways as long as equivalent quantities are added or subtracted to both sides of the equation, the order of steps taken will not matter.

$$\begin{array}{l} 3n+2=n-10 \\ \underline{-\quad 2 \;=\; -2} \\ 3n \quad = n-12 \\ \underline{-n \quad = -n} \\ 2n \quad = -12 \\ \quad n = -6 \end{array}$$

OR

$$\begin{array}{l} 3n+2=n-10 \\ \underline{+\quad 10 = +10} \\ 3n+12=n \\ \underline{-3n \quad = -3n} \\ 12=-2n \\ \quad n=-6 \end{array}$$

OR

$$\begin{array}{l} 3n+2=n-10 \\ \underline{-n \quad = -n} \\ 2n+2=-10 \\ \underline{\quad -2=-2} \\ 2n=-12 \\ n=-6 \end{array}$$

Connect the idea of adding two equations together as a means of justifying steps of solving a simple equation to the process of solving a system of equations. A system consisting of two linear functions such as $2x+3y=8$ and $x-3y=1$ can be solved by adding the equations together, and can be justified by exactly the same reason that solving the equation $2x-4=5$ can begin by adding the equation $4=4$.

Investigate the solutions to equations such as $3=x+\sqrt{2x-3}$. By graphing the two functions, $y=3$ and $y=x+\sqrt{2x-3}$ students can visualize that graphs of the functions only intersect at one point. However, subtracting $x=x$ from the original equation yields $3-x=\sqrt{2x-3}$ which when both sides are squared produces a quadratic equation that has two roots $x=2$ and $x=6$. Students should recognize that there is only one solution $(x=2)$ and that $x=6$ is generated when a quadratic equation results from squaring both sides; $x=6$ is extraneous to the original equation. Some rational equations, such as $\frac{x}{(x-2)}=\frac{2}{(x-2)}+\frac{5}{x}$ result in extraneous solutions as well.

Begin with simple, one-step equations and require students to write out a justification for each step used to solve the equation.

Ensure that students are proficient with solving simple rational and radical equations that have no extraneous solutions before moving on to equations that result in quadratics and possible solutions that need to be eliminated.

Provide visual examples of radical and rational equations with technology so that students can see the solution as the intersection of two functions and further understand how extraneous solutions do not fit the model.

It is very important that students are able to reason how and why extraneous solutions arise.
Computer software that generates graphs for visually examining solutions to equations, particularly rational and radical.
Examples of radical equations that do and do not result in the generation of extraneous solutions should be prepared for exploration.

Common Misconception:

Students may believe that solving an equation such as $3x + 1 = 7$ involves "only removing the 1," failing to realize that the equation $1 = 1$ is being subtracted to produce the next step.

Additionally, students may believe that all solutions to radical and rational equations are viable, without recognizing that there are times when extraneous solutions are generated and have to be eliminated.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

"Building and Solving Equations 2" – Mathematics Assessment Project

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.A
  - ~ Same Solutions?
  - ~ How does the solution change?

Domain: Reasoning with Equations and Inequalities (A.REI)

Cluster: *Solve equations and Inequalities in one variable.*

Standard: **A.REI.2**

Apply and extend previous understanding to solve equations, inequalities, and compound inequalities in one variable, including **literal equations** and inequalities. **(A.REI.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: A.CED.4

Explanations and Examples: See A.CED.4

Examples:

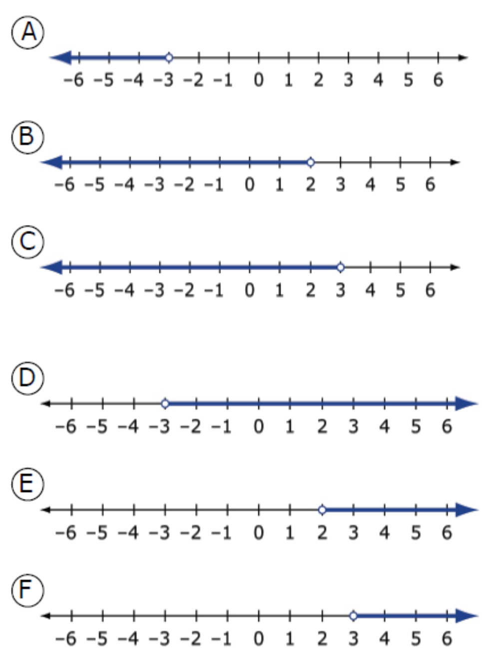
1. Solve for the variable:
   - $\frac{7}{3}y - 8 = 111$
   - $3x > 9$
   - $ax + 7 = 12$
   - $\frac{3+x}{7} = \frac{x-9}{4}$
   - Solve for x: $\frac{2}{3}x + 9 < 18$

2. Match each inequality in items 1 – 3 with the number line in items A – F that represent the solution to the inequality.

   1 $\boxed{-4x < -12}$

   2 $\boxed{2(x + 2) < 8}$

   3 $\boxed{5 - 2x < 2 - x}$



Solutions: **1.** F **2.** B **3.** F

Instructional Strategies:

There are two major reasons for discussing the topic of inequalities along with equations: First, there are analogies between solving equations and inequalities that help students understand them both. Second, the applications that lead to equations almost always lead in the same way to inequalities.

In grades 6-8, students solve and graph linear equations and inequalities. Graphing experience with inequalities is limited to graphing on a number line diagram. Despite this work, some students will still need more practice to be proficient. It may be beneficial to remind students of the most common solving techniques, such as converting fractions from one form to another, removing parentheses in the sentences, or multiplying both sides of an equation or inequality by the common denominator of the fractions. Students must be aware of what it means to check an inequality's solution. The substitution of the end points of the solution set in the original inequality should give equality regardless of the presence or the absence of an equal sign in the original sentence. The substitution of any value from the rest of the solution set should give a correct inequality.

Careful selection of examples and exercises is needed to provide students with meaningful review and to introduce other important concepts, such as the use of properties and applications of solving linear equations and inequalities. Stress the idea that the application of properties is also appropriate when working with equations or inequalities that include more than one variable, fractions and decimals. Regardless of the type of numbers or variables in the equation or inequality, students have to examine the validity of each step in the solution process.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(B + b)$ when solving for *b* is similar to solving an equation with one variable. Provide students with an opportunity to abstract from particular numbers and apply the same kind of manipulations to formulas as they did to equations. One of the purposes of doing abstraction is to learn how to evaluate the formulas in easier ways and use the techniques across mathematics and science.

Draw students' attention to equations containing variables with subscripts. The same variables with different subscripts (e.g., $x_1 and\ x_2$) should be viewed as different variables that cannot be combined as like terms. A variable with a variable subscript, such as a_n, must be treated as a single variable – the n^{th} term, where variables *a* and *n* have different meaning.

Common Misconceptions:

Some students may believe that for equations containing fractions only on one side, it requires "clearing fractions" (the use of multiplication) only on that side of the equation. To address this misconception, start by demonstrating the solution methods for equations similar to $\frac{1}{4}x + \frac{1}{5}x + \frac{1}{6}x + 46 = x$ and stress that the Multiplication Property of Equality is applied to both sides, which are multiplied by 60.

Students may confuse the rule of changing a sign of an inequality when multiplying or dividing by a negative number with changing the sign of an inequality when one or two sides of the inequality become negative (for ex., $3x > -15\ or\ x < -5$).

Some students may believe that subscripts can be combined as $b_1 + b_2 = b_3$ and the sum of different variables *d* and D is $2D(d + D = 2D)$.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.B
  - Integer Solutions to Inequality
- A-REI.A.1
  - Reasoning with linear inequalities

Domain: Reasoning with Equations and Inequalities (A.REI)

Cluster: *Understand solving equations as a process of reasoning and explaining the reasoning.*

Standard: **A.REI.8 (all)**

Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). **(A.REI.10)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: Functions Domain

Explanations and Examples:

Quoting from Teaching Mathematics in Grades 6-12:Developing Research Based Instructional Practices:

> "Research suggests that multiple representations of functions are not used to their fullest extent in traditional mathematics instruction. Knuth (2000) noted that traditional algebra instruction emphasizes **producing graphs from symbolic representations** of functions (e.g., the task, "Produce a graph of $y = x^2$"), but generally does not ask students to **reason from graphs back to symbolic representations.** To illustrate the detrimental effects of this practice, Knuth gave high school students tasks in which they were to determine the equation of a given linear graph. In one task, students were asked to determine the value of "?" in $?x + 3y = -6$. They were given a graph of the equation to use. Many of the students who gave a **correct solution** to the task used the **inefficient process** of calculating the slope of the graph provided, determining the y-intercept from the graph, writing the equation in slope-intercept form, and then converting it back to standard form. **Students did not seem to recognize that every point on the graph represented a solution** to $?x + 3y = -6$. If they understood this idea, called the **Cartesian connection,** they likely would have chosen any point (x, y) from the graph to substitute into the equation to determine the value of the "?" symbol. Although students in traditional algebra classes produce tables, graphs, and equations for functions, **the act of producing these representations becomes a rote process devoid of meaning** if problems that prompt them to recognize ideas such as the Cartesian connection are not included.
>
> Students who do not fully grasp the Cartesian connection may also **lack skill in choosing the most efficient representations for solving problems.** Slavit (1998) examined the algebraic problem solving strategies of students in a precalculus course where the instructor emphasized graphical representations. Despite this emphasis, some **students persisted in using equations and symbol manipulation even when it was inefficient to do so.** For example, when given a task requiring a solution to
> $-0.1x^2 + 3x + 80 = x$, one of the students interviewed first tried to factor. When factoring became difficult, she used the quadratic formula. Although she was prompted by the interviewer to discuss other solution strategies, **approaching the problem graphically never occurred to her.** A graphical

approach might involve locating the roots of the parabola $y = -0.1x^2 + 2x + 80$ or determining the intersection point of $y = -0.1x^2 + 3x + 80$ and *y* = *x*. Slavit partially attributed the lack of use of graphical representations to past instruction focusing heavily on symbol manipulation. **When instruction emphasizes only one representational system, some students come to see graphical and symbolic representations as two separate systems of procedures to follow rather than as representations that complement one another.**

When students view **function representations as complementary** to one another, they develop **better understanding of the attributes of functions**. Schwarz and Hershkowitz (1999) found that students who explored multiple representations with technology understood functions more deeply than those who did not. Students using the technology were able to generate a broader range of examples of functions and also understood multiple representations as different descriptions of the same function. The zooming and scaling features of the technology helped students develop better part-whole reasoning about function representations. **The results of the study strongly suggest that technology capable of generating multiple representations of functions should be a foundational part of algebra instruction."**

Groth, Randall E.. Teaching Mathematics in Grades 6 - 12: Developing Research-Based Instructional Practices (p. 225). SAGE Publications. Kindle Edition.

Examples:

1. Which of the following points is on the circle with equation $(x-1)^2 + (y+2)^2 = 5$?

   (a) (1, -2) (b) (2, 2) (c) (3, -1) (d) (3, 4)

2. Graph the equation and determine which of the following points are on the graph of $y = 3^x + 1$.

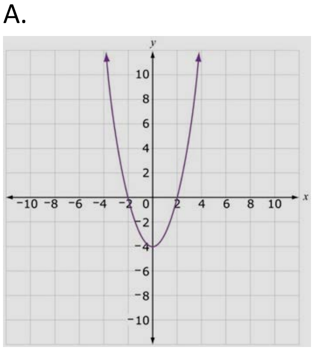
   (a) (2, 7) (b) $\left(-1, \frac{4}{3}\right)$ (c) (2, 10) (d) (0, 1)

$\sqrt{x-4}$

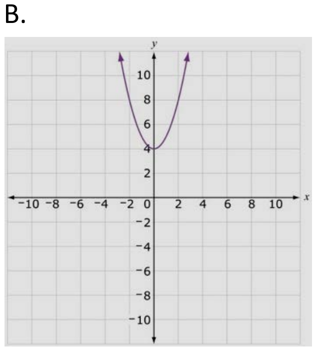
3. Which graph could represent the solution set of $y = \sqrt{x-4}$?

   Solution: B

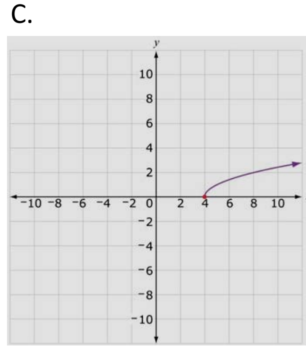
A.



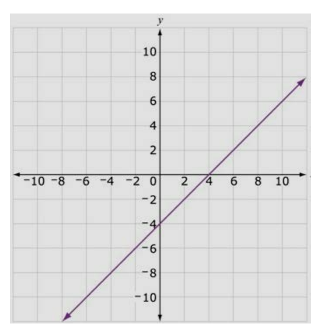
B.



C.



D.



Instructional Strategies:

Beginning with simple, real-world examples, help students to recognize a graph as a set of solutions to an equation. For example, if the equation $y = 6x + 5$ represents the amount of money paid to a babysitter (i.e., $5 for gas to drive to the job and $6/hour to do the work), then every point on the line represents an amount of money paid, given the amount of time worked.

Explore visual ways to solve an equation such as $2x + 3 = x - 7$ by graphing the functions $y = 2x + 3$ and $y = x - 7$. Students should recognize that the intersection point of the lines is at (-10, -17). They should be able to verbalize that the intersection point means that when *x* = -10 is substituted into both sides of the equation, each side simplifies to a value of -17. Therefore, -10 is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear or both.

Using technology, have students graph a function and use the trace function to move the cursor along the curve. Discuss the meaning of the ordered pairs that appear at the bottom of the calculator, emphasizing that every point on the curve represents a solution to the equation.

Begin with simple linear equations and how to solve them using the graphs and tables on a graphing calculator. Then, advance students to nonlinear situations so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions.

Use the table function on a graphing calculator to solve equations. For example, to solve the equation $x^2 = x + 12$, students can examine the equations $y = x^2$ *and* $y = x + 12$ and determine that they intersect when $x = 4$ and when $x = -3$ by examining the table to find where the *y*-values are the same.

Investigate real-world examples of two-dimensional inequalities. For example, students might explore what the graph would look like for money earned when a person earns *at least* $6 per hour. (The graph for a person earning *exactly* $6/hour would be a linear function, while the graph for a person earning at least $6/hour would be a half-plane including the line and all points above it.)

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

"Optimization Problems: Boomerangs" – Mathematics Assessment Project

Illustrative Mathematics High School Algebra tasks: **Scroll to the appropriate section to find named tasks.**

- A-REI.D.10
  - Collinear Points

High School – Functions

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation.*

Standard: **F.IF.1 (all)**

Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If *f* is a function and *x* is an element of its domain, then $f(x)$ denotes the output of *f* corresponding to the input *x*. The graph of *f* is the graph of the equation $y = f(x)$. **(F.IF.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: Functions and Algebra Domain

Explanations and Examples:

The goal with the functions domain is teach students a *process* for analyzing functions, rather than teaching each function family in isolation, causing students to miss the larger structure holding them all together. For that reason, several of the Function standards were chosen as ALL standards and should be present throughout all mathematics instruction. In fact, the five ALL functions standards so clearly define that process and are so intertwined that it is difficult to separate them into individual standards to discuss here. As a result, this section of the flipbook will discuss the common process here and address any individual nuances in each additional standard.

1. Functions should be **analyzed qualitatively** (from a global perspective) for key features (F.IF.4)
   a) Direction of change (increasing/decreasing)
   b) Type of change (constant linear or non-constant nonlinear)
   c) Minimum and maximum values.
   d) Symmetry
   e) End behavior
   f) Periodicity
2. Functions should be **analyzed quantitatively** (from a local perspective) by
   a) Identifying the domain and range and relating it back to the relationship
   b) Representing the function across all representations and identifying key features.
   c) Understanding function notation in the context of the situation.
   d) Understanding how the graph of the function is related to the equation.

The first concepts for students to wrestle with are:

- Defining a function as the rule that *assigns* an *every* element from the input set to *exactly one* element of the output set
- Naming the input set domain and the output set range
- Introducing notation naming the input *x* and the output *f(x)*
- Defining the graph of *f* as the graph of the equation *y=f(x)*

Define the function relationship

- The standard uses an "assignment" definition for a function using a rule to assign an input to an output.
- Inputs and outputs are named domain and range and are a fundamental part of the definition for a function.
- The definition does **not** require a function to be an equation, graph, or even numerical.
- Assignment requires two elements**:** (1) an element from the domain is assigned (2) to exactly one element of the range.

Identifying functions from a table

- It is **not correct** to describe a procedure for determining if a table (or set of ordered pairs) is a function by saying **"the x value cannot repeat."** This neglects an essential requirement for a function: a correspondence between two values.
- In a data table the input value might repeat but, if it does, it must ALWAYS be assigned to the same output value.

Identifying a function from a graph

- Focus on the definition: that every element of the domain is assigned to exactly one element of the range.
- The vertical line test provides an easy procedure to instruct students but there are numerous reasons to **avoid this strategy.**
- Once students learn the vertical line test, they tend to blindly apply it to all graphs rather than having a procedure that can grow with them through geometric transformations, inverse function, polar functions, etc.
- Focus on clearly identifying the input (independent variable or domain) and matching it to a single output (dependent variable or range). This also provides essential practice the students need with domain and range.

Function notation

- Asking students to **interpret function notation** in non-quantitative situations can help reinforce what it means for *x* to be the input value in a function rule and how *f(x)*, the output value, is the associated output.
- x is a number and a specific input, *f(x)* is a number and the corresponding output. (See N.Q.1)
- The name of the function is *f*, for example, not *f(x).*
- Dr. Bill McCallum wrote about how to interpret the statement *y=f(x).*

  As for the $y = f(x)$ notation, when we say something like "the function $y = x^2$" we are using abbreviated language for "the function defined by the equation $y = x^2$, where *x* is the independent variable and *y* is the dependent variable." You can't say that every time, so we have a shortened form, which depends on certain conventions: the dependent variable occurs on the left and an expression in the independent variable occurs on the right."

Examples:

1. Determine which of the following tables represent a function and explain why.

A	
x	***f(x)***
0	1
1	2
2	2
3	4

B	
x	***f(x)***
0	0
1	2
1	3
4	5

Solution: Table A represents a function because for each element in the domain there is exactly one element in the range. Table B does not represent a function because when x = 1, there are two values for *f*(*x*): 2 and 3.

2. For the functions a. through f. below:

- o List the algebraic operations in order of evaluation. What restrictions does each operation place on the domain of the function?
- o Give the function's domain.

a. $y = \frac{2}{x-3}$

b. $y = \sqrt{x-5} + 1$

c. $y = 4 - (x-3)^2$

d. $y = \frac{7}{4-(x-3)^2}$

e. $y = 4 - (x-3)^{\frac{1}{2}}$

f. $y = \frac{7}{4-(x-3)^{\frac{1}{2}}}$

3. Is a geometric transformation an example of a function? If not, why? If so, how does that support its use in formal proofs?

Viewing transformations as functions is essential to proving congruence through rigid transformations. We have to know that if I do a translation, for example, that there is exactly one guaranteed output. Once we have a guaranteed output, we can apply geometric reasoning to the sequence of transformations and know, beyond all doubt, that the image is congruent.

Instructional Strategies:

Provide applied contexts in which to explore functions. For example, examine the amount of money earned when given the number of hours worked on a job, and contrast this with a situation in which a single fee is paid by the "carload" of people, regardless of whether 1, 2, or more people are in the car.

Use diagrams to help students visualize the idea of a function machine. Students can examine several pairs of input and output values and try to determine a simple rule for the function.

Rewrite sequences of numbers in tabular form, where the input represents the term number (the position or index) in the sequence, and the output represents the number in the sequence.

Help students to understand that the word "domain" implies the set of all possible input values and that the integers are a set of numbers made up of {…-2, -1, 0, 1, 2, …}.

Distinguish between relationships that are not functions and those that are functions (e.g., present a table in which one of the input values results in multiple outputs to contrast with a functional relationship). Examine graphs of functions and non-functions, recognizing that if a vertical line passes through at least two points in the graph, then *y* (or the quantity on the vertical axis) is not a function of x (or the quantity on the horizontal axis).

Common Misconceptions:

Students may believe that all relationships having an input and an output are functions, and therefore, misuse the function terminology.

Students may also believe that the notation *f(x)* means to multiply some value *f* times another value *x*. The notation alone can be confusing and needs careful development. For example, *f*(2) means the output value of the function *f* when the input value is 2.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.A
  - Interpreting the graph
- F-IF.A.1
  - The Parking Lot
  - Your Father
  - Parabolas and Inverse Functions
  - Using Function Notation I
  - The Customers
  - Points on a graph
  - Domains

Domain: Interpreting Functions (F.IF)
Cluster: *Understand the concept of a function and use function notation.*

Standard: **F.IF.2 (all)**
Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context. **(F.IF.2)**

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF.1

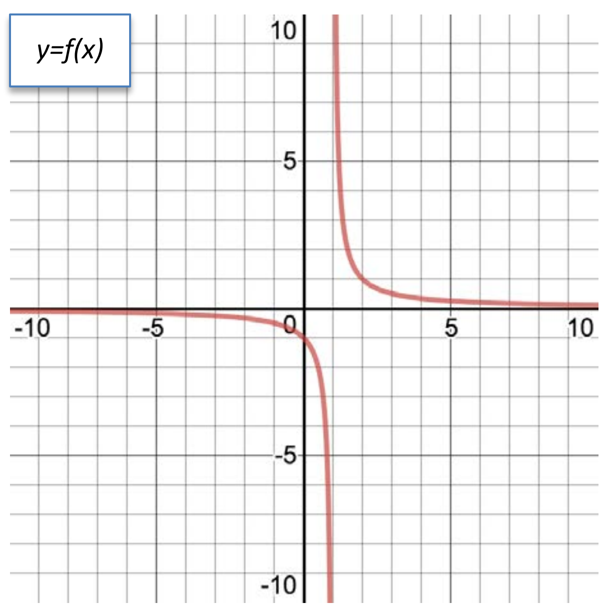
Explanations and Examples:
In addition to explanation about function notation discussed in N.Q.1 and F.IF.1, it is also important to analyze function notation across multiple contexts and representations.

Examples:

An Illustrated Math Task:
1. You put a yam in the oven. After 45 minutes, you take it out. Let f be the function that assigns to each minute after you placed the yam in the oven, its temperature in degrees Fahrenheit.
    a. Write a sentence explaining what $f(0) = 65$ means in everyday language
    b. Write a sentence explaining what $f(5) < f(10)$ means in everyday language.
    c. Write a sentence explaining what $f(40) = f(45)$ means in everyday language
    d. Write a sentence explaining what $f(45) > f(60)$ means in everyday language.

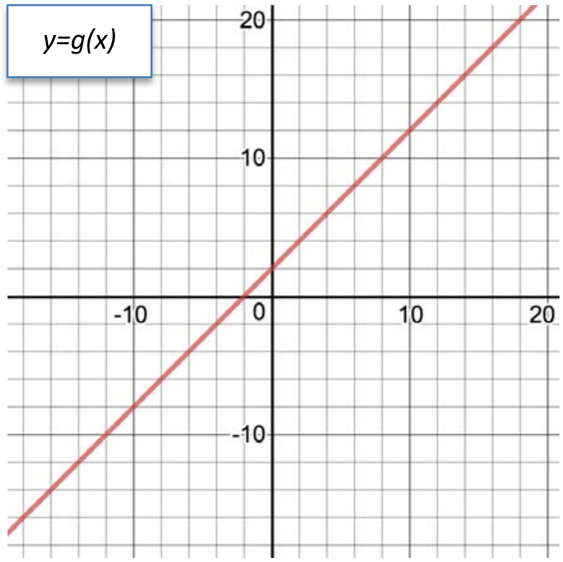
Use the table and graph to answer the questions below.
2. Find $f(4)$
3. If $f(x) = 2$, find x.

x	f(x)
-2	-4
-1	-1
0	2
1	5
2	8
3	11
4	14



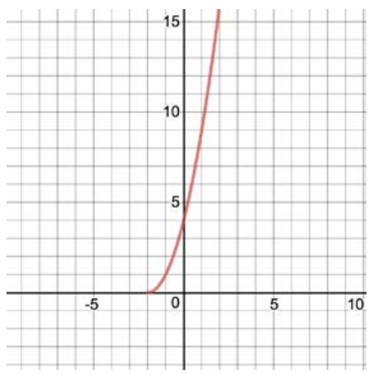
4. Use the table and/or the equation to perform the given function operation. Graph the result.

x	f(x)
-2	4
-1	1
0	0
1	1
2	4
3	9
4	16

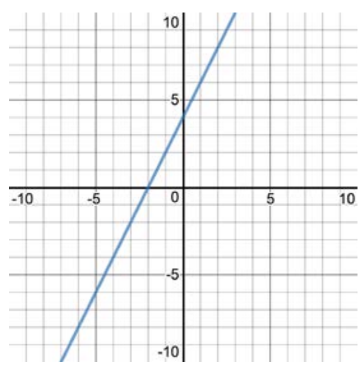


Solution:

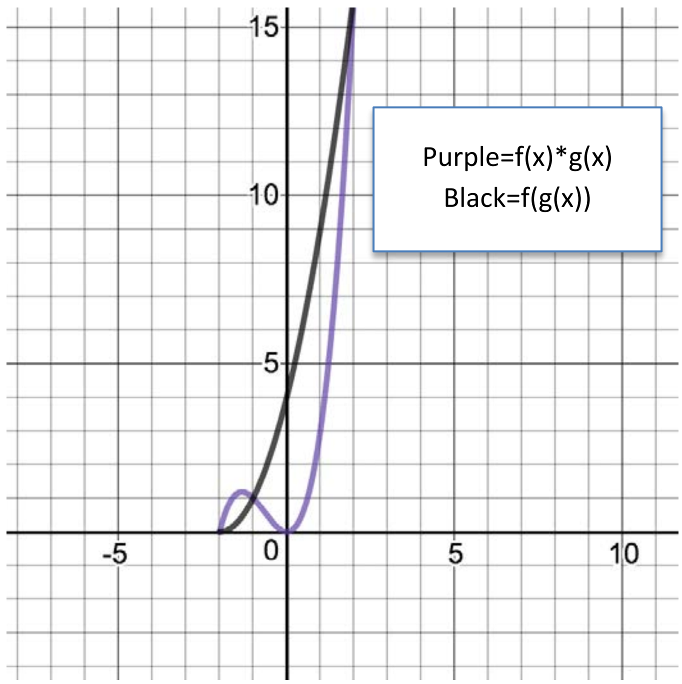
x	*f(x)*	*f(x+2)*
-2	4	*f(-2+2)=0*
-1	1	*f(-1+2)=1*
0	0	*f(0+2)=4*
1	1	*f(1+2)=9*
2	4	*f(2+2)=16*
3	9	
4	16	



x	*g(x)*	*2g(x)*
-2	0	2g(-2)=0
-1	1	2g(-1)=2
0	2	2g(0)=4
1	3	2g(1)=6
2	4	2g(2)=8
3	5	2g(3)=10
4	6	2g(4)=12



x	*f(x)*	*g(x)*	*f(x)*g(x)*	*f(g(x))*
-2	4	0	f(-2)*g(-2)=0	f(g(-2))=f(0)=0
-1	1	1	f(-1)*g(-1)=1	f(g(-1))=f(1)=1
0	0	2	f(0)*g(0)=0	f(g(0))=f(2)=4
1	1	3	f(1)*g(1)=3	f(g(1))=f(3)=9
2	4	4	f(2)*g(2)=16	f(g(2))=f(16)=2
3	9	5		
4	16	6		



Resources/Tools:
(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered “best practice” to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.A.2
    - Using Function Notation II
    - Yarn in the Oven
    - The Random Walk
    - Cell phones
    - Random Walk II

Domain: Interpreting Functions ★ (F.IF)

Cluster: *Understand the concept of a function and use function notation.*

Standard: F.IF.4 ★ (all)

For a function that models a relationship between two quantities, interpret key features of expressions, graphs and tables in terms of the quantities, and sketch graphs showing key features given a description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. **(F.IF.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: F.IF.7

Explanations and Examples:

This standard is often paired with F.IF.7 but F.IF.7 focuses on the key feature from the graph, while this standard focuses on identifying key features across all representations. Another key difference between the two standards is that F.IF.7 is not a modeling standard while F.IF.4 is a modeling standard. As a modeling standard, the focus should be on *interpreting the quantities* in context. This standard is an ALL standard because all the functions studied in high school have this standard applied and practice.

Examples:

- A rocket is launched from 180 feet above the ground at time *t* = 0. The function that models this situation is given by $h = -16t^2 + 96t + 180$, where *t* is measured in seconds and *h* is height above the ground measured in feet.
  - What is a reasonable domain restriction for *t* in this context?
  - Determine the height of the rocket two seconds after it was launched.
  - Determine the maximum height obtained by the rocket.
  - Determine the time when the rocket is 100 feet above the ground.
  - Determine the time at which the rocket hits the ground.
  - How would you refine your answer to the first question based on your response to the second and fifth questions?

- Compare the graphs of $y = 3x^2$and $y = 3x^3$.
- Let $R(x) = \dfrac{2}{\sqrt{x-2}}$. Find the domain of *R*(*x*). Also find the range, zeros, and asymptotes of *R*(*x*).
- Let $f(x) = x^2 - 5x + 1$. Graph the function and identify end behavior and any intervals of constancy, increase, and decrease.
- It started raining lightly at 5am, then the rainfall became heavier at 7am. By 10am the storm was over, with a total rainfall of 3 inches. It didn't rain for the rest of the day. Sketch a possible graph for the number of inches of rain as a function of time, from midnight to midday.

Instructional Strategies:

Flexibly move from examining a graph and describing its characteristics (e.g., intercepts, relative maximums, etc.) to using a set of given characteristics to sketch the graph of a function.

Examine a table of related quantities and identify features in the table, such as intervals on which the function increases, decreases, or exhibits periodic behavior.

Recognize appropriate domains of functions in real-world settings. For example, when determining a weekly salary based on hours worked, the hours (input) could be a rational number, such as 25.5. However, if a function relates the number of cans of soda sold in a machine to the money generated, the domain must consist of whole numbers.

Given a table of values, such as the height of a plant over time, students can estimate the rate of plant growth. Also, if the relationship between time and height is expressed as a linear equation, students should explain the meaning of the slope of the line. Finally, if the relationship is illustrated as a linear or non-linear graph, the student should select points on the graph and use them to estimate the growth rate over a given interval.

Begin with simple, linear functions to describe features and representations, and then move to more advanced functions, including non-linear situations.

Provide students with many examples of functional relationships, both linear and non-linear. Use real-world examples, such as the growth of an investment fund over time, so that students can not only describe what they see in a table, equation, or graph, but also can relate the features to the real-life meanings.

Allow students to collect their own data sets, such as the falling temperature of a glass of hot water when removed from a flame versus the amount of time, to generate tables and graphs for discussion and interpretation.

Common Misconceptions:

Students may believe that it is reasonable to input any *x*-value into a function, so they will need to examine multiple situations in which there are various limitations to the domains.

Students may also believe that the slope of a linear function is merely a number used to sketch the graph of the line. In reality, slopes have real-world meaning, and the idea of a rate of change is fundamental to understanding major concepts from geometry to calculus.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B
  - Pizza Place Promotion
  - F-IF.B.4
  - Influenza epidemic
  - Warming and Cooling
  - How is the weather?
  - Telling a Story with Graphs
  - Logistic Growth Model , Abstract Version
  - Logistic Growth Model , Explicit Version

Domain: Interpreting Functions ★ (F.IF)

Cluster: *Understand the concept of a function and use function notation..*

Standard: F.IF.5 (all) ★

Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.* ★ **(F.IF.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.6 Attend to precision.

Connections: F.IF.1

Explanations and Examples:

Given the graph if a function, determine the practical domain of the function as it relates to the numerical relationship it describes.

Students may explain orally or in written format, the existing relationships.

Examples:

- If the function $h(n)$ gives the number of person-hours it takes to assemble *n* engines in a factory, then the positive integers would be an appropriate domain for the function.
- A hotel has 10 stories above ground and 2 levels in its parking garage below ground. What is an appropriate domain for a function. $T(n)$ that gives the average number of times an elevator in the hotel stops at the n^{th} floor each day.
- Oakland Coliseum, home of the Oakland Raiders, is capable of seating 63,026 fans. For each game, the amount of money that the Raiders' organization brings in as revenue is a function of the number of people, n, in attendance. If each ticket costs $30, find the domain and range of this function.

Sample Response:

Let *r* represent the revenue that the Raider's organization makes, so that $r = f(n)$. Since *n* represents a number of people, it must be a nonnegative whole number. Therefore, since 63,026 is the maximum number of people who can attend a game, we can describe the domain of *f* as follows: $\{Domain = n: 0 \leq n \leq 63{,}026 \text{ } and \text{ } n \text{ } is \text{ } an \text{ } integer\}$.

The range of the function consists of all possible amounts of revenue that could be earned. To explore this question, note that $r = 0$ if nobody comes to the game, $r = 30$ if one person comes to the game, $r = 60$ if two people come to the game, etc. Therefore, *r* must be a multiple of 30 and cannot exceed $(30 \cdot 63{,}026) = 1{,}890{,}780$, so we see that $\{Range = r: 0 \leq r \leq 1{,}890{,}780 \text{ } and \text{ } r \text{ } is \text{ } an \text{ } integer \text{ } multiple \text{ } of \text{ } 30\}$.

Instructional Strategies:

The deceptively simple task above asks students to find the domain and range of a function from a given context. The function is linear and if simply looked at from a formulaic point of view, students might find the formula for the line and say that the domain and range are all real numbers. However, in the context of this problem, this answer does not make sense, as the context requires that all input and output values are non-negative integers, and imposes additional restrictions. This problem could serve different purposes. Its primary purpose is to illustrate that the domain of a function is a property of the function in a specific context and not a property of the formula that represents the function. Similarly, the range of a function arises from the domain by applying the function rule to the input values in the domain. A second purpose would be to illicit and clarify a common misconception, that the domain and range are properties of the formula that represent a function. Finally, the context of the task as written could be used to transition into a more involved modeling problem, finding the Raiders' profit after one takes into account overhead costs, costs per attendee, etc.

Resources/Tools:

(Most modeling questions and resources can be modified to support this standard. Some specific and explicit resources are shown here but it should not be considered "best practice" to ask these questions in isolation frequently.)

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.B.5
  - Oakland Coliseum
  - Average Cost

Domain: Interpreting Functions (F.IF)

Cluster: *Understand the concept of a function and use function notation*

Standard: **F.IF.9 (all)**

Compare properties of two functions using a variety of representations (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, a quantity increasing exponentially eventually exceeds a quantity increasing linearly.* **(F.IF.9)**

Suggested Standards for Mathematical Practice (MP):

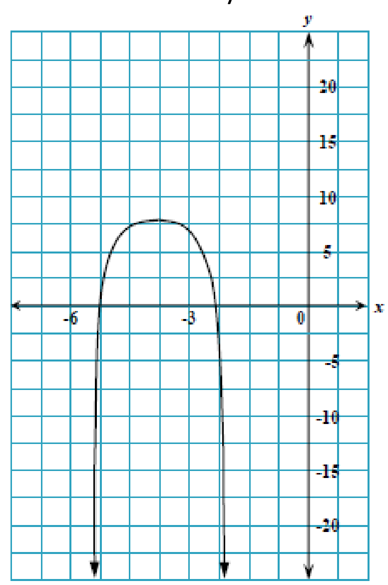
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF.4, F.IF.7

Explanations and Examples: See F.IF.1 and F.IF.4

Examples:

- Examine the functions below. Which function has the larger maximum? How do you know?



$$f(x) = 2x^2 - 8x + 20$$

Resources/Tools:

Illustrative Mathematics High School Functions tasks: **Scroll to the appropriate section to find named tasks.**

- F-IF.C.9
  - o Throwing Baseballs

High School – Geometry

Domain: Congruence (G.CO)

▶ **Cluster:** *Experiment with transformations in the plane.*

Standard: **G.CO.1**

(9/10) Verify experimentally (for example, using patty paper or geometry software) the properties of rotations, reflections, translations, and symmetry:

- G.CO.1a. **(9/10)** Lines are taken to lines, and line segments to line segments of the same length. **(8.G.1a)**
- G.CO.1b. **(9/10)** Angles are taken to angles of the same measure. **(8.G.1b)**
- G.CO.1c. **(9/10)** Parallel lines are taken to parallel lines. **(8.G.1c)**
- G.CO.1d. **(9/10)** Identify any line and/or rotational symmetry within a figure. **(G.CO.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: G.CO.1-8

Explanations and Examples:

Students will verify these properties experimentally, not formally prove them. Because this is the first time students will work with transformations, there are some important developmental reasons for beginning with experimental verification. The Van Hiele Levels of Geometric Development describe how students progress in their geometric thinking and highlight the importance of mastering this foundational standard.*

*Description below comes from here but the numbering has been changed to match the numbering found in the research referenced from here.

0. Pre-*any* geometric reasoning ability.
1. **Visualization/Recognition**:
    a. Content: Geometric Objects and patterns from shapes.
    b. Forms of Reasoning: Visual/tactile/sensory manipulation, observation and comparison
    c. Outcome: Students recognize, name, and compare shapes and patterns.
    d. At this level of reasoning, many students will misclassify shapes based on orientation.
    e. Researchers find the majority of students enter the course at level 1 or below (72-74%).
    f. Usiskin, 1982 first found these results but researchers have continued getting similar results.

2. **Description/Analysis:**
    a. Content: Properties of shapes and patterns.
    b. Form of Reasoning: Empirical description through observation, manipulation, construction, and measurement.
    c. Outcome: Students can characterize classes of geometric objects by describing their necessary properties.
3. **Informal Deduction:**
    a. Content: Relationships among the properties
    b. Forms of Reasoning: Logical connection and ordering of properties (which may have been discovered empirically) into short deductive chains.
    c. Outcome: Students can form abstract definitions, distinguish between necessary and sufficient conditions, understand and sometimes invent logical arguments.
4. **Formal Deductive Proof:**
    a. Content: A formal system of relationships among properties.
    b. Form of reasoning: Formal logical analysis
    c. Outcome: Students learn the system's traditional organization; classical results and their logical necessity; solve problems; and construct proofs.
    d. The goal for most high school geometry courses, at this level students can use deductive reasoning and formal construct proofs.
    e. A study of 1,596 HS geometry students found only 4% achieved this level of reasoning. Other studies have found similarly low results.
5. **Rigor:** Reasoning about alternative axiomatic systems.

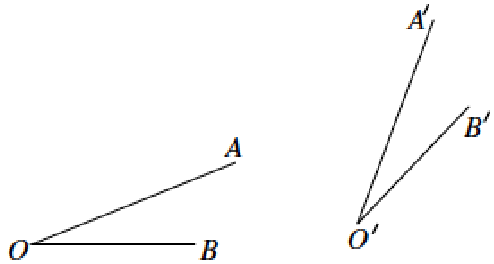
Two other important properties of this theory are that 1) the levels are discrete and linear, students will master one level before moving on to the next and 2) communication between levels is extremely difficult. The implications for this standard are that incoming geometry students might misclassify shapes based on orientation, which might be affected by reflections or rotations, and that the form of reasoning they are developmental ready for involves tactile manipulation. Therefore, asking students to verify experimentally the properties of transformation is a necessary step in their development.

Bridging the gap between this standard and future standards, teachers could follow up the initial experiments with investigations asking students to verify these properties experimentally using shapes. For example, translating a triangle is practice with translating lines and angles *and* the relationships between those lines and angles. Rotating parallel lines can allow students begin thinking about the component parts of parallelograms. As students move from Van Hiele Level 1 to Level 2, they will begin to reason about the component parts and properties of geometric objects.

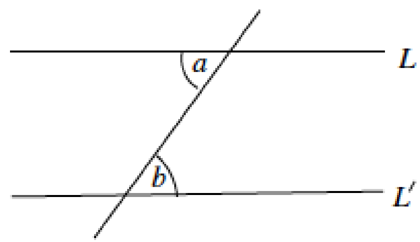
Examples:

From Engage NY Grade 8 Module 2:

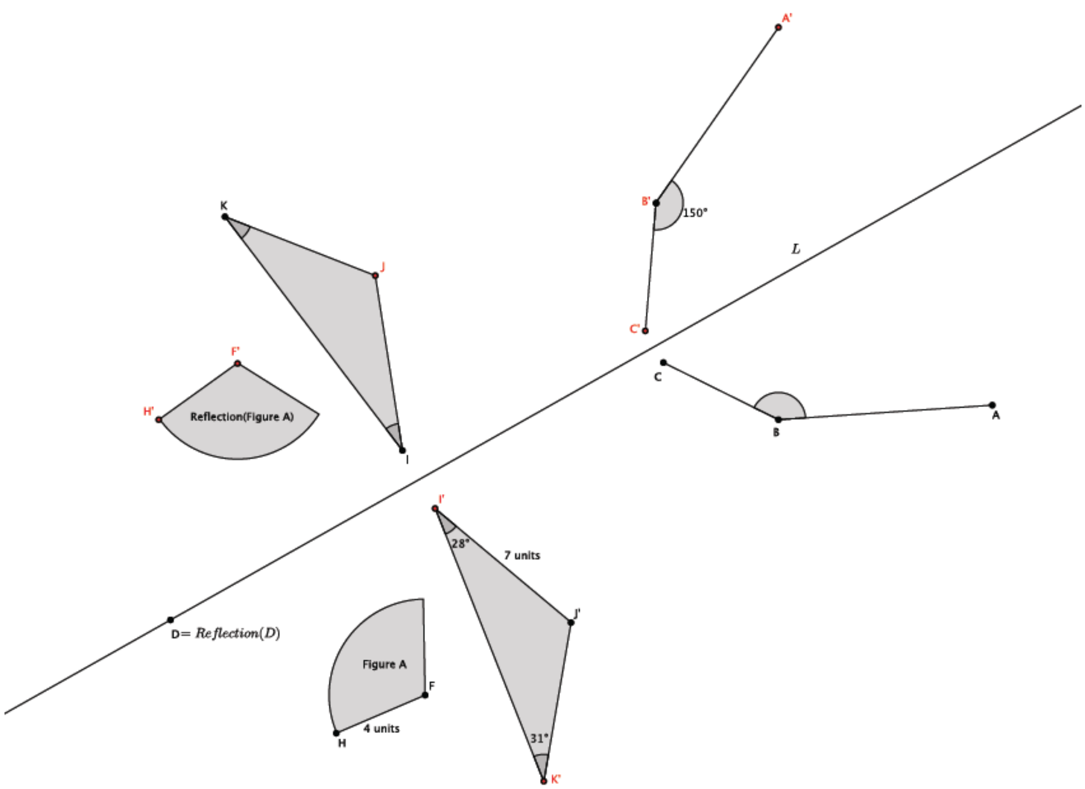
1. Given angles $\angle AOB$ and $\angle A'O'B'$, how can we tell whether they have the same degree without having to measure each angle individually?



2. If two lines L and L' are parallel, and they are intersected by another line, how can we tell if the angles $\angle a$ and $\angle b$ (as shown) have the same degree when measured?



3. Use the picture to label the unnamed points. (Letters in red are the answers.)



4. What is the measure of $\angle JKI$? $\angle KIJ$? $\angle ABC$? How do you know?

   $\boldsymbol{m\angle JKI = 31°}$, $\boldsymbol{m\angle KIJ = 28°}$, and $\boldsymbol{m\angle ABC = 150°}$. Reflections preserve angle measures.

5. What is the length of segment *reflection(FH)? IJ?* How do you know?

   $\boldsymbol{|Reflection(FH)| = 4\ units}$, and $= \boldsymbol{7\ units}$. Reflections preserve lengths of segments.

6. What is the location of *reflection(D)?* Explain.

   Point $\boldsymbol{D}$ and its image are in the same location on the plane. Point $\boldsymbol{D}$ was not moved to another part of the plane because it is on the line of reflection. The image of any point on the line of reflection will remain in the same location as the original point.

Instructional Strategies:

The Van Hiele theory also provides guidance on sequencing instruction to develop geometric reasoning.

- **Inquiry:** Conversation and collaborative learning focused on specific observations, questions, and emerging vocabulary.
- **Directed Orientation:** Exploration using carefully sequenced materials for short tasks which gradually reveal the structures.
- **Explication:** Students share and discuss their developing understanding with each other.
- **Free Orientation:** Exploration of more complex and open-ended tasks.
- **Integration:** Reviews and summaries, anchor charts, higher order questions.

One other instructional implication that results from Van Hiele's model is the decision on what materials to use. There are many dynamic geometry environments (DGE) but, perhaps, the best materials to use are hands-on materials that can be physically measured or manipulated. Patty paper, thin wax paper cut into squares the size of hamburger patties, are an inexpensive and useful resource for investigating transformations. Because patty paper can be "flipped," moved, and rotated while keeping the shape visible, it is exceptionally well suited for representing the concept. Constructions can replicate the same figure, by design, but risk obscuring the "movement" from the student as they are focused on simply drawing the new shape.

Resources/Tools:

Engage NY Grade 8, Module 2, Topic A

Illustrative Mathematics **Grade 8** tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.A.1
  - Origami Silver Rectangle
  - Reflections, Rotations, and Translations

Illinois Teach & Talk

For additional teacher learning about transformations and proofs:

- Teaching Geometry by H. Wu
- How Transformations Help Us Think About Geometry by J. King
- Transformational Geometry by R.G. Brown

Domain: Congruence (G.CO)

▶ **Cluster:** *Experiment with transformations in the plane.*

Standard: **G.CO.2**

(9/10) Recognize transformations as functions that take points in the plane as inputs and give other points as outputs and describe the effect of translations, rotations, and reflections on two-dimensional figures. *For example,* (x, y) *maps to* $(x + 3, y - 5)$*; reflecting triangle ABC(input) across the line of reflection maps the triangle to exactly one location, A'B'C'(output).* **(G.CO.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: G.CO.1-8, F.IF.1

Explanations and Examples:

There are two separate expectations included here:

1. Recognize transformations as a function and
2. Describe the effect of the transformations.

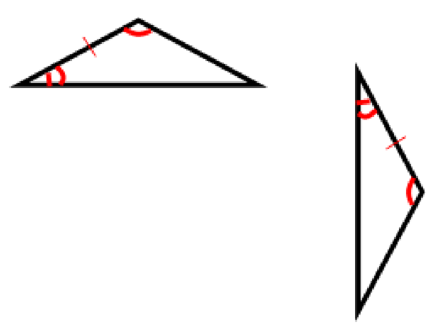
Viewing a transformation as a function is more than an attempt to make an arbitrary connection to the function standards, it is an essential requirement for using transformations in a proof. First, students visually verify that the transformations do not change the line, angle, etc. Then, students learn that not only is the output unchanged but that there is *exactly one* output. We can know, beyond all doubt, where and how the geometric figure will be after the transformation.

The second expectation for this standard asks students to describe the effect of the transformation. This does not require descriptions in the coordinate plane but that is a useful and algebraic connection to make with students. Another way to explore this standard is to ask students to draw the effect of the transformation, in other words, a visual description.

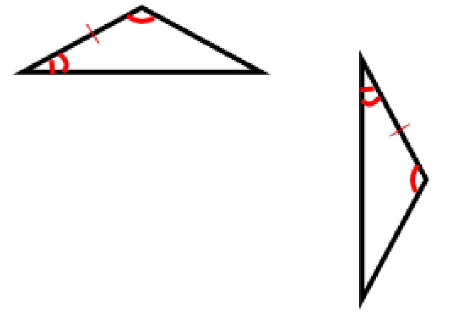
Examples:

Example 1:

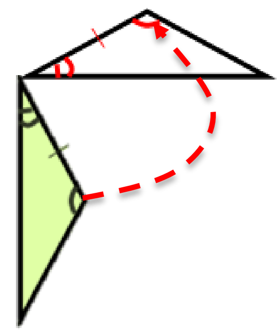
This pair of triangles are congruent. You want to use transformations to illustrate the congruence.



If the triangle is viewed as the input in the translation function seen in red, how can know that the result will be the triangle in green?

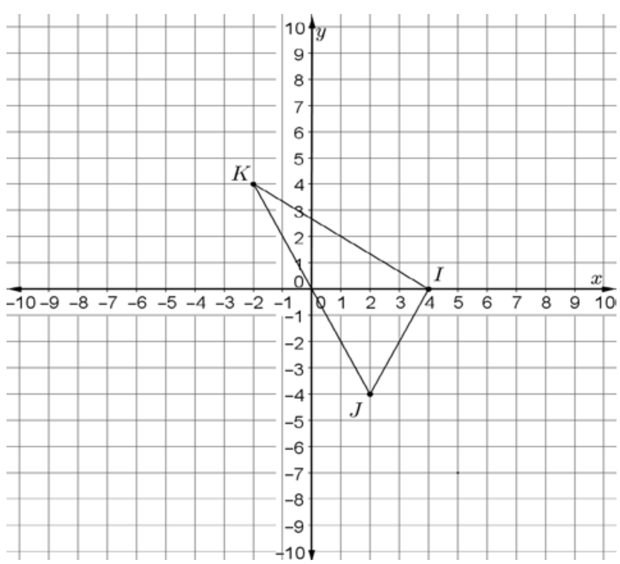


Now the triangle is rotated with the center of rotation where the two triangles meet. You want to rotate the triangle until the other two vertexes of the triangle meet, is this possible? How do you know?



Example 2:

Find the coordinates of the vertices of the image of ΔIJKafter the translation $T(x, y) \rightarrow T(x-3, y+5)$.

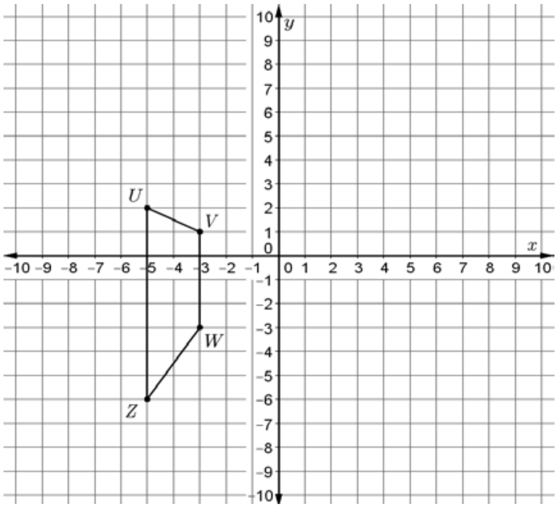


I'______

J'______

K'______

Find the coordinates of the vertices of the image of trapezoid *UVWZ* after a reflection across the *x*-axis.



U'________

V'________

W'_______

Instructional Strategies: See G.CO.1

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.A.2
  - Fixed points of rigid motions
  - Dilations and Distances
  - Horizontal Stretch of the Plane

Domain: Congruence (G.CO)

- **Cluster:** *Understand congruence in terms of rigid motions*

Standard: G.CO.3

(9/10) Given two congruent figures, describe a **sequence** of rigid motions that exhibits the congruence (**isometry**) between them using coordinates and the non-coordinate plane. **(8.G.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: G.CO.1-8

Explanations and Examples:

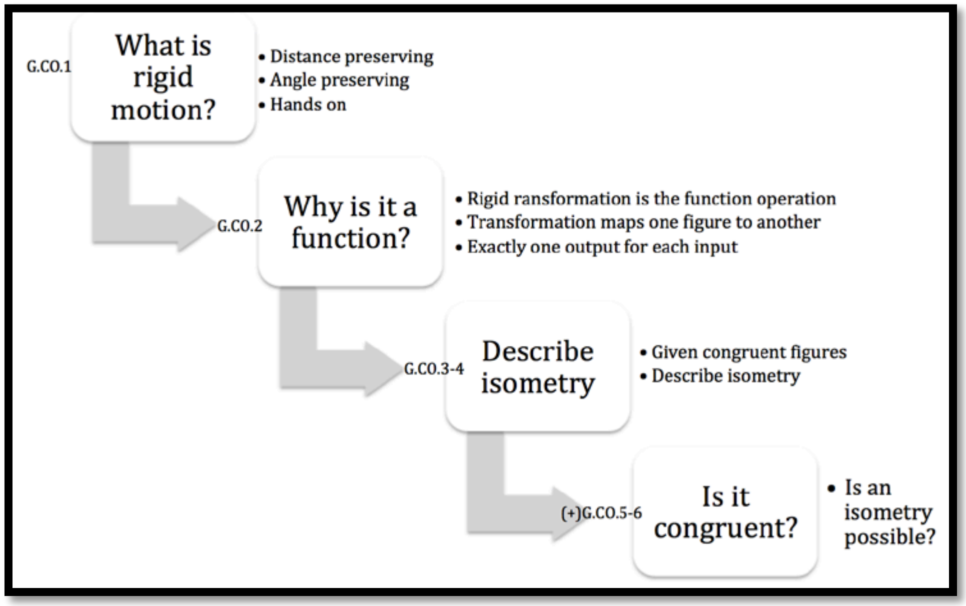
Before discussing this particular standard, it might be helpful to review the progression of congruence using rigid motion. First, students are *given two congruent figures* and are asked to describe how to transform the figure so that one figure *coincides* with the other (G.CO.3). These standards are working in Van Hiele Levels 1 and 2, asking students to manipulate the figures and explore the properties of transformations. Then students advance in their geometric reasoning to more formal deductive thinking, focusing on the *definition* of congruence in terms of rigid motion (G.CO.4). The key expectation to note for both of these standards is that students are given congruent figures and asked to show the congruence between them.

The additional two transformation standards (G.CO5-6) are (+) standards that might be included for *some* students. These standards ask students to *determine if the triangles are congruent* using rigid motion. These proofs are more complicated because of the level of reasoning required. Additional information can be found in the 11/+ flipbook. G.CO.9 also addresses triangle congruence but does not ask students to prove.

Looking deeper into the word “isometry”, Wolfram MathWorld states:

- Two geometric figures related by an isometry are said to be geometrically congruent (Coxeter and Greitzer 1967, p. 80).
- Two geometric figures are said to exhibit geometric congruence (or "be geometrically congruent") iff one can be transformed into the other by an isometry (Coxeter and Greitzer 1967, p. 80).

But what *is* an isometry? An isometry is a mapping that maps one figure to another figure and is distance preserving. In other words, this one word encompasses both G.CO.1 (*rigid* transformation) and G.CO.2 (rigid transformation is a mapping function). If we know the figures are congruent, then there *must* be an isometry relating the two (G.CO.3-4). If an isometry can be found between two figures, then the figures are congruent (G.CO.5-6).

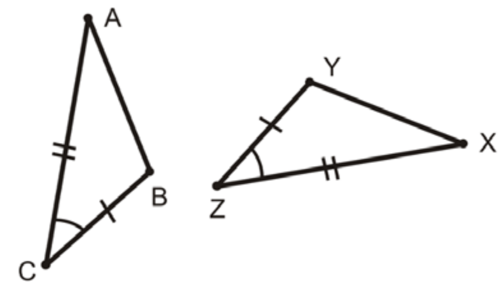


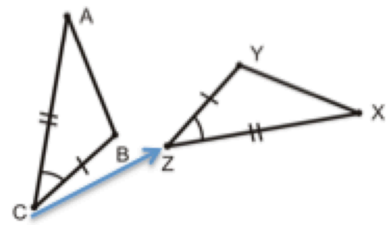
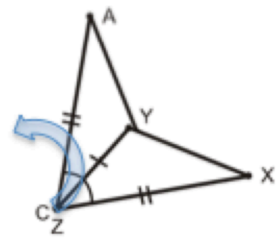
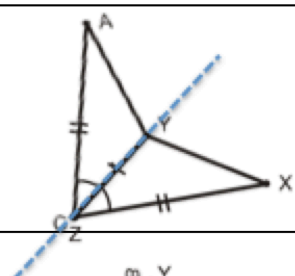
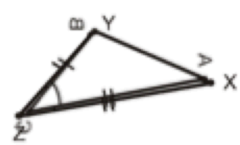
Examples:

The following examples will illustrate this standard. The same examples will be used in the next few congruence standards so that you can see the progression of student expectations. Remember three important thoughts as you read through these examples. First, reading is not the best modality for processing congruence through rigid motion. These examples will show what a student might write as an example but will not encompass the entirety of instruction, which will surely include hands on activities. Second, there is a reason that you are encouraged to explore G.CO.1 prior to introducing these standards. Developmentally, you and your students need to explore the transformations with lines, angles, and parallel lines so that you can observe critical features that will be necessary for this step in the progression. Finally, rigid motion is intended to make an abstract idea more concrete and *easier* for students. We want students to use appropriate tools strategically. If an isometry makes the process *more* complex, then do not use that method. Regular non-rigid motion congruence proofs are in other standards and can be used within any proof to explain the student’s line of reasoning.

Example 1

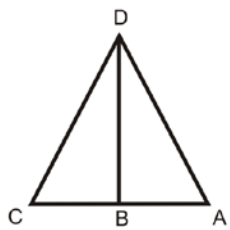
Given: $\triangle ABC \cong \triangle XYZ$ with SAS congruence as marked in the diagram. Describe a sequence of rigid motions that demonstrates the congruence between the two triangles.



Translate $\triangle ABC$ so that **point C corresponds with point Z.**	
With the center of rotation at point C, rotate $\triangle ABC$ until $\overrightarrow{CB}$ corresponds with $\overline{ZY}$.	
Reflect $\triangle ABC$ across $\overrightarrow{CB}$.	
Because all three points of the triangles coincide, then the triangles occupy the same space, demonstrating that $\triangle ABC \cong \triangle XYZ$.	

Example 2:

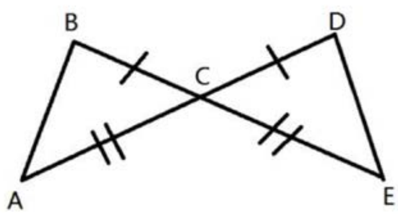
Given $\triangle BCD \cong \triangle BAD$ and $\overline{DB} \perp \overline{CA}$, describe a sequence of rigid motions that will demonstrate the congruence between the two triangles.



Since two points of the three already coincide, you just need to use transformations to get point C and point A to coincide. Reflecting across $\overleftrightarrow{BD}$, point C will coincide with point. This demonstrates $\triangle BCD \cong \triangle BAD$ using transformations.

Example 3:

Both triangle pairs below look similar but use different transformations to demonstrate congruency. Describe the transformation each pair will use and why the other transformation will *not* demonstrate congruency.

<table>
<tr>
<td>


To demonstrate congruence between these two triangles, you would do a reflection across a line that goes through point C so that $\angle ACB$ coincides with $\angle ECD$.

If you rotate this pair of triangles, then $\overline{CD}$ will coincide with $\overline{AC}$. Since those sides are not congruent, then we can't be know if point A and Point D will coincide with one another.
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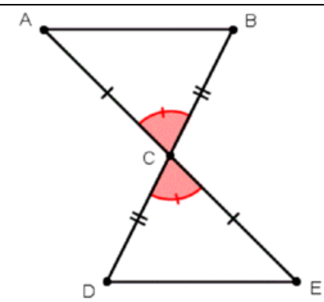
<td>


To demonstrate congruence between these two triangles, you would do a rotation about point C and rotate the triangles until $\overline{AC}$ coincides with $\overline{EC}$.

If you reflect across a line through point C, then $\overline{CD}$ will coincide with $\overline{AC}$. Since those sides are not congruent, then we can't be know if point A and Point D will coincide with one another.
</td>
</tr>
</table>

Instructional Strategies:

Using hands-on manipulatives is critical at this stage of reasoning. Using patty paper or transparencies will help students follow the path of the transformations. Another strategy, that is useful throughout proving congruency, is to color code corresponding parts and/or the given information.

Resources/Tools: See G.CO.1

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.A.2
  - Congruent Segments
  - Circle Sandwich
  - Congruent Rectangles
  - Cutting a rectangle into two congruent triangles
  - Congruent Triangles
  - Triangle congruence with coordinates

Domain: Congruence (G.CO)

◆ **Cluster:** *Understand congruence in terms of rigid motions*

Standard: G.CO.4

(9/10) Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent. **(G.CO.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: G.CO.1-6

Explanations and Examples:

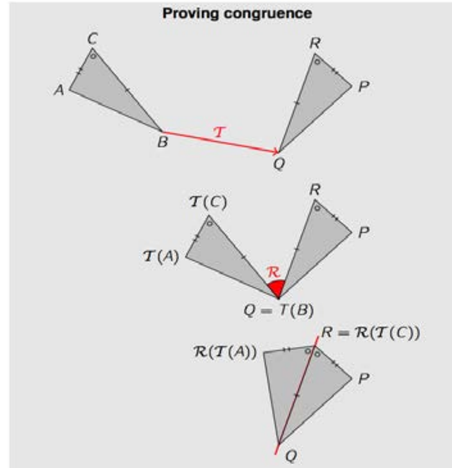
From the Geometry Progression

This section from the progression illustrates how the standards move from the focus on manipulation informally in G.CO.3 to the more formal descriptions required in G.CO.4. Also, notice the addition of notation. The standards are silent about notation requirements but might be considered to ensure clarity and reinforce that a transformation is a function. (Reference to the example in the margin has been placed below.)

Understand congruence in terms of rigid motions Two figures are defined to be congruent if there is a sequence of rigid motions carrying one onto the other.[G-CO.6] It is important to be wary of circularity when using this definition to establish congruence. For example, you cannot assume that if two triangles have corresponding sides of equal length and corresponding angles of equal measure then they are congruent; this is something that must be proved using the definition of congruent, as shown in the margin.[G-CO.7]

Notice that the argument in the margin does not in fact use every equality of corresponding sides and angles. It only uses $|BC| = |QR|$, $m\angle ACB = m\angle PRQ$, and $|CA| = |RP|$ (along with the fact that rigid motions preserve all of these equalities). These equalities are indicated with matching hash marks in the figures.

Suppose that the corresponding sides and corresponding angles of $\triangle ABC$ and $\triangle PQR$ are equal. First translate $\triangle ABC$ along the line segment BQ, so that $\mathcal{T}(B) = Q$. Then rotate clockwise about Q through the angle $\angle \mathcal{T}(C)QR$. Because translation and rotation preserve distance, we have $R = \mathcal{R}(\mathcal{T}(C))$. Now reflect across the line through R and Q. Because the rigid motions preserve angles, the line through R and P coincides with the reflection of the line through R and $\mathcal{R}(\mathcal{T}(A))$, and because they preserve distance the point P coincides with the reflection of $\mathcal{R}(\mathcal{T}(A))$. Now all three points of the triangle coincide, so we have produced a sequence of rigid motions that maps $\triangle ABC$ onto $\triangle PQR$, and they are therefore congruent.



While thinking about avoiding circular reasoning, wording of this standard needs to be considered carefully.

Standard	Use the definition	of congruence in terms of rigid motions	to show that	two triangles are congruent	if and only if	corresponding pairs of sides and corresponding pairs of angles are congruent.
Explanation	Students are moving further in their reasoning skills to formally use the definition (next column)	The definition of congruence, defined in G.CO.3, is <u>two figures are congruent iff they can be related by an isometry</u> (a mapping that maps one figure to another figure and is distance preserving).	Show, not prove. Using the definition is more formal than G.CO.3 but not as formal as G.CO.5-6	You are given congruent triangles, show that the corresponding parts have isometry. -or- You are given congruent parts and need to show that the shapes have isometry. The goal is to reinforce that transforming a shape moves corresponding parts, orientations, and *all* relationships in the same way (or vice versa).		

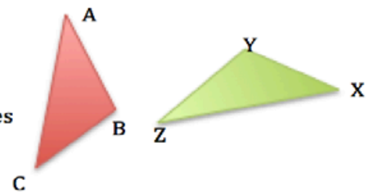
A word about congruence: Congruence is NOT defined as "same shape and size." That lacks precision. A rectangle that is 1×8 is the same shape and size as another rectangle that is 2×4 but they aren't congruent. Another example of the lack of clarity can be found when looking at congruent angles. How can two angles be the "same size" when they infinite length?

Examples:

Example 1:

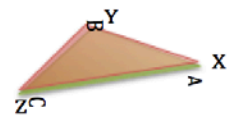
Given: $\triangle ABC \cong \triangle XYZ$

Because the triangles are congruent, there is an isometry between the triangles that moves $\triangle ABC$ to coincide with $\triangle XYZ$.



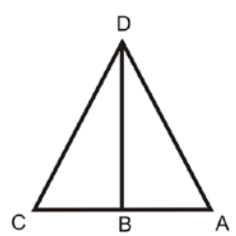
Because $\triangle ABC$ to coincides with $\triangle XYZ$

and because transformations are distance and angle preserving, then all the corresponding parts are also congruent.



Example 2:

Given: $\overline{DC} \cong \overline{DA}$, $\overline{CB} \cong \overline{AB}$, $\angle DCB \cong \angle DAB$, $\overline{DB} \perp \overline{CA}$



Reflect $\triangle DAB$ across the line $\overline{DB}$.

Because points A, B, and C are collinear and perpendicular to the line of reflection, then after the reflection $\overline{AB}$ will coincide with $\overline{CB}$. (Discovered during investigations for G.CO.1)

Because $\overline{CB} \cong \overline{AB}$, then points C and A will also coincide.

Because all three points for each triangle coincide, then the triangles demonstrate isometry and are congruent.

Instructional Strategies: See G.CO.4

Resources/Tools: See G.CO.1

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.B
  - Are the Triangles Congruent?

Domain: Congruence (G.CO)

▶ **Cluster:** *Construct arguments about geometric theorems using rigid transformations and/or logic.*

Standard: **G.CO.7**

(9/10) Construct arguments about lines and angles using theorems. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints. (Building upon standard in 8th grade Geometry.) **(G.CO.9)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.

Connections: G.CO, 8.G

Explanations and Examples:

There are some important changes to note with this standard in the 2017 Kansas Math Standards. First, the wording "construct arguments" was intentionally chosen to remove rigorous deductive proof as an expectation *for all* students. Proofs are at the discretion of the teacher based on instructional time to adequately teach the required standards and the needs of the students. "Construct arguments" means that the expectation for all students is that they can operate at Van Hiele level 3:

3. **Informal Deduction:**
   a. Content: Relationships among the properties
   b. Forms of Reasoning: Logical connection and ordering of properties (which may have been discovered empirically) into short deductive chains.
   c. Outcome: Students can form abstract definitions, distinguish between necessary and sufficient conditions, understand and sometimes invent logical arguments.

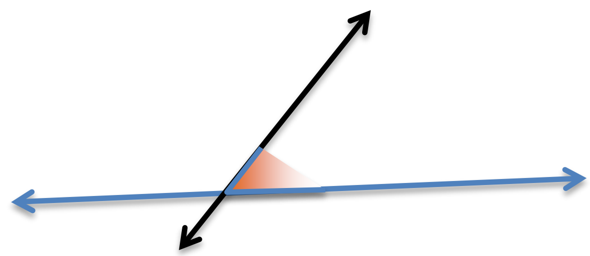
Classroom teachers and mathematics education researchers agree that students have a hard time learning how to do geometric proofs. An article by Battista and Clements (1995) provides information for teachers to explain this shift. The most significant implication for instructional strategies for proof is stated in their conclusion.

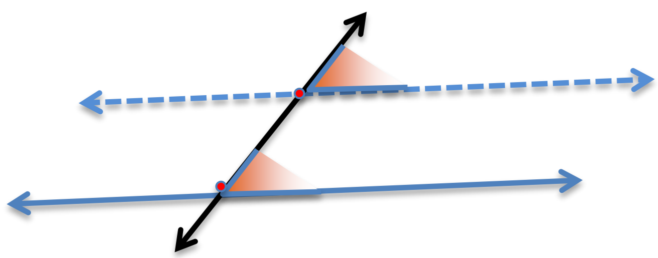
> "Ironically, the most effective path to engendering meaningful use of proof in secondary school geometry is to avoid formal proof for much of students' work. By focusing instead on justifying ideas while helping students build the visual and empirical foundation for higher levels of geometric thought, we can lead students to appreciate the need for formal proof. Only then will we be able to use it meaningfully as a mechanism for justifying ideas."

Another key change is that the Theorems listed are no longer *italicized.* This means that the standards listed are *required* for all students. This list does not necessarily encompass every possible Theorem about lines and angles that a teacher might teach but it does represent the list of possible Theorems that could be *tested*.

To maintain coherence, using the properties of rigid transformations should be considered if it makes the argument easier and more intuitive. For example, vertical angles can be seen as a reflection or a rotation. Another example of a theorem made easier with transformations is corresponding angles of parallel lines are congruent.

Imagine you start with a pair of intersecting lines and consider one angle formed by the intersection line.





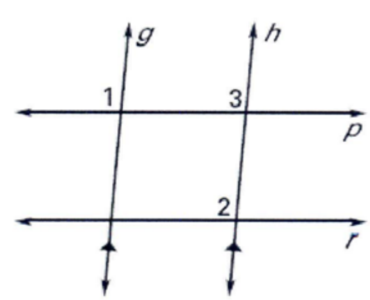
If the point of intersection is translated one line along the other line, a second parallel is created.

However, for alternate interior angles it might be easier to use the theorem of corresponding angles and vertical angles to construct an argument than it would be to combine those transformations.

Other than thinking about using transformations to scaffold student thinking, the instruction for these theorems should proceed as normal.

Examples:

Given: $g \parallel h$, $\angle 1 \cong \angle 2$
Construct an argument to show that $p \parallel r$.



- Because $g \parallel h$, then $\angle 1 \cong \angle 3$ because corresponding angles of parallel lines are congruent.
- Since $\angle 1 \cong \angle 3$ and $\angle 1 \cong \angle 2$ then $\angle 2 \cong \angle 3$ because of the transitive property.
- Because $\angle 2 \cong \angle 3$ and they are corresponding angles for lines *p* and *r* cut by the transversal *h*.
- If corresponding angles are congruent, then $p \parallel r$.

An alternate argument using transformations:

- Angle 1 can be translated along line *p* to coincide with angle 3 because $g \parallel h$ and corresponding angles are congruent.
- Because $\angle 1 \cong \angle 2$, we can translate angle 1 (currently superimposed on angle 3) down line *h* to coincide with angle 2.
- Translating an angle along a transversal creates a parallel line so we know that $p \parallel r$.

Instructional Strategies:

Encourage multiple ways to illustrate the arguments; such as in narrative paragraphs, using flow diagrams, in two-column format, and using diagrams without words. Students should be encouraged to focus on the validity of the underlying reasoning while exploring a variety of formats for expressing that reasoning.

Resources/Tools: See G.CO.1

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.C.9
  - Points equidistant from two points in the plane
  - Tangent Lines and the Radius of a Circle

Domain: Congruence (G.CO)

▶ **Cluster:** *Construct arguments about geometric theorems using rigid transformations and/or logic.*

Standard: G.CO.8

(9/10) Construct arguments about the relationships within one triangle using theorems. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point; angle sum and exterior angle of triangles. **(G.CO.10)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.

Connections: G.CO.7

Explanations and Examples: See G.CO.7

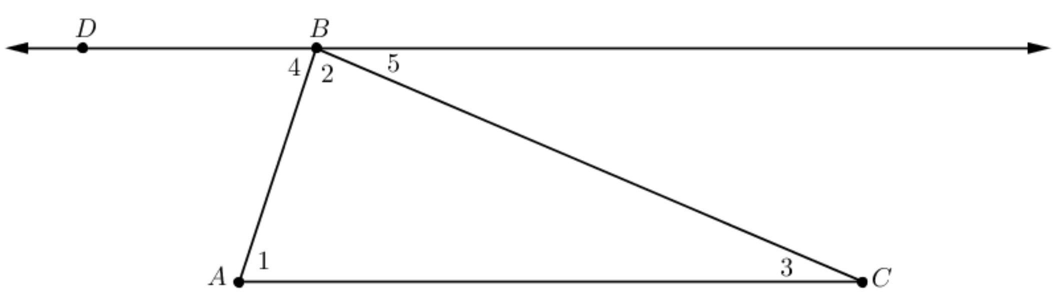
As with G.CO.7, *all* students should "construct arguments" and not "prove," which is left to the discretion of teachers.

Examples:

The following examples are adapted from CPALMS (www.cpalms.org).

Example 1:

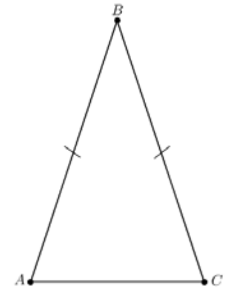
The diagram below shows ΔABC in which $\overline{AC}$ is parallel to line $\overleftrightarrow{BD}$.



In the space below, show that the sum of the interior angles of ΔABC is 180°, that is, $m\angle 1 + m\angle 2 + m\angle 3 = 180°$.

Example 2:

The diagram below shows isosceles ΔABC with $\overline{AB} \cong \overline{BC}$.



In the space below, construct an argument to show that $\angle A \cong \angle C$.

Instructional Strategies: See G.CO.1

Resources/Tools: See G.CO.1

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.C.10
  - Classifying triangles

Domain: Congruence (G.CO)

▶ **Cluster:** *<u>Construct arguments about geometric theorems using rigid transformations and/or logic.</u>*

Standard: **G.CO.9**

(9/10) Construct arguments about the relationships between two triangles using theorems. Theorems include: SSS, SAS, ASA, AAS, and HL. **(2017)**

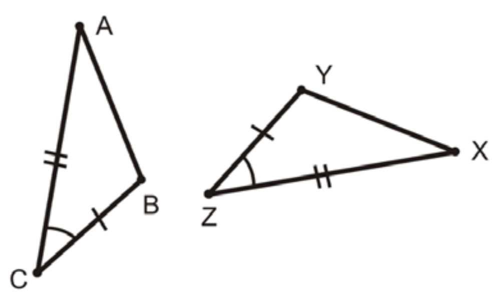
Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: G.CO.3-6

Explanations and Examples:

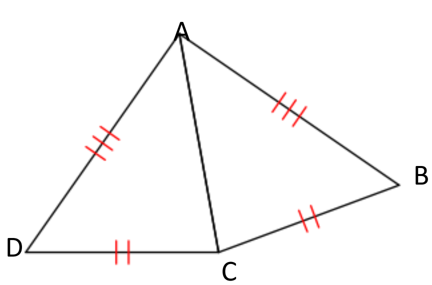
There are a couple of points to make about this new standard. First, recall that "prove" triangles congruent is a plus standard. This is a similar standard but only asks *for all students* to construct arguments. Additionally, the cluster says that arguments "use rigid transformations and/or logic." This gives the teacher and student flexibility in how an argument is approached. **Do not approach triangle congruence through rigid transformations unless it makes the argument more intuitive or is more natural for a student's academic strengths.**

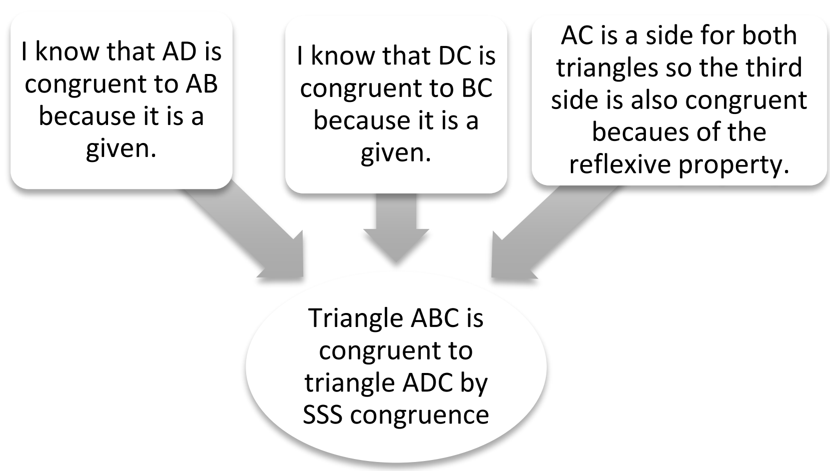


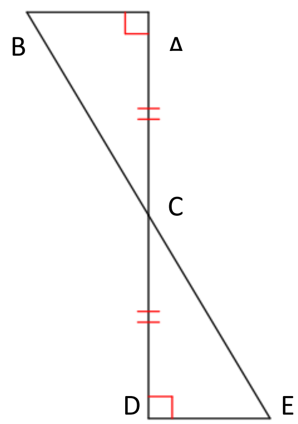
For example, once SAS congruence is established the argument for the previous example can simply be $\triangle ABC \cong \triangle XYZ$ because of SAS congruence.

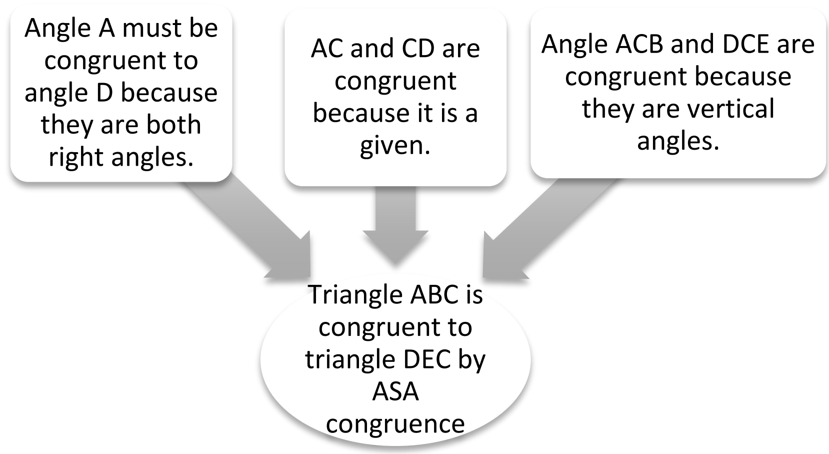
Examples:

Example 1:

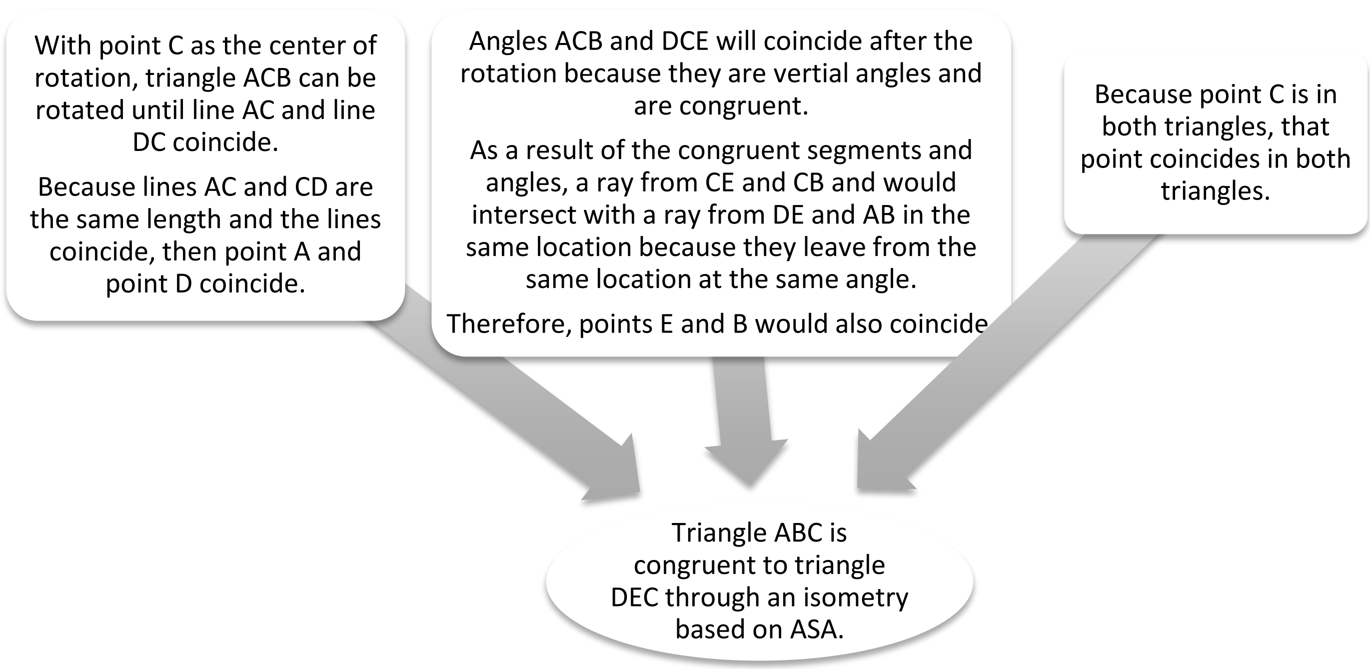






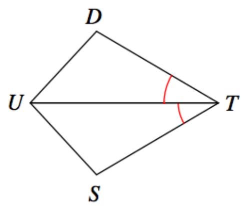


Or an alternate argument using rigid transformations, which might be used to establish ASA congruence:



While the second example looks more complicated than the first, when considering the Van Hiele levels of geometric reasoning, it is actually less abstract than the first. Its complexity comes from the written form. When shown with physically movement, the argument becomes much simpler to understand than the first abstract proof. When constructing an argument, one strategy is to ask students to convince a classmate that there is enough information to guarantee congruence.

Example 3:



What additional information do you need to argue that $\triangle DTU \cong \triangle STU$?

If I was going to reflect $\triangle DTU$ across line $\overline{TU}$ I would need to know that point D and point S are in the same location. To know that information, $\angle DUT \cong \angle SUT$ would guarantee that the lines would intersect at D and S. Alternatively, if $\overline{TS} \cong \overline{TD}$, we would know that D and S were in the same location.

Instructional Strategies:

Similar to other congruent proofs, using colors to highlight the corresponding parts being investigated can help students focus on the relevant component parts. Additionally, focus on developing reasoning and persuasion skills by encouraging students to critically think about why these are the necessary and sufficient conditions for triangle congruence.

While most resources that could support this section will focus on proofs, the questions can easily be adapted to "construct an argument." Teaching this topic provides an opportunity for differentiation because students can be given the figure with some being asked to describe the sequence of transformations, some being asked to construct an argument, and others being asked for deductive proof.

Resources/Tools:

Engage NY Geometry Module 1: Congruence, Proof, and Constructions

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.B.8
  - Why does SAS work?
  - Why does SSS work?
  - Why Does ASA Work?
  - When Does SSA Work to Determine Triangle Congruence?
  - SSS Congruence Criterion

Domain: Congruence (G.CO)

▶ **Cluster:** *Construct arguments about geometric theorems using rigid transformations and/or logic.*

Standard: **G.CO.10**

(9/10) Construct arguments about parallelograms using theorems. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals. (Building upon prior knowledge in elementary and middle school.) **(G.CO.11)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.

Connections: See G.CO.7

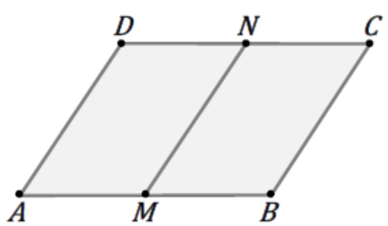
Explanations and Examples:

Students may use geometric simulations (computer software or graphing calculator) to explore theorems about parallelograms.

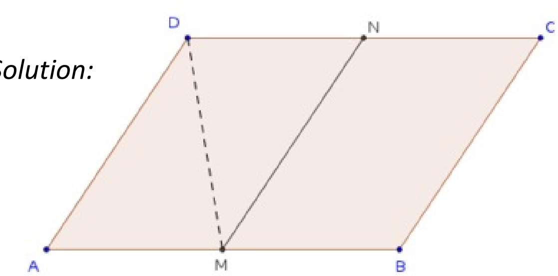
Examples:

Suppose that *ABCD* is a parallelogram, and that *M* and *N* are the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively.

Prove that $MN = AD$, and that the line $\overleftrightarrow{MN}$ is parallel to $\overleftrightarrow{AD}$.



Solution:



Solution:

The diagram above consists of the given information, and one additional line segment, $\overline{MD}$, which we will use to demonstrate the result. We claim that triangles $\Delta AMD$and ΔNDM are congruent by SAS:

We have $\overline{MD} \cong \overline{DM}$ by reflexivity.

We have $\angle AMD \cong \angle NDM$ since they are opposite interior angles of the transversal *MD* through parallel lines *AB* and *CD.*

We have $MA = ND$, since *M* and *N* are midpoints of their respective sides, and opposite sides of parallelograms are congruent $MA = \frac{1}{2}(AB) = \frac{1}{2}(CD) = ND$

Now since corresponding parts of congruent triangles are congruent, we have $\overleftrightarrow{DA} \cong \overleftrightarrow{NM}$, as desired. Similarly, we have congruent opposite interior angles $\angle DMN \cong \angle MDA, so\ MN \parallel AD$.

Instructional Strategies: See G.CO.7-10

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.C.11
    - Is this a parallelogram?
    - Parallelograms and translations
    - Midpoints of the sides of a parallelogram
    - Congruence of parallelograms

Domain: Congruence (G.CO)

Cluster: *Make geometric constructions.*

Standard: G.CO.11

(9/10) Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line. **(G.CO.12)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

Connections: 8.G.6

Explanations and Examples:

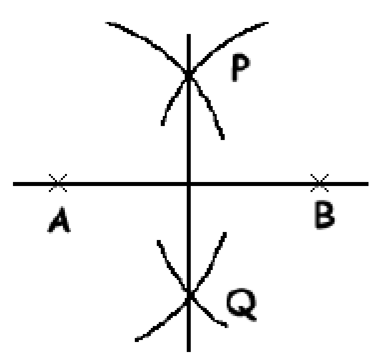
The expectation is to build on prior student experience with simple constructions. Emphasize the ability to formalize and explain how these constructions result in the desired objects. Some of these constructions are closely related to previous standards and can be introduced with them.

Students may use geometric software to make geometric constructions.

Examples:

- Construct a triangle given the lengths of two sides and the measure of the angle between the two sides.
- Construct the circumcenter of a given triangle.
- Construct the perpendicular bisector of a line segment.

This construction can also be used to construct a 90 degree angle or to find the midpoint of a line.

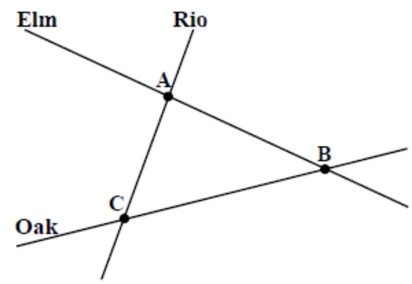


1. Mark two points on your line, A and B - this construction will give you a straight line which passes exactly half way between these two points and is perpendicular (at right angles) to the line.
2. Open your compasses to a distance more than half way between A and B.
3. With the point of the compass on one of the points, draw circular arcs above and below the line, at P and Q.
4. Keeping the compasses set to exactly the same distance, repeat with the compass point on your other point.
5. Draw a line through points P and Q.
6. $\overline{PQ}$ is the perpendicular bisector of $\overline{AB}$ - check that the angles are exactly 90 degrees and that it does indeed halve the distance between A and B.

You have been asked to place a warehouse so that it is an equal distance from the three roads indicated on the following map. Find this location and show your work.

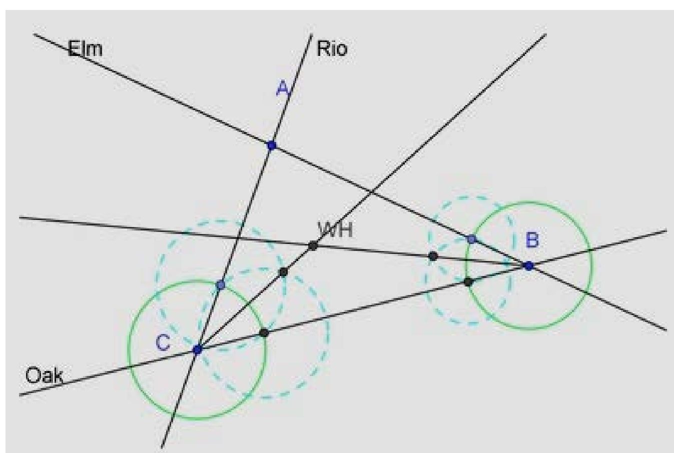
Show how to fold your paper to physically construct this point as an intersection of two creases.

Explain why the above construction works and, in particular, why you only needed to make two creases,



Solution: (This task connects to standard G.C.3)

1. Fold and crease the paper so that Oak lies on top of Rio. Do the same so that Oak lies on top of Elm. The point of intersection of the two creases is the point an equal distance from the three sides.
2. Since the desired location should be an equal distance from three sides of triangle ABC, we are looking for the center of the circle inscribed in the triangle. The center of the inscribed circle, called the incenter, can be found by constructing the angle bisectors of the three interior angles of the triangle, as in the diagram below. Since these angle bisectors are concurrent, it is sufficient to construct two of the angle bisectors (and hence only make two creases in part (a)).



Now we show the concurrence of the three angle bisectors: It is easy to see that the distance from the warehouse *W* = *WH* to Rio equals the distance from *W* to Oak. Namely, draw perpendiculars from *W* to both Rio and Oak, with respective intersection points *X* and *Y*.

The triangles $\triangle WXC$ and $\triangle WYC$ are congruent since they are right triangles with $\angle WCX = \angle WCY$ and sharing side $\overline{WC}$. So $WX = WY$. Similarly, drawing a perpendicular to Elm through *W* meeting Elm at *Z*, we have $WY = WZ$. Combining the two equalities, we learn that $WX = WZ$, so that *W* is on the angle bisector and the three angle bisectors are concurrent.

Students should analyze each listed construction in terms of what simpler constructions are involved (e.g., constructing parallel lines can be done with two different constructions of perpendicular lines).

Using congruence theorems, ask students to prove that the constructions are correct.

Provide meaningful problems (e.g. constructing the centroid or the incenter of a triangle) to offer students practice in executing basic constructions.

Challenge students to perform the same construction using a compass and string. Use paper folding to produce a reflection; use bisections to produce reflections.

Ask students to write “how-to” manuals, giving verbal instructions for a particular construction. Offer opportunities for hands-on practice using various construction tools and methods.

Compare dynamic geometry commands to sequences of compass-and-straightedge steps. Prove, using congruence theorems, that the constructions are correct.

Instructional Strategies:

Students should analyze each listed construction in terms of what simpler constructions are involved (e.g., constructing parallel lines can be done with two different constructions of perpendicular lines).

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Compare dynamic geometry commands to sequences of compass-and-straightedge steps. Prove, using congruence theorems, that the constructions are correct.

Resources/Tools:

Mathematics Assessment Project:

- "Inscribing and Circumscribing Right Triangles" – This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. In particular, it will help you identify and help students who have difficulty:
    - Decomposing complex shapes into simpler ones in order to solve a problem.
    - Bringing together several geometric concepts to solve a problem.
    - Finding the relationship between radii of inscribed and circumscribed circles of right triangles.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-CO.D.12
    - Reflected Triangles
    - Locating Warehouse
    - Construction of perpendicular bisector
    - Bisecting an angle
    - Angle bisection and midpoints of line segments
    - Origami equilateral triangle
    - Origami regular octagon
    - Origami Silver Rectangle

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

Cluster: *Understand similarity in terms of similarity transformation.*

Standard: **G.SRT.1**

(9/10) Use geometric constructions to verify the properties of dilations given by a center and a scale factor:

G.SRT.1a. **(9/10)** A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. **(G.SRT.1a)**

G.SRT.1b. **(9/10)** The dilation of a line segment is longer or shorter in the ratio given by the scale factor. **(G.SRT.1b)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: G.SRT.1-3

Explanations and Examples:

Students should understand that a dilation is a transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

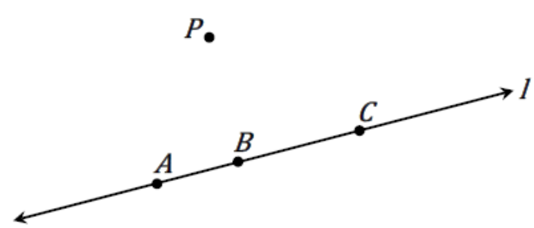
Perform a dilation with a given center and scale factor on a figure in the coordinate plane. Verify that when a side passes through the center of dilation, the side and its image lie on the same line. Verify that corresponding sides of the preimage and images are parallel. Verify that a side length of the image is equal to the scale factor multiplied by the corresponding side length of the preimage.

Students may use geometric simulation software to model transformations. Students may observe patterns and verify experimentally the properties of dilations.

Examples:

Example 1:

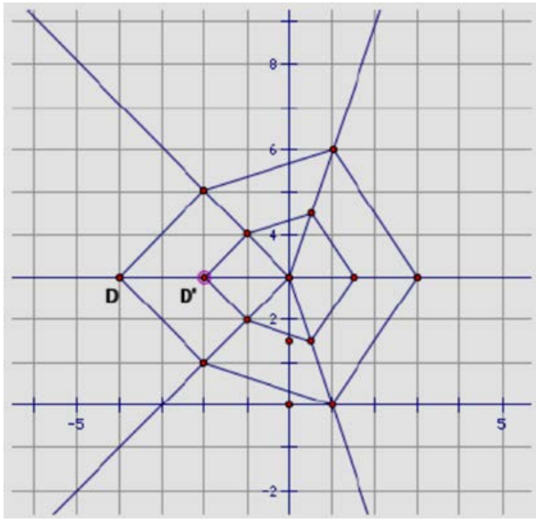
Suppose we apply a dilation by a factor of 2, centered at the point *P* to the figure.



a. In the picture, locate the images *A'*, *B'*, and *C'* of the points A, B, C under this dilation.
b. Based on you picture in part a., what do you think happens to the line *l* when we perform the dilation?
c. Based on your picture in part a., what appears to be the relationship between the distance *A'B'* and the distance *AB?*
d. Can you prove your observations in part c?

Example 2:
Draw a polygon. Pick a point and construct a dilation of the polygon with that point as the center. Identify the scale factor that you used.

Example Response:



Instructional Strategies:

Allow adequate time and hands-on activities for students to explore dilations visually and physically.

Use graph paper and rulers or dynamic geometry software to obtain images of a given figure under dilations having specified centers and scale factors. Carefully observe the images of lines passing through the center of dilation and those not passing through the center, respectively. A line segment passing through the center of dilation will simply be shortened or elongated but will lie on the same line, while the dilation of a line segment that does not pass through the center will be parallel to the original segment (this is intended as a clarification of Standard 1a).

Illustrate two-dimensional dilations using scale drawings and photocopies.

Measure the corresponding angles and sides of the original figure and its image to verify that the corresponding angles are congruent and the corresponding sides are proportional (i.e. stretched or shrunk by the same scale factor). Investigate the SAS and SSS criteria for similar triangles.

Use graph paper and rulers or dynamic geometry software to obtain the image of a given figure under a combination of a dilation followed by a sequence of rigid motions (or rigid motions followed by dilation).

Work backwards – given two similar figures that are related by dilation, determine the center of dilation and scale factor. Given two similar figures that are related by a dilation followed by a sequence of rigid motions, determine the parameters of the dilation and rigid motions that will map one onto the other.

Using the theorem that the angle sum of a triangle is 180°, verify that the AA criterion is equivalent to the AAA criterion. Given two triangles for which AA holds, use rigid motions to map a vertex of one triangle onto the corresponding vertex of the other in such a way that their corresponding sides are in line. Then show that dilation will complete the mapping of one triangle onto the other.

Students may be interested in scale models or experiences with blueprints and scale drawings (perhaps in a work-related situation) to illustrate similarity.

Some students often do not recognize that congruence is a special case of similarity. Similarity with a scale factor equal to 1 becomes a congruency.

Students may not realize that similarities preserve shape, but not size. Angle measures stay the same, but side lengths change by a constant scale factor.

Students may incorrectly apply the scale factor. For example students will multiply instead of divide with a scale factor that reduces a figure or divide instead of multiply when enlarging a figure.

Some students often do not list the vertices of similar triangles in order. However, the order in which vertices are listed is preferred and especially important for similar triangles so that proportional sides can be correctly identified.

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-SRT.A.1
  - "Dilating a line"

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

● **Cluster:** *Understand similarity in terms of similarity transformations.*

Standard: G.SRT.2

(9/10) Recognize transformations as functions that take points in the plane as inputs and give other points as outputs and describe the effect of dilations on two-dimensional figures. **(8.G.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: F.IF and G.CO.2

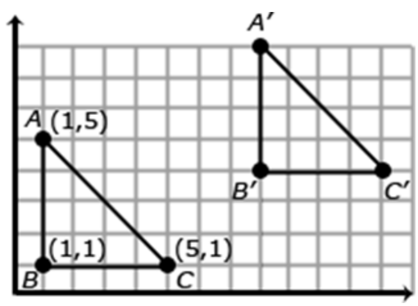
Explanations and Examples:

Students identify resulting coordinates from translations, reflections, and rotations (90°, 180°and 270° both clockwise and counterclockwise), recognizing the relationship between the coordinates and the transformation. For example, a translation of 5 left and 2 up would subtract 5 from the *x*-coordinate and add 2 to the y-coordinate. $D(-4,-3) \rightarrow D'(-9,-1)$. A reflection across the *x*-axis would change $(6,-8) \rightarrow B'(6,8)$.

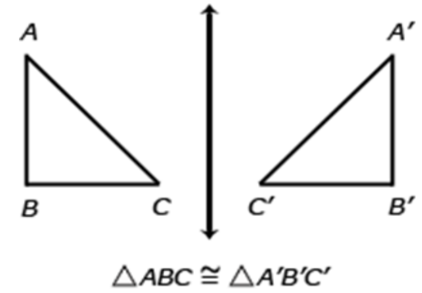
Additionally, students recognize the relationship between the coordinates of the pre-image, the image and the scale factor following a dilation from the origin. Dilations are non-rigid transformations that enlarge (scale factors greater than one) or reduce (scale factors less than one) the size of a figure using a scale factor.

A dilation is a transformation that moves each point along a ray emanating from a fixed center, and multiplies distances from the center by a common scale factor. In dilated figures, the dilated figure is *similar* to its pre-image.

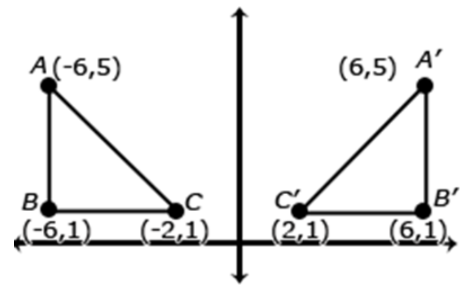
Translation: A translation is a transformation of an object that moves the object so that every point of the object moves in the same direction as well as the same distance. In a translation, the translated object is *congruent* to its pre-image. ΔABC has been translated 7 units to the right and 3 units up. To get from A (1,5) to A' (8,8), move A 7 units to the right (from *x* = 1 to *x* = 8) and 3 units up (from *y* = 5 to *y* = 8). Points B and C also move in the same direction (7 units to the right and 3 units up).



Reflection: A reflection is a transformation that flips an object across a line of reflection (in a coordinate grid the line of reflection may be the x or y axis). In a rotation, the rotated object is *congruent* to its pre-image.

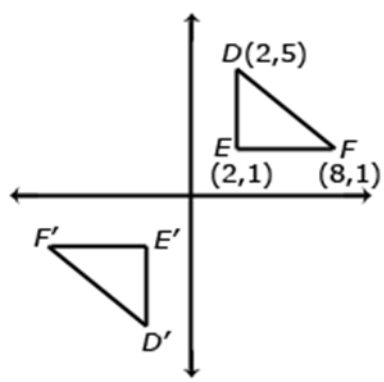


When an object is reflected across the y axis, the reflected x coordinate is the opposite of the pre-image x coordinate.



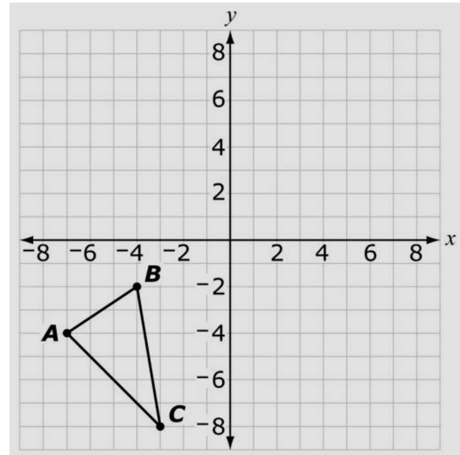
Rotation: A rotated figure is a figure that has been turned about a fixed point. This is called the center of rotation. A figure can be rotated up to 360˚. Rotated figures are congruent to their pre-image figures.

Consider when ΔDEF is rotated 180˚ clockwise about the origin. The coordinates of ΔDEF are D(2,5), E(2,1), and F(8,1). When rotated 180˚, $\Delta D'E'F'$ has new coordinates D'(-2,-5), E'(-2,-1) and F'(-8,-1). Each coordinate is the opposite of its pre-image.



Examples:

Triangle ABC is shown on this coordinate grid.



Part A

$\triangle ABC$ is rotated 180 degrees clockwise about the origin to form $\triangle DEF$.

What are the coordinates of the vertices of $\triangle$ DEF?

D(☐ ☐), E(☐ ☐), F(☐ ☐)

Part B

What conjecture can be made about the relationship between the coordinates of the vertices of an original shape and the coordinates of the vertices of the image of the shape when it is rotated 180 degrees clockwise about the origin?

You must demonstrate that the conjecture is always true or that there is at least one example in which the conjecture is not true.

Sample Response:

Part A

$D(7{,}4), E(4{,}2). F(3{,}8)$

Part B

The conjecture is that the coordinates of the vertices of the image will have the opposite sign of the coordinates of the vertices of the original shape. When a point is rotated 180 degrees clockwise about the origin, if a line is drawn through the original point and the origin, the image of the point will also be on the line, and it will be the same distance from the origin that the original point was, but on the opposite side of the origin. When two points on the same line are the same distance from the origin and on opposite sides of the origin, the coordinates of the points have opposite signs, because the slope from each coordinate to the origin is the same, but to move from the origin to each point to get its coordinates, you must move in opposite directions. So if you move right from the origin to get to one point, you will move left to get to the other, and if you move up from the origin to get to one point, you will move down to get to the other. So the coordinates of the vertices of the image will have the opposite sign of the coordinates of the vertices of the original shape.

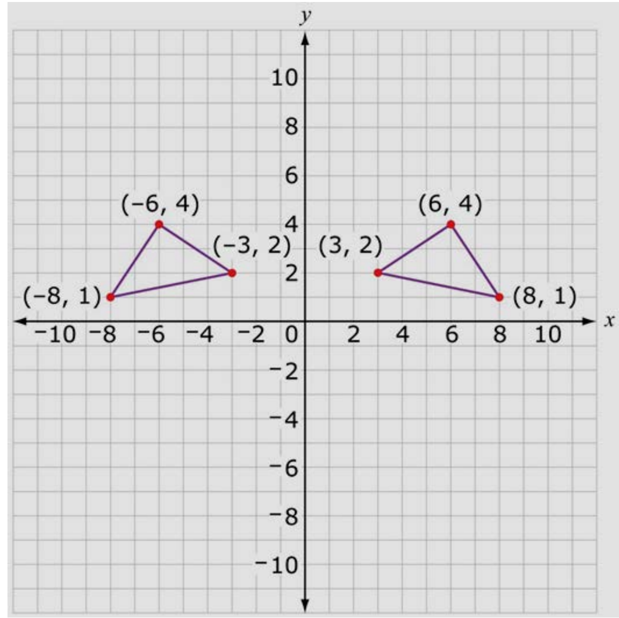
A student made this conjecture about reflections on an *xy*-coordinate plane.

When a polygon is reflected over the *y*-axis, the *x*-coordinates of the corresponding vertices of the polygon and its image are opposite, but the *y*-coordinates are the same.

Develop a chain of reasoning to justify or refute the conjecture. You must demonstrate that the conjecture is always true or that there is at least one example in which the conjecture is not true. You may include one or more graphs in your response.

Sample Response:

When a polygon is reflected over the *y*-axis, each vertex of the reflected polygon will end up on the opposite side of the *y*-axis but the same distance from the *y*-axis. So, the *x*- coordinates of the vertices will change from positive to negative or negative to positive, but the absolute value of the number will stay the same, so the *x*-coordinates of the corresponding vertices of the polygon and its image are opposites. Since the polygon is being reflected over the *y*-axis, the image is in a different place horizontally but it does not move up or down, which means the *y*-coordinates of the vertices of the image will be the same as the *y*-coordinates of the corresponding vertices of the original polygon.
As an example, look at the graph below, and notice that the *x*-coordinates of the corresponding vertices of the polygon and its image are opposites but the *y*-coordinates are the same.
This means the conjecture is correct.



Instructional Strategies: See. G.CO

Resources/Tools:

Illustrative Mathematics **Grade 8** tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.A.3
  - Reflecting reflections
  - Triangle congruence with coordinates
  - Point Reflection
  - Effects of Dilations on Length, Area, and Angles

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

Cluster: *Understand similarity in terms of similarity transformations.*

Standard: **G.SRT.3**

(9/10) Given two similar figures, describe a **sequence** of transformations that exhibits the similarity between them using coordinates and the non-coordinate plane. **(8.G.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

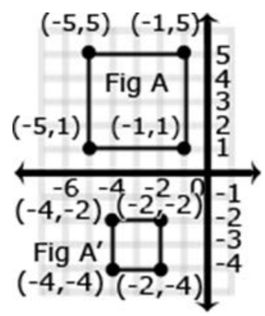
Connections: G.SRT.1, G.CO.3-4

Explanations and Examples:

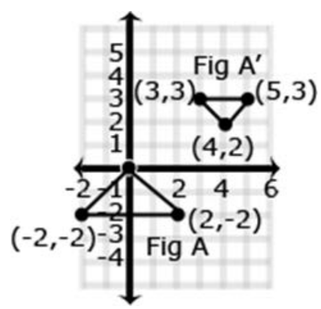
This is the students' introduction to similarity and similar figures. Students understand similar figures have angles with the same measure and sides that are proportional. Similar figures are produced from dilations. Students describe the sequence that would produce similar figures, including the scale factors. Students understand that a scale factor greater than one will produce an enlargement in the figure, while a scale factor less than one will produce a reduction in size.

Examples:

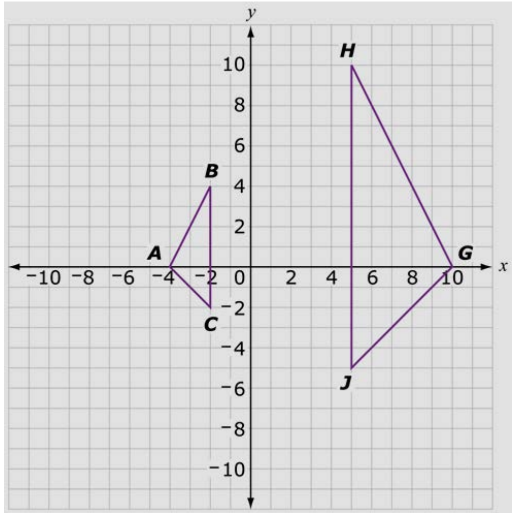
Is Figure A similar to Figure A'? Explain how you know.



Describe the sequence of transformations that results in the transformation of Figure A to Figure A'.



A transformation is applied to $\triangle ABC$ to form $\triangle DEF$ (not shown). Then, a transformation is applied to $\triangle DEF$ to form $\triangle GHJ$.



Part A
Graph ΔDEF on the xy-coordinate plane.

Part B
Describe the transformation applied to $\triangle DEF$ to form $\triangle ABC$.

Part C
Describe the transformation applied to $\triangle DEF$ to form $\triangle GHJ$.

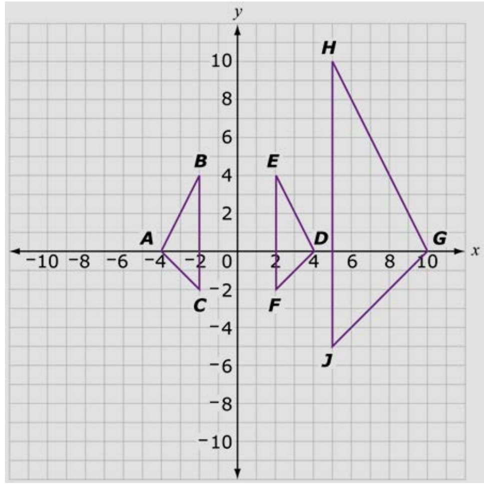
Part D
Select one statement that applies to the relationship between $\triangle$ GHJ and $\triangle$ ABC.

- $\triangle GHJ$ is congruent to $\triangle ABC$.
- $\triangle GHJ$ is similar to $\triangle ABC$.
- $\triangle GHJ$ is neither congruent nor similar to $\triangle ABC$.

Explain your reasoning.

***Sample Response*:**

Part A



Part B

A reflection over the *y*-axis

Part C

A dilation with a scale factor of 2.5 about the origin.

Part D

$\triangle GHJ$ is similar to $\triangle ABC$.

A dilation followed by a congruence, or a congruence followed by a dilation, is a similarity.
So, $\triangle GHJ$ is similar to $\triangle ABC$.

Instructional Strategies: See G.CO

Resources/Tools:

Illustrative Mathematics **Grade 8** tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.A.4
  - Are They Similar?
  - Creating Similar Triangles

Also see engageNY Modules: https://www.engageny.org/resource/grade-8-mathematics

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

● **Cluster:** *Understand similarity in terms of similarity transformations.*

Standard: **G.SRT.4**

(9/10) Understand the meaning of similarity for two-dimensional figures as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides. **(G.SRT.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: G.SRT.1-2

Explanations and Examples:

Use the idea of dilation transformations to develop the definition of similarity. Understand that a similarity transformation is a rigid motion followed by a dilation.

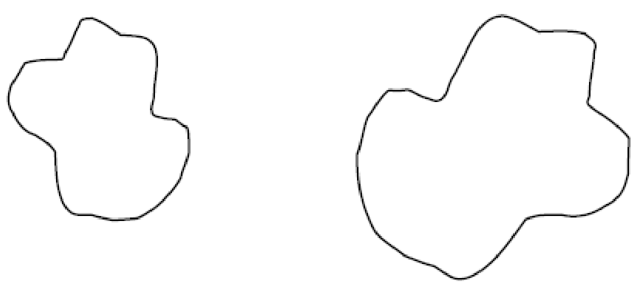
Demonstrate that in a pair of similar triangles, corresponding angles are congruent (angle measure is preserved) and corresponding sides are proportional.

Determine that two figures are similar by verifying that angle measure is preserved and corresponding sides are proportional.

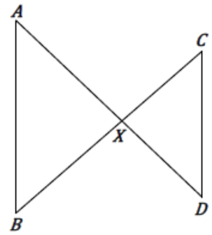
Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:

Are these two figures similar? Explain why or why not



In the picture below, line segments *AD* and *BC* intersect at *X*. Line segments *AB* and *CD* are drawn, forming two triangles $\triangle AXB$ and $\triangle$ CXD*.*



In each part a-d below, some additional assumptions about the picture are given. In each problem, determine whether the given assumptions are enough to prove that the two triangles are similar, and if so, what the correct correspondence of vertices is. If the two triangles must be similar, prove this result by describing a sequence of similarity transformations that maps one variable to the other. If not explain why not.

a. The lengths of $\overline{AX}$and $\overline{AD}$ satisfy the equation $2\overline{AX} = 3\overline{XD}$.
b. The lengths $\overline{AX}, \overline{BX}, \overline{CX}, and\ \overline{DX},$ satisfy the equation $\frac{\overline{AX}}{\overline{BX}} = \frac{\overline{DX}}{\overline{CX}}$
c. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
d. $\angle XAB \cong \angle XC$

Solution:

a. We are given that 2AX = 3XD. This is not enough information to prove similarity. To see that in a simple way draw an arbitrary triangle △AXB. Extend AX and choose a point D on the extended line so that 2AX=3XD. Extend BX and choose a point C on the extended line so that 2BX=XC. Now triangles AXB and CXD satisfy the given conditions but are not similar. (If you are extremely unlucky, AXB and CXD might be similar by a different correspondence of sides. If this happens, rotate the line BC a little bit. The lengths of AX, XD, BX, XC remain the same but the triangles are no longer similar.)

b. We are given that $\frac{AX}{BX} = \frac{DX}{CX}$Rearranging this proportion gives $\frac{AX}{DX} = \frac{BX}{CX}$Let $k = \frac{AX}{DX}$Suppose we rotate the triangle DXC 180 degrees about point X, as in part (a), so that the angle DXC coincides with angle AXB. Then dilate the triangle DXC by a factor of k about the center X. This dilation moves the point D to A, since k(DX) = AX, and moves C to B, since k(CX) = BX. Then since the dilation fixes X, and dilations take line segments to line segments, we see that the triangle DXC is mapped to triangle AXB. So the original triangle DXC is similar to AXB. (Note that we state the similarity so that the vertices of each triangle are written in corresponding order.)

c. Again, rotate triangle DXC so that angle DXC coincides with angle AXB. Then the image of the side CD under this rotation is parallel to the original side CD, so the new side CD is still parallel to side AB. Now, apply a dilation about point X that moves the vertex C to point B. This dilation moves the line CD to a line through B parallel to the previous line CD. We already know that line AB is parallel to CD, so the dilation must move the line CD onto the line AB. Since the dilation moves D to a point on the ray XA and on the line AB, D must move to A. Therefore, the rotation and dilation map the triangle DXC to the triangle AXB. Thus DXC is similar to AXB.

d. Suppose we draw the bisector of angle AXC, and reflect the triangle CXD across this angle bisector. This maps the segment XC onto the segment XA; and since reflections preserve angles, it also maps segment XD onto segment XB. Since angle XCD is congruent to angle XAB, we also know that the image of side CD is parallel to side AB. Therefore, if we apply a dilation about the point X that takes the new point C to A, then the new line CD will be mapped onto the line AB, by the same reasoning used in (c). Therefore, the new point D is mapped to B, and thus the triangle XCD is mapped to triangle XAB. So triangle XCD is similar to triangle XAB. (Note that this is not the same correspondence we had in parts (b) and (c)!)

Instructional Strategies: G.CO

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-SRT.A.2
  - Are They Similar?

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

◆ **Cluster:** *Construct arguments about theorems involving similarity.*

Standard: **G.SRT.5**

(9/10) Construct arguments about triangles using theorems. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity, and AA. **(G.SRT.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.

Connections: G.SRT.4-5, 8.G.7-9

Explanations and Examples:

Use AA, SAS, SSS similarity theorems to prove triangles are similar.

Use triangle similarity to prove other theorems about triangles

- Prove a line parallel to one side of a triangle divides the other two proportionally, and its converse
- Prove the Pythagorean Theorem using triangle similarity.

Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:

Example 1:

Prove that if two triangles are similar, then the ratio of corresponding attitudes is equal to the ratio of corresponding sides.

Example 2:

How does the Pythagorean Theorem support the case for triangle similarity?

- o View the video below and create a visual proving the Pythagorean Theorem using similarity. http://www.youtube.com/watch?v=LrS5_l-gk94

Example 3:

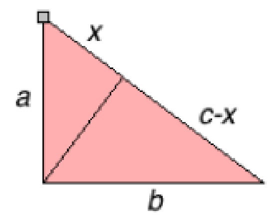
To prove the Pythagorean Theorem using triangle similarity:

We can cut a right triangle into two parts by dropping a perpendicular onto the hypotenuse.

Since these triangles and the original one have the same angles, all three are similar.

Therefore

$$\frac{x}{a} = \frac{a}{c}, \frac{c-x}{b} = \frac{b}{c}$$
$$x = \frac{a^2}{c}, c - x = \frac{b^2}{c}$$
$$x = (c - x) = c$$
$$\frac{a^2}{c} + \frac{b^2}{c} = c$$
$$a^2 + b^2 = c^2$$



http://www.math.ubc.ca/~cass/euclid/java/html/pythagorassimilarity.html

Instructional Strategies:

Review triangle congruence criteria and similarity criteria, if it has already been established.
Review the angle sum theorem for triangles, the alternate interior angle theorem and its converse, and properties of parallelograms. Visualize it using dynamic geometry software.

Using SAS and the alternate interior angle theorem, prove that a line segment joining midpoints of two sides of a triangle is parallel to and half the length of the third side. Apply this theorem to a line segment that cuts two sides of a triangle proportionally.

Generalize this theorem to prove that the figure formed by joining consecutive midpoints of sides of an arbitrary quadrilateral is a parallelogram. (This result is known as the Midpoint Quadrilateral Theorem or Varignon's Theorem.)

Use cardboard cutouts to illustrate that the altitude to the hypotenuse divides a right triangle into two triangles that are similar to the original triangle. Then use AA to prove this theorem. Then, use this result to establish the Pythagorean relationship among the sides of a right triangle ($a^2 + b^2 = c^2$) and thus obtain an algebraic proof of the Pythagorean Theorem.

Prove that the altitude to the hypotenuse of a right triangle is the geometric mean of the two segments into which its foot divides the hypotenuse.

Prove the converse of the Pythagorean Theorem, using the theorem itself as one step in the proof. Some students might engage in an exploration of Pythagorean Triples (e.g., 3-4-5, 5-12-13, etc.), which provides an algebraic extension and an opportunity to explore patterns.

Resources/Tools:

Mathematics Assessment Project:

- "Solving Geometry Problems: Floodlights" – This lesson unit is intended to help you assess how well students are able to identify and use geometrical knowledge to solve a problem. In particular, this unit aims to identify and help students who have difficulty in:
  - Making a mathematical model of a geometrical situation.
  - Drawing diagrams to help with solving a problem.
  - Identifying similar triangles and using their properties to solve problems.
  - Tracking and reviewing strategic decisions when problem-solving.
- "Proofs of the Pythagorean Theorem" – This lesson unit is intended to help you assess how well students are able to produce and evaluate geometrical proofs. In particular, this unit is intended to help you identify and assist students who have difficulties in:
  - Interpreting diagrams.
  - Identifying mathematical knowledge relevant to an argument.
  - Linking visual and algebraic representations.
  - Producing and evaluating mathematical arguments.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-SRT.B.4
  - Joining two midpoints of sides of a triangle
  - Pythagorean Theorem

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

◆ **Cluster:** *Construct arguments about theorems involving similarity.*

Standard: **G.SRT.6**

(9/10) Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. **(G.SRT.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.

Connections: G.CO.3-8, G.SRT.3-4

Explanations and Examples:

Similarity postulates include SSS, SAS, and AA.

Congruence postulates include SSS, SAS, ASA, AAS, and H-L.

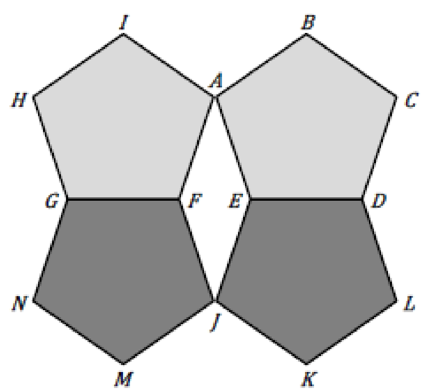
Apply triangle congruence and triangle similarity to solve problem situations (e.g., indirect measurement, missing sides/angle measures, side splitting).

Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:

This diagram is made up of four regular pentagons that are all the same size.

- Find the measure of $\angle AEJ$
- Find the measure of $\angle BJF$
- Find the measure of $\angle KJM$



Instructional Strategies: See G.SRT.5

Resources/Tools:

Mathematics Assessment Project:

- "Analyzing Congruence Proofs" –This lesson unit is intended to help you assess how well students are able to:
  - Work with concepts of congruency and similarity, including identifying corresponding sides and corresponding angles within and between triangles.
  - Identify and understand the significance of a counter-example.
  - Prove, and evaluate proofs in a geometric context.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-SRT.B.5
  - Bank Shot
  - Extensions, Bisections and Dissections in a Rectangle
  - Folding a square into thirds
  - Tangent Line to Two Circles
  - Congruence of parallelograms

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

◆ **Cluster:** *Define trigonometric ratios and solve problems involving right triangles.*

Standard: **G.SRT.7**

(9/10) Show that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles. **(G.SRT.6)**

Suggested Standards for Mathematical Practice (MP):

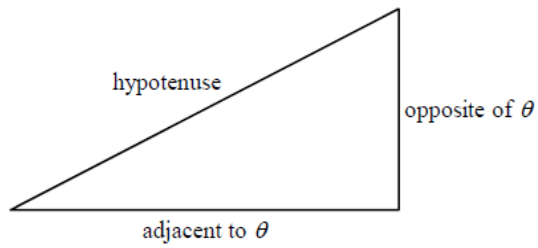
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: G.SRT5-7

Explanations and Examples:

Students may use applets to explore the range of values of the trigonometric ratios as θ ranges from 0 to 90 degrees.

Use the characteristics of similar figures to justify trigonometric ratios.



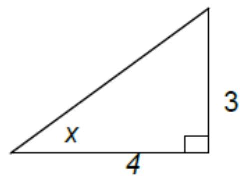
$$\text{sine of } \theta = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \text{cosecant of } \theta = \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

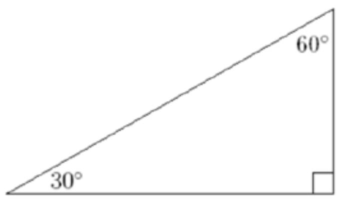
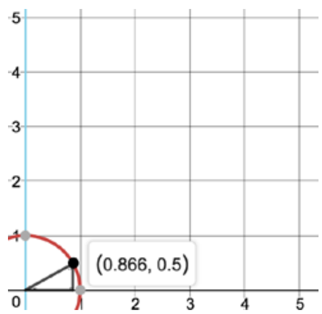
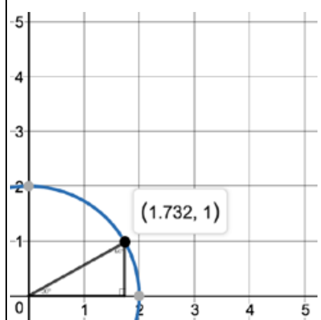
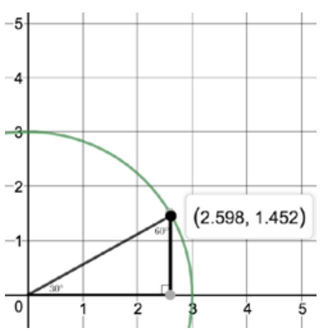
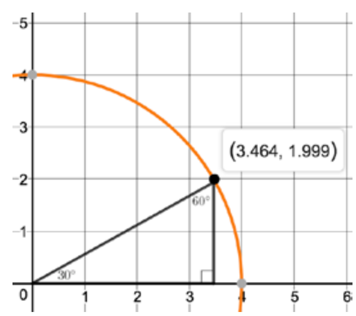
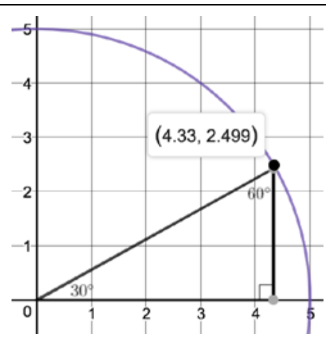
$$\text{cosine of } \theta = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{secant of } \theta = \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\text{tangent of } \theta = \tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \text{cotangent of } \theta = \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Examples:

Find the sine, cosine, and tangent of *x*.



	Each of the triangles below are similar by AAA similarity. In this case, the angles are 30-60-90. Let's calculate the side ratio of the measured legs.
 $a = 0.866$ $b = 0.5$ $c = 1$ $\frac{a}{b} = \frac{0.866}{0.5} = 1.732$	 $a = 1.732$ $b = 1$ $c = 2$ $\frac{a}{b} = \frac{1.732}{1} = 1.732$
 $a = 2.598$ $b = 1.452$ $c = 3$ $\frac{a}{b} = \frac{2.598}{1.452} = 1.789$	 $a = 3.464$ $b = 1.999$ $c = 4$ $\frac{a}{b} = \frac{3.464}{1.999} = 1.732$
 $a = 4.33$ $b = 2.499$ $c = 5$ $\frac{a}{b} = \frac{4.33}{2.499} = 1.732$	So the ratio of the legs is 1.732, regardless of the size of the triangle, because the triangles are similar. For this example, the ratio is $\tan 30° = 1.732$ for every right triangle with a $30°$ angle.

Instructional Strategies:

Review vocabulary (opposite and adjacent sides, legs, hypotenuse and complementary angles) associated with right triangles.

Make cutouts or drawings of right triangles or manipulate them on a computer screen using dynamic geometry software and ask students to measure side lengths and compute side ratios. Observe that when triangles satisfy the AA criterion, corresponding side ratios are equal. Side ratios are given standard names, such as sine, cosine and tangent. Allow adequate time for students to discover trigonometric relationships and progress from concrete to abstract understanding of the trigonometric ratios.

Show students how to use the trigonometric function keys on a calculator. Also, show how to find the measure of an acute angle if the value of its trigonometric function is known.

Investigate sines and cosines of complementary angles, and guide students to discover that they are equal to one another. Point out to students that the "co" in cosine refers to the "sine of the complement."

Observe that, as the size of the acute angle increases, sines and tangents increase while cosines decrease.

Stress trigonometric terminology by the history of the word "sine" and the connection between the term "tangent" in trigonometry and tangents to circles.

Have students make their own diagrams showing a right triangle with labels showing the trigonometric ratios. Although students like mnemonics such as SOHCAHTOA, *these are not a substitute for conceptual understanding*. Some students may investigate the reciprocals of sine, cosine, and tangent to discover the other three trigonometric functions.

Use the Pythagorean Theorem to obtain exact trigonometric ratios for 30°, 45°, and 60° angles.

Use cooperative learning in small groups for discovery activities and outdoor measurement projects.

Have students work on applied problems and project, such as measuring the height of the school building or a flagpole, using clinometers and the trigonometric functions.

Resources/Tools:

Mathematics Assessment Project:

- "Geometry Problems: Circles and Triangles" – This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. In particular, the lesson will help you identify and help students who have the following difficulties:
  - Solving problems by determining the lengths of the sides in right triangles.
  - Finding the measurements of shapes by decomposing complex shapes into simpler ones.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G.SRT.C
  - Finding the Area of an Equilateral Triangle
  - Mt. Whitney to Death Valley

Domain: Similarity, Right Triangles, and Trigonometry ★ (G.SRT)

◆ **Cluster:** *Define trigonometric ratios and solve problems involving right triangles.*

Standard: **G.SRT.8 ★**

(9/10) Explain and use the relationship between the sine and cosine of complementary angles. **(G.SRT.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.

Connections: G.SRT.7

Explanations and Examples:

Calculate sine and cosine ratios for acute angles in a right triangle when given two side lengths.

Use a diagram of a right triangle to explain that for a pair of complimentary angles *A* and *B*, the sine of angle *A* is equal to the cosine of angle *B* and the cosine of angle *A* is equal to the sine of angle *B.*

Geometric simulation software applets and graphing calculators can be used to explore the relationship between sine and cosine.

Examples:

Example1:

What is the relationship between cosine and sine in relation to complementary angles?

- Construct a table demonstrating the relationship between sine and cosine of complementary angles.

Example2:

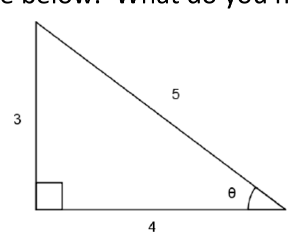
Find the second acute angle of a right triangle given that the first acute angles has a measure of 39°.

Example 3:

Complete the following statement: If $\sin 30^{\circ} = \frac{1}{2}$ then the $\cos \qquad = \frac{1}{2}$.

Example 4:

Find the sine and cosine of angle θ in the triangle below. What do you notice?



Instructional Strategies: See G.SRT.6

Resources/Tools: See G.SRT.6

▶ **Major Clusters** ◆ **Supporting Clusters** ● **Additional Clusters**

Domain: Similarity, Right Triangles, and Trigonometry (G.SRT)

◆ **Cluster:** *Define trigonometric ratios and solve problems involving right triangles.*

Standard: G.SRT.9

(9/10) Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.★ **(G.SRT.8)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

Connections: G.SRT.7

Explanations and Examples:

Use angle measures to estimate side lengths (e.g., The side across from a 33° angle will be shorter than the side across from a 57° angle).

Use side lengths to estimate angle measures (e.g., The angle opposite of a 10 cm side will be larger than the angle across from a 9 cm side).

Draw right triangles that describe real world problems and label the sides and angles with their given measures.

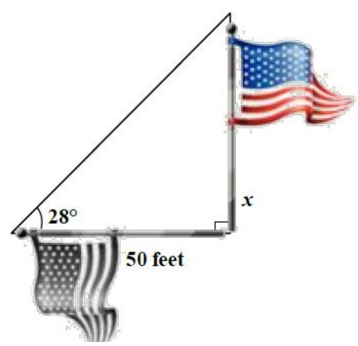
Solve application problems involving right triangles, including angle of elevation and depression, navigation, and surveying.

Students may use graphing calculators or programs, tables, spreadsheets, or computer algebra systems to solve right triangle problems.

Examples:

Example1:

Find the height of a flagpole to the nearest tenth if the angle of elevation of the sun is 28° and the shadow of the flagpole is 50 feet.



Example2:

A teenager whose eyes are 5 feet above ground level is looking into a mirror on the ground and can see the top of a building that is 30 feet away from the teenager. The angle of elevation from the center of the mirror to the top of the building is 75°. How tall is the building? How far away from the teenager's feet is the mirror?

Example3:

While traveling across flat land, you see a mountain directly in front of you. The angle of elevation to the peak is 3.5°. After driving 14 miles closer to the mountain, the angle of elevation is 9°24'36". Explain how you would set up the problem, and find the approximate height of the mountain.

Instructional Strategies: See G.SRT.7

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-SRT.C.8
  - Shortest line segment from a point P to a line L
  - Ask the Pilot

Domain: Circles (G.C)

● **Cluster:** *Understand and apply theorems about circles.*

Standard: **G.C.1**
(9/10) Construct arguments that all circles are similar. **(G.C.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.

Connections: G.C.1-2

Explanations and Examples:

Using the fact that the ratio of diameter to circumference is the same for circles, prove that all circles are similar.

Prove that all circles are similar by showing that for a dilation centered at the center of a circle, the preimage and the image have equal central angle measures.

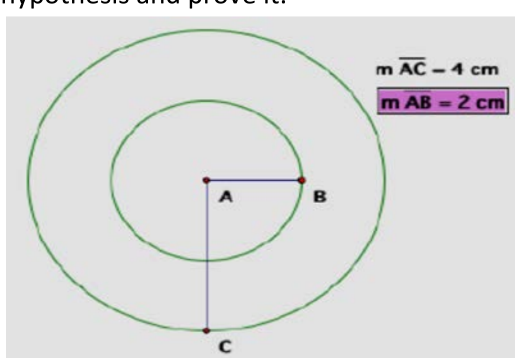
Students may use geometric simulation software to model transformations and demonstrate a sequence of transformations to show congruence or similarity of figures.

Examples:

Example 1:
Show that the two given circles are similar by stating the necessary transformations from ***C*** to ***D***.

C: center (2, 3) radius 5
D: center (–1, 4) radius 10

Example2:
Draw or find examples of several different circles. In what ways are they related? How can you describe this relationship in terms of geometric ideas? Form a hypothesis and prove it.



Instructional Strategies:

Given any two circles in a plane, show that they are related by dilation. Guide students to discover the center and scale factor of this dilation and make a conjecture about all dilations of circles.

Starting with the special case of an angle inscribed in a semicircle, use the fact that the angle sum of a triangle is 180° to show that this angle is a right angle. Using dynamic geometry, students can grab a point on a circle and move it to see that the measure of the inscribed angle passing through the endpoints of a diameter is always 90°. Then extend the result to any inscribed angles. For inscribed angles, proofs can be based on the fact that the measure of an exterior angle of a triangle equals the sum of the measures of the nonadjacent angles. Consider cases of acute or obtuse inscribed angles.

- Use properties of congruent triangles and perpendicular lines to prove theorems about diameters, radii, chords, and tangent lines.
- Use formal geometric constructions to construct perpendicular bisectors of the sides and angle bisectors of a given triangle. Their intersections are the centers of the circumscribed and inscribed circles, respectively.
- Dissect an inscribed quadrilateral into triangles, and use theorems about triangles to prove properties of these quadrilaterals and their angles.
- Challenge students to generalize the results about angle sums of triangles and quadrilaterals to a corresponding result for *n*-gons.
- Students sometimes confuse inscribed angles and central angles. For example they will assume that the inscribed angle is equal to the arc like a central angle.
- Students may think they can tell by inspection whether a line intersects a circle in exactly one point. It may be beneficial to formally define a tangent line as the line perpendicular to a radius at the point where the radius intersects the circle.
- Students may confuse the segment theorems. For example, they will assume that the inscribed angle is equal to the arc like a central angle.

Resources/Tools:

Mathematics Assessment Project:

- "Inscribing and Circumscribing Right Triangles" – This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. In particular, it will help you identify and help students who have difficulty:
    - Decomposing complex shapes into simpler ones in order to solve a problem.
    - Bringing together several geometric concepts to solve a problem.
    - Finding the relationship between radii of inscribed and circumscribed circles of right triangles.
- "Geometry Problems: Circles and Triangles" – This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. The lesson will help you identify and help students who have the following difficulties:
    - Solving problems by determining the lengths of the sides in right triangles.
    - Finding the measurements of shapes by decomposing complex shapes into simpler ones.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-C.A.1
    - Similar Circles

Domain: Circles (G.C)

● **Cluster:** *Understand and apply theorems about circles.*

Standard: **G.C.2**

(9/10) Identify and describe relationships among **inscribed** angles, radii, and chords. Include the relationship between central, inscribed, and **circumscribed** angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle. **(G.C.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

Connections: G.C.1, 8.G

Explanations and Examples:

Identify central angles, inscribed angles, circumscribed angles, diameters, radii, chords, and tangents.

Describe the relationship between a central angle and the arc it intercepts.

Describe the relationship between an inscribed angle and the arc it intercepts.

Describe the relationship between a circumscribed angle and the arcs it intercepts.

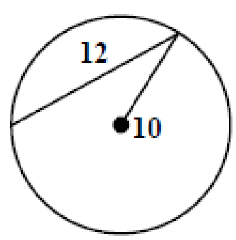
Recognize that an inscribed angle whose sides intersect the endpoints of the diameter of a circle is a right angle.

Recognize that the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

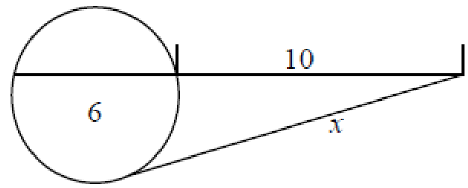
Examples:

Example 1:

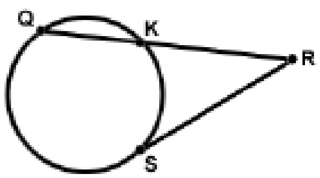
Given the circle below with radius of 10 and chord length of 12, find the distance from the chord to the center of the circle.



Example 2:
Find the unknown length in the picture below.



Solution:
The theorem for a secant segment and a tangent segment that share an endpoint not on the circle states that for the picture below secant segment QR and the tangent segment SR share and endpoint R, not on the circle. Then the length of SR squared is equal to the product of the lengths of QR and KR.



$x^2 = 16 \cdot 10$
So for the example above $x^2 = 160$
$x = \sqrt{160} = 4\sqrt{10} \approx 12.6$

How does the angle between a tangent to a circle and the line connecting the point of tangency and the center of the circle change as you move the tangent point?

Instructional Strategies: See G.C.1

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-C.A.2
    - Right triangles inscribed in circles I
    - Right triangles inscribed in circles II
    - Tangent Lines and the Radius of a Circle
    - Neglecting the Curvature of the Earth

Domain: Circles (G.C)

Cluster: *Understand and apply theorems about circles.*

Standard: **G.C.3**

(9/10) Construct arguments using properties of polygons inscribed and circumscribed about circles. **(G.C.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.

Connections: G.C.1-5

Explanations and Examples:

This standard was split into two standards for the 2017 Kansas Math Standards. The expectation *for all* students is to "construct arguments using properties of polygons…" but, based on teacher discretion, will students construct inscribed and circumscribed circles.

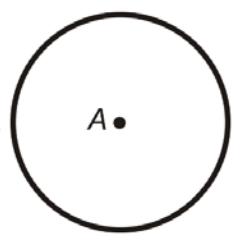
An example lesson:

Inscribed Quadrilaterals in Circles An inscribed polygon is a polygon where every vertex is on a circle. Note, that not every quadrilateral or polygon can be inscribed in a circle. Inscribed quadrilaterals are also called cyclic quadrilaterals. For these types of quadrilaterals, they must have one special property. We will investigate it here.

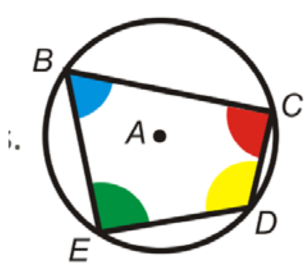
Investigation: Inscribing Quadrilaterals

Tools Needed: pencil, paper, compass, ruler, colored pencils, and scissors

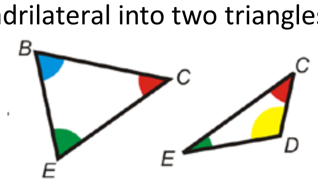
1. Draw a circle. Mark the center point A.



2. Place four points on the circle. Connect them to form a quadrilateral. Color the 4 angles of the quadrilateral 4 different colors.



3. Cut out the quadrilateral. Then cut the quadrilateral into two triangles, by cutting on a diagonal.

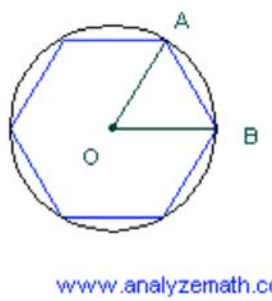


4. Line up ∠B and ∠D so that they are adjacent angles. What do you notice? What does this show?

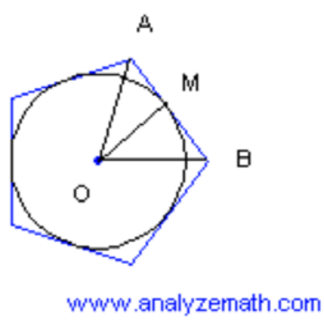
This investigation shows that the opposite angles in an inscribed quadrilateral are supplementary. By cutting the quadrilateral in half, through the diagonal, we were able to show that the other two angles (that we did not cut through) formed a linear pair when matched up.

Inscribed Quadrilateral Theorem: A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

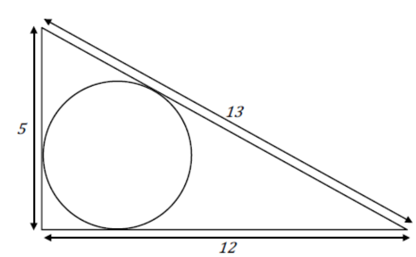
Example 1:
A 6 sided regular polygon (hexagon) is inscribed in a circle of radius 10 cm, find the length of one side of the hexagon.



Example 2:
A circle of radius 6 cm is inscribed in a 5-sided regular polygon (pentagon), find the length of one side of the pentagon.

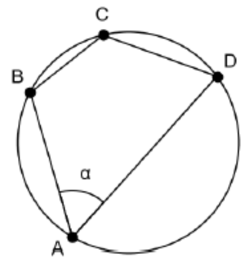


Example 3:
The following diagram shows a circle that just touches the sides of a right triangle whose sides are 5 units, 12 units, and 13 units long. Draw radius lines as in the previous task and find the radius of the circle in this 5, 12, 13 right triangle. Explain your work and show your calculations.



Example 4:

Given the inscribed quadrilateral below show that $\angle B$ is supplementary to $\angle D$.



Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-C.A.3
    - Inscribing a triangle in a circle
    - Inscribing a circle in a triangle I
    - Inscribing a circle in a triangle II
    - Circumcenter of a triangle
    - Warehouse
    - Placing a Fire Hydrant

Domain: Expressing Geometric Properties with Equations (G.GPE)

● **Cluster:** *Translate between the geometric description and the equation for a conic section.*

Standard: **G.GPE.1**

(9/10) Write the equation of a circle given the center and radius or a graph of the circle; use the center and radius to graph the circle in the coordinate plane. **(G.GPE.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: 8.G.7-9, G.SRT.5

Explanations and Examples:

Because completing the square is an 11th grade expectation, students in 9/10 are no longer expected to derive the equation for a circle or complete the square to find the equation. Instead, students will be given the center and radius and should write the equation for the circle in standard form or graph the circle.

Examples:

- Write an equation for a circle with a radius of 2 units and center at (1, 3).
- Write and equation for a circle given that the endpoints of the diameter are (–2, 7) and (4, –8).
- Graph a circle with a center at (0, -5) and a radius of 3 units.

Instructional Strategies:

Compare and contrast features of the graph and equation for parabolas and circles so that students can synthesize both types of equations. This is also one of the first equations that the students will study that is not a function. This is also an opportunity to work on N.Q.1 and appropriately scaling the graph. Many technology-created graphs will create a graph that does not look circular.

Resources/Tools:

Mathematics Assessment Project:

- "Equations of Circles 1" – This lesson unit is intended to help you assess how well students are able to:
  - Use the Pythagorean Theorem to derive the equation of a circle.
  - Translate between the geometric features of circles and their equations.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-GPE.A.1
  - Slopes and Circles
  - Explaining the equation for a circle

Domain: Expressing Geometric Properties with Equations (G.GPE)

▶ **Cluster:** *Use coordinates to prove simple geometric theorems algebraically.*

Standard: **G.GPE.6**

(9/10) Use coordinates to prove simple geometric theorems algebraically, including the use of slope, distance, and midpoint formulas *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle.* **(G.GPE.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: 8.EE.5, 8.G.7-9, 7.SP.4, G.GPE.7

Explanations and Examples:

Represent the vertices of a figure in the coordinate plane using variables.

Use coordinates to prove or disprove a claim about a figure.

For example: use slope to determine if sides are parallel, intersecting, or perpendicular; use the distance formula to determine if sides are congruent or to decide if appoint is inside a circle, outside a circle, or on the circle; use the midpoint formula or the distance formula to decide if a side has been bisected.
Students may use geometric simulation software to model figures and prove simple geometric theorems.

Examples:

- Use slope and distance formula to verify the polygon formed by connecting the points (–3, –2), (5, 3), (9, 9), (1, 4) is a parallelogram.
- Prove or disprove that triangle *ABC* with coordinates *A* (–1, 2), *B* (1, 5), *C* (–2, 7) is an isosceles right triangle.
- Take a picture or find a picture which includes a polygon. Overlay the picture on a coordinate plane (manually or electronically). Determine the coordinates of the vertices. Classify the polygon. Use the coordinates to justify the classification.

Instructional Strategies:

Review the concept of slope as the rate of change of the *y*-coordinate with respect to the *x*-coordinate for a point moving along a line, and derive the slope formula.

Use similar triangles to show that every nonvertical line has a constant slope.

Review the point-slope, slope-intercept and standard forms for equations of lines.

Pay special attention to the slope of a line and its applications in analyzing properties of lines.

Allow adequate time for students to become familiar with slopes and equations of lines and methods of computing them.

Use slopes and the Euclidean distance formula to solve problems about figures in the coordinate plane such as:

- Given three points, are they vertices of an isosceles, equilateral, or right triangle?
- Given four points, are they vertices of a parallelogram, a rectangle, a rhombus, or a square?
- Given the equation of a circle and a point, does the point lie outside, inside, or on the circle?
- Given the equation of a circle and a point on it, find an equation of the line tangent to the circle at that point.
- Given a line and a point not on it, find an equation of the line through the point that is parallel to the given line.
- Given a line and a point not on it, find an equation of the line through the point that is perpendicular to the given line.
- Given the equations of two non-parallel lines, find their point of intersection.
- Given two points, use the distance formula to find the coordinates of the point halfway between them. Generalize this for two arbitrary points to derive the midpoint formula.

Use linear interpolation to generalize the midpoint formula and find the point that partitions a line segment in any specified ratio.

Given the vertices of a triangle or a parallelogram, find the equation of a line containing the altitude to a specified base and the point of intersection of the altitude and the base. Use the distance formula to find the length of that altitude and base, and then compute the area of the figure.

Resources/Tools:

Mathematics Assessment Project:

- "Finding Equations of Parallel and Perpendicular Lines" – This lesson unit is intended to help you assess how well students are able to understand the relationship between the slopes of parallel and perpendicular lines and, in particular, to help identify students who find it difficult to:
  - Find, from their equations, lines that are parallel and perpendicular.
  - Identify and use intercepts.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-GPE.B
  - Is this a rectangle?

Domain: Expressing Geometric Properties with Equations (G.GPE)

▶ **Cluster:** *Use coordinates to prove simple geometric theorems algebraically.*

Standard: **G.GPE.7**
(9/10) Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (*e.g. find the equation of a line parallel or perpendicular to a given line that passes through a given point*). **(G.GPE.5)**

Suggested Standards for Mathematical Practice (MP):
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: G.GPE.6, 8.G.5, G.CO.1, G.CO.7, A.REI.6

Explanations and Examples:
Relate work on parallel lines to standard A.REI.5 involving systems of equations having no solution or infinitely many solutions.

Lines can be horizontal, vertical or neither.

Prove that the slopes of parallel lines are equal.

Prove that the product of the slopes of perpendicular lines is -1.

Write the equation of a line parallel or perpendicular to a given line, passing through a given point.

Students may use a variety of different methods to construct a parallel or perpendicular line to a given line and calculate the slopes to compare relationships.

Examples:
- Find the equation of a line perpendicular to $3x + 5y = 15$ through the point (–3. 2).
- Find an equation of a line perpendicular to $y = 3x - 4$ that passes through (3, 4).
- Verify that the distance between two parallel lines is constant. Justify your answer.

Instructional Strategies:

Allow students to explore and make conjectures about relationships between lines and segments using a variety of methods. Discuss the role of algebra in providing a precise means of representing a visual image.

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-GPE.B.5
  - A Midpoint Miracle
  - Triangles inscribed in a circle
  - Unit Squares and Triangles
  - Equal Area Triangles on the Same Base I
  - Equal Area Triangles on the Same Base II

Domain: Expressing Geometric Properties with Equations ★ (G.GPE)

▶ **Cluster:** *Use coordinates to prove simple geometric theorems algebraically.*

Standard: **G.GPE.8 ★**

(9/10) Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, including the use of the distance and midpoint formulas.★**(G.GPE.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision

Connections: G.GPE.7, 7.G.4-6

Explanations and Examples:

This standard provides practice with the distance formula and its connection with the Pythagorean Theorem.

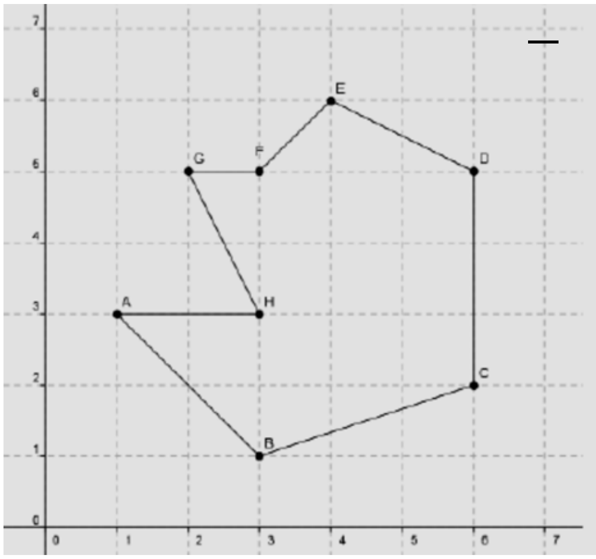
Use the coordinates of the vertices of a polygon graphed in the coordinate plane and use the distance formula to compute the perimeter.

Use the coordinates of the vertices of triangles and rectangles graphed in the coordinate plane to compute the area. Students may use geometric simulation software to model figures.

Examples:

- Find the perimeter and area of a rectangle with vertices at C (–1, 1), D (3, 4), E (6, 0), F (2, –3). Round your answer to the nearest hundredth when necessary.

- Find the area and perimeter for the figure below.



- Calculate the area of triangle *ABC* with altitude *CD*, given *A* (–4, –2), *B* (8, 7), *C* (1, 8) and *D* (4, 4).

Instructional Strategies:

Graph polygons using coordinates. Explore perimeter and area of a variety of polygons, including convex, concave, and irregularly shaped polygons.

Given a triangle, use slopes to verify that the length and height are perpendicular. Find the area.

Find the area and perimeter of a real-world shape using a coordinate grid and Google Earth. Select a shape (yard, parking lot, school, etc.). Use the tool menu to overlay a coordinate grid. Use coordinates to find the perimeter and area of the shape selected. Determine the scale factor of the picture as related to the actual real-life view. Then find the actual perimeter and area.

Resources/Tools:

Mathematics Assessment Project:

- "Finding Equations of Parallel and Perpendicular Lines" – This lesson unit is intended to help you assess how well students are able to understand the relationship between the slopes of parallel and perpendicular lines and, in particular, to help identify students who find it difficult to:
    - Find, from their equations, lines that are parallel and perpendicular.
    - Identify and use intercepts.

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-GPE.B
    - Is this a rectangle?

Domain: Modeling with Geometry ★ (G.MG)

▶ **Cluster:** *Apply geometric concepts in modeling situations.*

Standard: **G.MG.1 ★**

(9/10) Use geometric shapes, their measures, and their properties to describe objects *(e.g. modeling a tree trunk or a human torso as a cylinder).* **★(G.MG.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: 7.G.4-6, 8.G.10-12

Explanations and Examples:

Focus on situations that require relating two- and three- dimensional objects.

Estimate measures (circumference, area, perimeter, volume) of real-world objects using comparable geometric shapes or three-dimensional objects.

Apply the properties of geometric figures to comparable real-world objects (e.g., The spokes of a wheel of a bicycle are equal lengths because they represent the radii of a circle).

Students may us simulation software and modeling software to explore which model best describes a set of data or situation.

Examples:

Example 1:

How can you model objects in your classroom as geometric shapes?

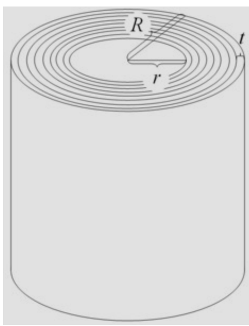
Example2:

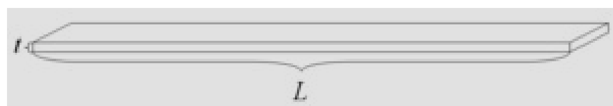
Picture a roll of toilet paper; assume that the paper in the roll is very tightly rolled. Assuming that the paper in the roll is very thin, find a relationship between the thickness of the paper, the inner and outer radii of the roll, and the length of the paper in the roll.

Express your answer as an algebraic formula involving the four listed variables.

The purpose of this task is to engage students in geometric modeling, and in particular to deduce algebraic relationships between variables stemming from geometric constraints. The modeling process is a challenging one, and will likely elicit a variety of attempts from the students. Teachers should expect to spend time guiding students away from overly complicated models. Similarly, the task presents one solution, but alternatives abound: For example, students could imagine slicing the roll along a radius, unraveling the cross-section into a sequence of trapezoids whose area can be computed.

Solution:





We begin by labeling the variables, for which the above diagrams may be useful. Let *t* denote the thickness of the paper, let *r* denote the inner radius, let *R* denote the outer radius and let *L* denote the length of the paper, all measured in inches. We now consider the area *A*, measured in square inches, of the annular cross-section displayed at the top of the first image, consisting of concentric circles. Namely, we see that this area can be expressed in two ways: First, since this area is the area of the circle of radius *R* minus the area of the circle of radius *r,* we learn that $A = \pi(R^2 - r^2)$.

Second, if the paper were unrolled, laid on a (very long) table and viewed from the side, we would see a very long thin rectangle. When the paper is rolled up, this rectangle is distorted, but -- assuming r is large in comparison to *t* -- the area of the distorted rectangle is nearly identical to that of the flat one. As in the second figure, the formula for the area of a rectangle now gives $A = t \cdot L$.

Comparing the two formulas for A, we find that the four variables are related by: $t \cdot L = \pi(R^2 - r^2)$.

Instructional Strategies:

Genuine mathematical modeling typically involves more than one conceptual category. For example, modeling a herd of wild animals may involve geometry, measurement, proportional reasoning, estimation, probability and statistics, functions, and algebra. It would be somewhat misleading to try to teach a unit with the title of "modeling with geometry." Instead, these standards can be woven into other content clusters.

A challenge for teaching modeling is finding problems that are interesting and relevant to high school students and, at the same time, solvable with the mathematical tools at the students' disposal. The resources listed below are a beginning for addressing this difficulty.

Resources/Tools:

Mathematics Assessment Project:

- "Modeling: Rolling Cups" – This lesson unit is intended to help you assess how well students are able to:
    - Choose appropriate mathematics to solve a non-routine problem.
    - Generate useful data by systematically controlling variables.
    - Develop experimental and analytical models of a physical situation

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-MG.A
    - Coins in a circular pattern
    - Eratosthenes and the circumference of the earth
    - Regular Tessellations of the plane
    - Running around a track I
    - Running around a track II
    - Paper Clip
- G-MG.A.1
    - The Lighthouse Problem
    - Hexagonal Pattern of Beehives
    - Tilt of earth's axis and the four seasons
    - Solar Eclipse
    - Toilet Roll
    - Global Positioning System II
    - How far is the horizon?

Domain: Modeling with Geometry ★ (G.MG)

Cluster: *Apply geometric concepts in modeling situations.*

Standard: **G.MG.2 ★**

(9/10) Apply concepts of density and displacement based on area and volume in modeling situations *(e.g. persons per square mile, BTUs per cubic foot)*. ★**(G.MG.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

Connections: 7.G.4-6, 8.G.10-12

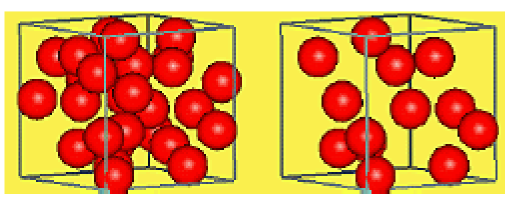
Explanations and Examples:

Decide whether it is best to calculate or estimate the area or volume of a geometric figure and perform the calculation or estimation.

Students may us simulation software and modeling software to explore which model best describes a set of data or situation.

Examples:

- Wichita, Kansas has 344,234 people within 165.9 square miles. What is Wichita's population density?
- Consider the two boxes below. Each box has the same volume. If each ball has the same mass, which box would weight more? Why



Block I	Block II
Mass = 79.4 grams	Mass = 25.4 grams
Volume = 29.8 cubic cm	Volume = 29.8 cubic cm

- A King Size waterbed has the following dimensions 72 in. X 84 in. X 9.5in. It takes 240.7 gallons of water to fill it which would weigh 2071 pounds. What is the weight of a cubic foot of water?

Instructional Strategies: See G.MG.1

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G-MG.A.1
  - How many cells are in the human body?
  - How many leaves on a tree?
  - How many leaves on a tree? Version 2
  - How thick is a soda can? I
  - How thick is a soda can? II
  - Archimedes and the King's crown
  - Indiana Jones and the Golden Statue

Domain: Modeling with Geometry ★ (G.MG)

▶ **Cluster:** *Apply geometric concepts in modeling situations.*

Standard: **G.MG.3** ★

(9/10) Apply geometric methods to solve design problems *(e.g. designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on* ***ratios****).* ★**(G.MG.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Solve problems and persevere in solving them.
- ✓ MP.4 Model with mathematics
- ✓ MP.5 Use appropriate tools strategically.

Connections: See G.MG.1-2

Explanations and Examples:

Create a visual representation of a design problem and solve using a geometric model (graph, equation, table, formula).

Interpret the results and make conclusions based on the geometric model.

Students may us simulation software and modeling software to explore which model best describes a set of data or situation.

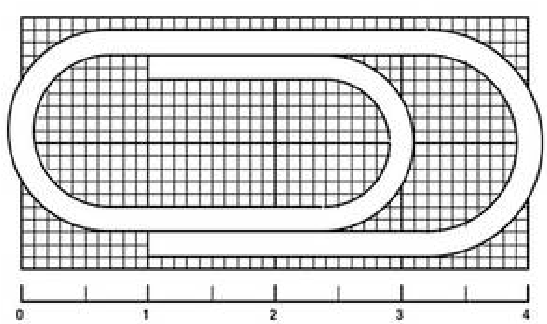
Examples:

Example1:

Given one geometric solid, design a different geometric solid that will hold the same amount of substance (e.g., a cone to a prism).

Example 2:

This paper clip is just over 4 cm long.

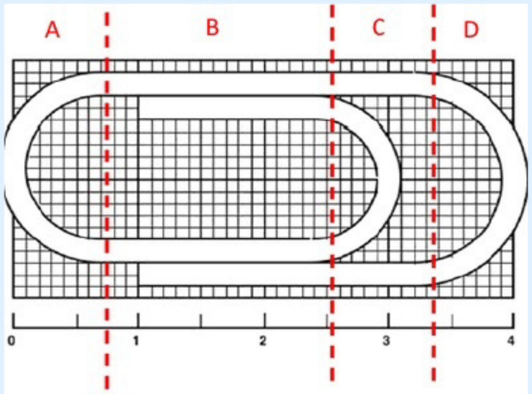


How many paper clips like this may be made from a straight piece of wire 10 meters long?

In this task, a typographic grid system serves as the background for a standard paper clip. A metric measurement scale is drawn across the bottom of the grid and the paper clip extends in both directions slightly beyond the grid. Students are given the approximate length of the paper clip and determine the number of like paper clips made from a given length of wire. Extending the paper clip beyond the grid provides an opportunity to include an estimation component in the problem. In the interest of open-ended problem solving, no scaffolding or additional questions are posed in this task. The paper clip modeled in this problem is an actual large standard paper clip.

Sample Response:

One approach is to divide the paper clip into vertical regions, and then to use the measurement grid to determine the length of the straight sections and estimate the length of the curved sections using a string or thin wire in conjunction with the measurement scale provided. One such division is accomplished using three vertical dividers splitting the paper clip into four distinct regions as shown.



Regions	Number of Linear Sections	Number of Curved Sections
Region A	0	1
Region B	4	0
Region C	2	1
Region D	0	1

The lengths of the linear sections were determined using the gridlines. The estimations of the lengths of the curved sections were determined using a string or thin wire in conjunction with the measurement scale provided.

Regions	Measurement of Linear Sections (listed from top to bottom)	Estimated Measurement of Curved Sections
Region A		2.5cm
Region B	1.8cm, 1.5cm, 1.8cm, 1.5cm	
Region C	0.8cm, 0.8cm	1.6cm
Region D		2.5cm

The length of wire needed to manufacture one paper clip is now approximately:

$$1.8\text{ cm} + 1.5\text{ cm} + 1.8\text{ cm} + 1.5\text{ cm} + 0.8\text{ cm} + 0.8\text{ cm} + 2.5\text{ cm} + 1.6\text{ cm} + 2.5\text{ cm} = 14.8\text{ cm}$$

The length of the straight piece of wire is 10 meters. Since 1 meter is the same as 100 centimeters, 10 meters is $10 \cdot 100 = 1000$ centimeters. Finally, we find that at 14.8 cm per paper clip, 1000 centimeters will produce approximately

$$\frac{1000}{14.8} \approx 67.6 \text{ paper clips.}$$

Since we can only make a whole number of paper clips, we conclude that approximately 67 paper clips may be manufactured from a straight piece of wire 10 meters in length.

Instructional Strategies: See G.MG.1

Resources/Tools:

Illustrative Mathematics High School Geometry tasks: **Scroll to the appropriate section to find named tasks.**

- G.MG.A
    - Paper Clip
- G.MG.A.3
    - Ice Cream Cone
    - Satellite

APPENDIX: TABLE 1 *The Properties of Operations*

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c=a+(b+c)$	$(78+25)+75=78+(25+75)$
Commutative	$a+b=b+a$	$2+98=98+2$
Additive Identity	$a+0=a$ *and* $0+a=a$	$9875+0=9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a+-a=-a+a=0$	$-47+47=0$
Properties of Multiplication		
Associative	$(a\times b)\times c=a\times(b\times c)$	$(32\times 5)\times 2=32\times(5\times 2)$
Commutative	$a\times b=b\times a$	$10\times 38=38\times 10$
Multiplicative Identity	$a\times 1=a$ *and* $1\times a=a$	$387\times 1=387$
Multiplicative Inverse	*For every real number a,* $a\neq 0$*, there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$\frac{8}{3}\times\frac{3}{8}=1$
Distributive Property of Multiplication over Addition		
Distributive	$a\times(b+c)=a\times b+a\times c$	$7\times(50+2)=7\times 50+7\times 2$

(Variables a, b, and c represent real numbers.)
Excerpt from NCTM's ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

TABLE 2. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for a in any expression containing a.	$If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 3. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.
$If\ a > b\ and\ b > c\ then\ a > c.$
$If\ a > b, then\ b < a.$
$If\ a > b, then - a < -b.$
$If\ a > b, then\ a \pm c > b \pm c.$
$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$
$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$
$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$
$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

Table 4. Cognitive Rigor Matrix/Depth of Knowledge (DOK)

Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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