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2017 Kansas Mathematics Standards

Flip Book

8th Grade

This project used work created by the Departments of Education in Ohio, North Carolina, Georgia and resources created by Achieve the Core, EngageNY, Illustrative Mathematics, and NCTM.

About the Flip Books

This project attempts to organize some of the most valuable resources that help develop the intent, understanding and implementation of the 2017 Kansas Mathematics Standards. These documents provide a starting point for teachers and administrators to begin discussion and exploration into the standards. It is not the only resource to support implementation of the 2017 Kansas Mathematics Standards.

This project is built on the previous work started in the summer of 2012 from Melisa Hancock (Manhattan, KS), Debbie Thompson (Wichita, KS) and Patricia Hart (Wichita, KS) who provided the initial development of the "flip books." The "flip books" are based on a model that Kansas had for earlier standards; however, this edition specifically targets the Kansas Mathematics Standards that were adopted in the summer of 2017. These flip books incorporate the resources from other state departments of education, the mathematics learning progressions, and other reliable sources including The National Council of Teachers of Mathematics and the National Supervisors of Mathematics. In addition, mathematics educators across the country have suggested changes/additions that could or should be made to further enhance its effectiveness. The document is posted on the KSDE Mathematics website at http://community.ksde.org/Default.aspx?tabid=5646 and will continue to undergo changes periodically. When significant changes/additions are implemented, the modifications will be posted and dated.

Planning Advice - Focus on the Clusters

The (mathematics standards) call for a greater focus. Rather than racing to cover topics in today's mile-wide, inch-deep curriculum, we need to use the power of the eraser and significantly narrow and deepen how time and energy is spent in the mathematics classroom. There is a necessity to focus deeply on the major work of each grade to enable students to gain strong foundations: solid conceptual understanding, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems both in and out of the mathematics classroom. (www.achievethecore.org)

Not all standards should have the same instructional emphasis. Some groups of standards require a greater emphasis than others. In order to be intentional and systematic, priorities need to be set for planning, instruction, and assessment. "Not everything in the Standards should have equal priority" (Zimba, 2011). Therefore, there is a need to elevate the content of some standards over that of others throughout the K-12 curriculum.

When the Standards were developed the following were considerations in the identification of priorities: 1) the need to be qualitative and well-articulated; 2) the understanding that some content will become more important than other; 3) the creation of a focus means that some essential content will get a greater share of the time and resources "while the remaining content is limited in scope"; 4) a "lower" priority does not imply exclusion of content, but is usually intended to be taught in conjunction with or in support of one of the major clusters.

"The Standards are built on the progressions, so priorities have to be chosen with an eye to the arc of big ideas in the Standards. A prioritization scheme that respects progressions in the Standards will strike a balance between the journey and the endpoint. If the endpoint is everything, few will have enough wisdom to walk the path; if the endpoint is nothing, few will understand where the journey is headed. Beginnings and the endings both need particular care. … It would also be a mistake to identify such standard as a locus of emphasis. (Zimba, 2011)

The important question in planning instruction is: "What is the mathematics you want the student to walk away with?" In order to accomplish this, educators need to think about "grain size" when planning instruction. Grain size corresponds to the knowledge you want the student to know. Mathematics is simplest at the right grain size. According to Phil Daro (*Teaching Chapters, Not Lessons—Grain Size of Mathematics*), strands are too vague and too large a grain size, while lessons are too small a grain size. Units or chapters produce about the right "grain size." In the planning process educators should attend to the clusters, and think of the standards as the ingredients of a cluster. Coherence of mathematical ideas and concepts exists at the cluster level across grades.

A caution--Grain size is important, but can result in conversations that do not advance the intent of this structure. Extended discussions among teachers where it is argued for "2 days" instead of "3 days" on a topic because it is a lower priority can detract from the overall intent of suggested priorities. The reverse is also true. As Daro indicates, focusing on lessons can provide too narrow a view which compromises the coherence value of closely related standards.

The video clip Teaching Chapters, Not Lessons—Grain Size of Mathematics presents Phil Daro further explaining grain size and the importance of it in the planning process. (Click on photo to view video.)

Along with "grain size", clusters have been given **priorities** which have important implications for instruction. These priorities should help guide the focus for teachers as they determine allocation of time for both planning and instruction. The priorities provided help guide the focus for teachers as they determine distribution of time for both planning and instruction, helping to assure that students really understand mathematics before moving on. Each cluster has been given a priority level. As professional educators begin planning, developing and writing units, these priorities provide guidance in assigning time for instruction and formative assessment within the classroom.

Each cluster within the standards has been given a priority level influenced by the work of Jason Zimba. The three levels are referred to as — **Major, Supporting** and **Additional**. Zimba suggests that about 70% of instruction should relate to the **Major** clusters. The lower two priorities (**Supporting** and **Additional**) can work together by supporting the **Major** priorities. You can find the grade Level Focus Documents for the 2017 Kansas Math Standards at: http://community.ksde.org/Default.aspx?tabid=6340.

Recommendations for Cluster-Level Priorities

Appropriate Use:

- Use the priorities as guidance to inform instructional decisions regarding time and resources spent on clusters by varying the degrees of emphasis.
- Focus should be on the major work of the grade in order to open up the time and space to bring the Standards for Mathematical Practice to life in mathematics instruction through sense-making, reasoning, arguing and critiquing, modeling, etc.
- Evaluate instructional materials by taking the cluster level priorities into account. The major work of the grade must be presented with the highest possible quality; the additional work of the grade should support the major priorities and not detract from them.
- Set priorities for other implementation efforts such as staff development, new curriculum development, and revision of existing formative or summative testing at the state, district or school level.

Things to Avoid:

- Neglecting any of the material in the standards. Seeing Supporting and Additional clusters as optional.
- Sorting clusters (from Major to Supporting to Additional) and then teaching the clusters in order. This would remove the coherence of mathematical ideas and create missed opportunities to enhance the major work of the grade with the other clusters.
- Using the cluster headings as a replacement for the actual standards. All features of the standards matter—from the practices to surrounding text, including the particular wording of the individual content standards. Guidance for priorities is given at the cluster level as a way of thinking about the content with the necessary specificity yet without going so far into detail as to comprise the coherence of the standards (grain size).

Mathematics Teaching Practices

(High Leverage Teacher Actions)

[National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: National Council of Teachers of Mathematics.]

The eight Mathematics Teaching Practices should be the foundation for mathematics instruction and learning. This framework was informed by over twenty years of research and presented in *Principles to Actions* by the National Council of Teachers of Mathematics (NCTM). If teachers are guided by this framework, they can move "toward improved instructional practice" and support "one another in becoming skilled at teaching in ways that matter for ensuring successful mathematics learning for all students" (NCTM, 2014, p. 12).

1. **Establish mathematics goals to focus learning.**
   Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.
2. **Implement tasks that promote reasoning and problem solving.**
   Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.
3. **Use and connect mathematical representations.**
   Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.
4. **Facilitate meaningful mathematical discourse.**
   Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.
5. **Pose purposeful questions.**
   Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
6. **Build procedural fluency from conceptual understanding.**
   Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.
7. **Support productive struggle in learning mathematics.**
   Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.
8. **Elicit and use evidence of student thinking.**
   Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.

Standards for Mathematical Practice in Grade 8

The Standards for Mathematical Practice are practices expected to be integrated into every mathematics lesson for all students Grades K-12. Below are a few examples of how these Practices may be integrated into tasks that Grade 8 students complete.

Practice	Explanation and Example
1) Make sense of problems and persevere in solving them.	Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They solve real world problems through application of algebraic and geometric concepts. They see the meaning of a problem and look for efficient ways to represent and solve it. They check their thinking by asking themselves, “What is the most efficient way to solve the problem?”, “Does this make sense?”, and “Can I solve the problem in a different way?” They understand the approaches of others to solving complex problems and identify correspondences between the different approaches. Example: to understand why a 20% discount followed by a 20% markup does not return an item to its original price, a MS student might translate the situation into a tape diagram or a general equation; or they might first consider the result for an item priced at $1.00 or $10.00.
2) Reason abstractly and quantitatively.	Mathematically proficient students make sense of quantities and their relationships in problem situations. They represent a wide variety of real world contexts through the use of real numbers and variables in mathematical expressions, equations, and inequalities. They examine patterns in data and assess the degree of linearity of functions. They *contextualize* to understand the meaning of the number or variable as related to the problem. They *decontextualize* to manipulate symbolic representations by applying properties of operations. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. Examples: 1)They apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems, 2) they solve problems involving unit rates by representing the situations in equation form, and 3) they use properties of operation to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.
3) Construct viable arguments and critique the reasoning of others.	Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They construct arguments using verbal or written explanations accompanied by expressions, equations, inequalities, models, and graphs, tables, and other data displays (i.e. box plot, dot plots, histograms, etc.) Example: Use of numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that 5 – 2x is equivalent to 3x. Proficient MS students progress from arguing exclusively through concrete referents such as physical objects and pictorial referents, to also including symbolic representations such as expressions and equations.

4) Model with mathematics.	Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. They analyze relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the mode if it has not served its purpose. Examples: MS students might apply proportional reasoning to plan a school event or analyze a problem in the community, or they can roughly fit a line to a scatter plot to make predictions and gather experimental data to approximate a probability.
5) Use appropriate tools strategically.	Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil/paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, a statistical package, or dynamic geometry software. They are sufficiently familiar with tools appropriate for their grade to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. They are able to use technological tools to explore and deepen their understanding of concepts. Examples: Use graphs to model functions, algebra tiles to see how properties of operations apply to equations, and dynamic geometry software to discover properties of parallelograms.
6) Attend to precision.	Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussions with others and in their *own* reasoning. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. Examples: 1) MS students can use the definition of rational numbers to explain why a number is irrational, and describe congruence and similarity in terms of transformations in the plane and 2) they accurately apply scientific notation to large numbers and use measures of center to describe data sets.
7) Look for and make use of structure.	Mathematically proficient students look for and notice patterns and then articulate what they *see*. They can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. Additional examples: 1) MS students might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes, and see the equation $3x = 2y$ represents a proportional relationship with a unit rate of 3/2 = 1.5, 2) they might recognize how the Pythagorean theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism.
8) Look for and express regularity in repeated reasoning.	Mathematically proficient students notice if calculations are repeated and look both for general methods and for shortcuts. By paying attention to the calculation of slope as they repeatedly check whether po9ints are on the line through (1,2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results. Examples: 1)By working with tables of equivalent ratios, middle school students can deduce the corresponding multiplicative relationships and connections to unit rates, 2) they notice the regularity with which interior angle sums increase with the number of sides in a polygon leads to a general formula for the interior angle sum of an *n*-gon, 3) MS students learn to see subtraction as addition of opposite, and use this in a general purpose tool for collecting like terms in linear expressions.

Implementing Standards for Mathematical Practice

This guide was created to help educators implement these standards into their classroom instruction. These are the practices for the **students,** and the teacher can assist students in using them efficiently and effectively.

#1 – Make sense of problems and persevere in solving them.

Summary of this Practice:

- Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem.
- Plan a solution pathway instead of jumping to a solution.
- Monitor their progress and change the approach if necessary.
- See relationships between various representations.
- Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another.
- Continually ask themselves, "Does this make sense?"
- Understand various approaches to solutions.

Student Actions	Teacher Actions
• Actively engage in solving problems and thinking is visible (doing mathematics vs. following steps or procedures with no understanding). • Relate current "situation" to concepts or skills previously learned, and checking answers using different methods. • Monitor and evaluate their own progress and change course when necessary. • Always ask, "Does this make sense?" as they are solving problems.	• Allow students time to initiate a plan; using question prompts as needed to assist students in developing a pathway. • Constantly ask students if their plans and solutions make sense. • Question students to see connections to previous solution attempts and/or tasks to make sense of the current problem. • Consistently ask students to defend and justify their solution(s) by comparing solution paths.

What questions develop this Practice?

- How would you describe the problem in your own words? How would you describe what you are trying to find?
- What do you notice about...?
- What information is given in the problem? Describe the relationship between the quantities.
- Describe what you have already tried. What might you change? Talk me through the steps you've used to this point.
- What steps in the process are you most confident about? What are some other strategies you might try?
- What are some other problems that are similar to this one?
- How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

What are the characteristics of a good math task for this Practice?

- Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available.
- Encourages multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.

#2 – Reason abstractly and quantitatively.

Summary of this Practice:

- Make sense of quantities and their relationships.
- Decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships.
- Understand the meaning of quantities and are flexible in the use of operations and their properties.
- Create a logical representation of the problem.
- Attend to the meaning of quantities, not just how to compute them.

Student Actions	Teacher Actions
• Use varied representations and approaches when solving problems. • Represent situations symbolically and manipulating those symbols easily. • Give meaning to quantities (not just computing them) and making sense of the relationships within problems.	• Ask students to explain the meaning of the symbols in the problem and in their solution. • Expect students to give meaning to all quantities in the task. • Question students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

What questions develop this Practice?

- What do the numbers used in the problem represent? What is the relationship of the quantities?
- How is_____related to______?
- What is the relationship between _____ and _____?
- What does ______ mean to you? (e.g. symbol, quantity, diagram)
- What properties might you use to find a solution?
- How did you decide that you needed to use _____? Could we have used another operation or property to solve this task? Why or why not?

What are the characteristics of a good math task for this Practice?

- Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- Consistently expects students to convert situations into symbols in order to solve the problem; and then requires students to explain the solution within a meaningful situation.
- Contains relevant, realistic content.

#3 – Construct viable arguments and critique the reasoning of others.

Summary of this Practice:

- Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments.
- Justify conclusions with mathematical ideas.
- Listen to the arguments of others and ask useful questions to determine if an argument makes sense.
- Ask clarifying questions or suggest ideas to improve/revise the argument.
- Compare two arguments and determine correct or flawed logic.

Student Actions	Teacher Actions
• Make conjectures and exploring the truth of those conjectures. • Recognize and use counter examples. • Justify and defend all conclusions and using data within those conclusions. • Recognize and explain flaws in arguments, which may need to be demonstrated using objects, pictures, diagrams, or actions.	• Encourage students to use proven mathematical understandings, (definitions, properties, conventions, theorems etc.), to support their reasoning. • Question students so they can tell the difference between assumptions and logical conjectures. • Ask questions that require students to justify their solution and their solution pathway. • Prompt students to respectfully evaluate peer arguments when solutions are shared. • Ask students to compare and contrast various solution methods • Create various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.)

What questions develop this Practice?

- What mathematical evidence would support your solution? How can we be sure that...? How could you prove that...?
- Will it still work if...?
- What were you considering when...? How did you decide to try that strategy?
- How did you test whether your approach worked?
- How did you decide what the problem was asking you to find? (What was unknown?)
- Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not?
- What is the same and what is different about...? How could you demonstrate a counter-example?

What are the characteristics of a good math task for this Practice?

- Structured to bring out multiple representations, approaches, or error analysis.
- Embeds discussion and communication of reasoning and justification with others.
- Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- Expects students to give feedback and ask questions of others' solutions.

#4 – Model with mathematics.

Summary of this Practice:

- Understand reasoning quantitatively and abstractly (able to decontextualize and contextualize).
- Apply the math they know to solve problems in everyday life.
- Simplify a complex problem and identify important quantities to look at relationships.
- Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation.
- Reflect on whether the results make sense, possibly improving/revising the model.
- Ask themselves, "How can I represent this mathematically?"

Student Actions	Teacher Actions
• Apply mathematics to everyday life. • Write equations to describe situations. • Illustrate mathematical relationships using diagrams, data displays, and/or formulas. • Identify important quantities and analyzing relationships to draw conclusions.	• Demonstrate and provide students experiences with the use of various mathematical models. • Question students to justify their choice of model and the thinking behind the model. • Ask students about the appropriateness of the model chosen. • Assist students in seeing and making connections among models.

What questions develop this Practice?

- What number model could you construct to represent the problem?
- How can you represent the quantities?
- What is an equation or expression that matches the diagram..., number line..., chart..., table...?
- Where did you see one of the quantities in the task in your equation or expression?
- What math do you know that you could use to represent this situation?
- What assumptions do you have to make to solve the problem?
- What formula might apply in this situation?

What are the characteristics of a good math task for this Practice?

- Structures represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.

#5 – Use appropriate tools strategically.

Summary of this Practice:

- Use available tools recognizing the strengths and limitations of each.
- Use estimation and other mathematical knowledge to detect possible errors.
- Identify relevant external mathematical resources to pose and solve problems.
- Use technological tools to deepen their understanding of mathematics.
- Use mathematical models for visualize and analyze information

Student Actions	Teacher Actions
• Choose tools that are appropriate for the task. • Know when to use estimates and exact answers. • Use tools to pose or solve problems to be most effective and efficient.	• Demonstrate and provide students experiences with the use of various math tools. A variety of tools are within the environment and readily available. • Question students as to why they chose the tools they used to solve the problem. • Consistently model how and when to estimate effectively, and requiring students to use estimation strategies in a variety of situations. • Ask student to explain their mathematical thinking with the chosen tool. • Ask students to explore other options when some tools are not available.

What questions develop this practice?

- What mathematical tools could we use to visualize and represent the situation?
- What information do you have?
- What do you know that is not stated in the problem? What approach are you considering trying first?
- What estimate did you make for the solution?
- In this situation would it be helpful to use...a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use...?
- What can using a_____show us that _____may not?
- In what situations might it be more informative or helpful to use...?

What are the characteristics of a good math task for this Practice?

- Lends itself to multiple learning tools. (Tools may include; concrete models, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- Requires students to determine and use appropriate tools to solve problems.
- Asks students to estimate in a variety of situations:
  - a task when there is no need to have an exact answer
  - a task when there is not enough information to get an exact answer
  - a task to check if the answer from a calculation is reasonable

#6 – Attend to precision.

Summary of this Practice:

- Communicate precisely with others and try to use clear mathematical language when discussing their reasoning.
- Understand meanings of symbols used in mathematics and can label quantities appropriately.
- Express numerical answers with a degree of precision appropriate for the problem context.
- Calculate efficiently and accurately.

Student Actions	Teacher Actions
• Use mathematical terms, both orally and in written form, appropriately. • Use and understanding the meanings of math symbols that are used in tasks. • Calculate accurately and efficiently. • Understand the importance of the unit in quantities.	• Consistently use and model correct content terminology. • Expect students to use precise mathematical vocabulary during mathematical conversations. • Question students to identify symbols, quantities and units in a clear manner.

What questions develop this Practice?

- What mathematical terms apply in this situation? How did you know your solution was reasonable?
- Explain how you might show that your solution answers the problem.
- Is there a more efficient strategy?
- How are you showing the meaning of the quantities?
- What symbols or mathematical notations are important in this problem?
- What mathematical language..., definitions..., properties can you use to explain...?
- How could you test your solution to see if it answers the problem?

What are the characteristics of a good math task for this Practice?

- Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- Expects students to use symbols appropriately.
- Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

#7 – Look for and make use of structure.

Summary of this Practice:

- Apply general mathematical rules to specific situations.
- Look for the overall structure and patterns in mathematics.
- See complicated things as single objects or as being composed of several objects.

Student Actions	Teacher Actions
• Look closely at patterns in numbers and their relationships to solve problems. • Associate patterns with the properties of operations and their relationships. • Compose and decompose numbers and number sentences/expressions.	• Encourage students to look for something they recognize and having students apply the information in identifying solution paths (i.e. compose/decompose numbers and geometric figures, identify properties, operations, etc.) • Expect students to explain the overall structure of the problem and the big math idea used to solve the problem.

What questions develop this Practice?

- What observations do you make about...? What do you notice when...?
- What parts of the problem might you eliminate..., simplify...?
- What patterns do you find in...?
- How do you know if something is a pattern?
- What ideas that we have learned before were useful in solving this problem?
- What are some other problems that are similar to this one? How does this relate to...?
- In what ways does this problem connect to other mathematical concepts?

What are the characteristics of a good math task for this Practice?

- Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value; working with properties; etc.)
- Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. Building on the structure of equal sharing, students connect the understanding to the traditional division algorithm. When "unit size" cannot be equally distributed, it is necessary to break down into a smaller "unit size". (example below)

$4\overline{)351}$ $\underline{-32}$	3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.
31	There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.
$\underline{-28}$ 3	This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

- Expects students to recognize and identify structures from previous experience(s) and apply this understanding in a new situation. i.e. $7 \times 8 = (7 \times 5) + (7 \times 3)$ OR $7 \times 8 = (7 \times 4) + (7 \times 4)$ new situations could be, distributive property, area of composite figures, multiplication fact strategies.

#8 – Look for and express regularity in repeated reasoning.

Summary of this Practice:

- See repeated calculations and look for generalizations and shortcuts.
- See the overall process of the problem and still attend to the details.
- Understand the broader application of patterns and see the structure in similar situations.
- Continually evaluate the reasonableness of their intermediate results.

Student Actions	Teacher Actions
• Notice if processes are repeated and look for both general methods and shortcuts. • Evaluate the reasonableness of intermediate results while solving. • Make generalizations based on discoveries and constructing formulas when appropriate.	• Ask what math relationships or patterns can be used to assist in making sense of the problem. • Ask for predictions about solutions at midpoints throughout the solution process. • Question students to assist them in creating generalizations based on repetition in thinking and procedures.

What questions develop this Practice?

- Will the same strategy work in other situations?
- Is this always true, sometimes true or never true? How would we prove that...?
- What do you notice about...?
- What is happening in this situation? What would happen if...?
- Is there a mathematical rule for...?
- What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

What are the characteristics of a good math task for this Practice?

- Present several opportunities to reveal patterns or repetition in thinking, so students can make a generalization or rule.
- Requires students to see patterns or relationships in order to develop a mathematical rule.
- Expects students to discover the underlying structure of the problem and come to a generalization.
- Connects to a previous task to extend learning of a mathematical concept.

Critical Areas for Mathematics in 8th Grade

In Grade 8, instructional time should focus on **three** critical areas:

1. **Formulating and reasoning about expressions, equations, and inequalities including modeling an association in bivariate data with a linear equation, and solving linear equations and inequalities.**
   Students use linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ($\frac{y}{x} = m \text{ or } y = mx$) as special linear equations ($y = mx + b$), understanding that the constant of proportionality (m) is the slope, and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change shifting from an informal approach of counting rise over run to the meaningful use of a formula for slope. Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and y-intercept) in terms of the situation.

   Students strategically choose and efficiently implement procedures to solve linear equations and inequalities in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students use linear equations, linear inequalities, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

2. **Grasping the concept of a function and using functions to describe quantitative relationships.**
   Students grasp the concept of a function as a rule that assigns to each input exactly one output. They understand that functions describe situations where one quantity determines another. They can translate among representations and partial representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.

3. **Analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem.**
   Students use ideas about distance and angles, relationships about angles formed by intersecting lines, informal geometric constructions, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students learn to measure angles. They develop important ideas related to the concepts of angles, spanning a wide range of angle relationships and theorems (particularly when parallel lines are cut by a transversal), and use them to solve problems. Students understand the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds true, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume and surface area by exploring and generalizing volume and surface area for cone and pyramids and by solving problems involving pyramids, cones, and spheres.

Dynamic Learning Maps (DLM) and Essential Elements

The Dynamic Learning Maps and Essential Elements are knowledge and skills linked to the grade-level expectations identified in the Common Core State Standards. The purpose of the Dynamic Learning Maps Essential Elements is to build a bridge from the content in the Common Core State Standards to academic expectations for students with the most significant cognitive disabilities.

For more information please visit the Dynamic Learning Maps and Essential Elements website.

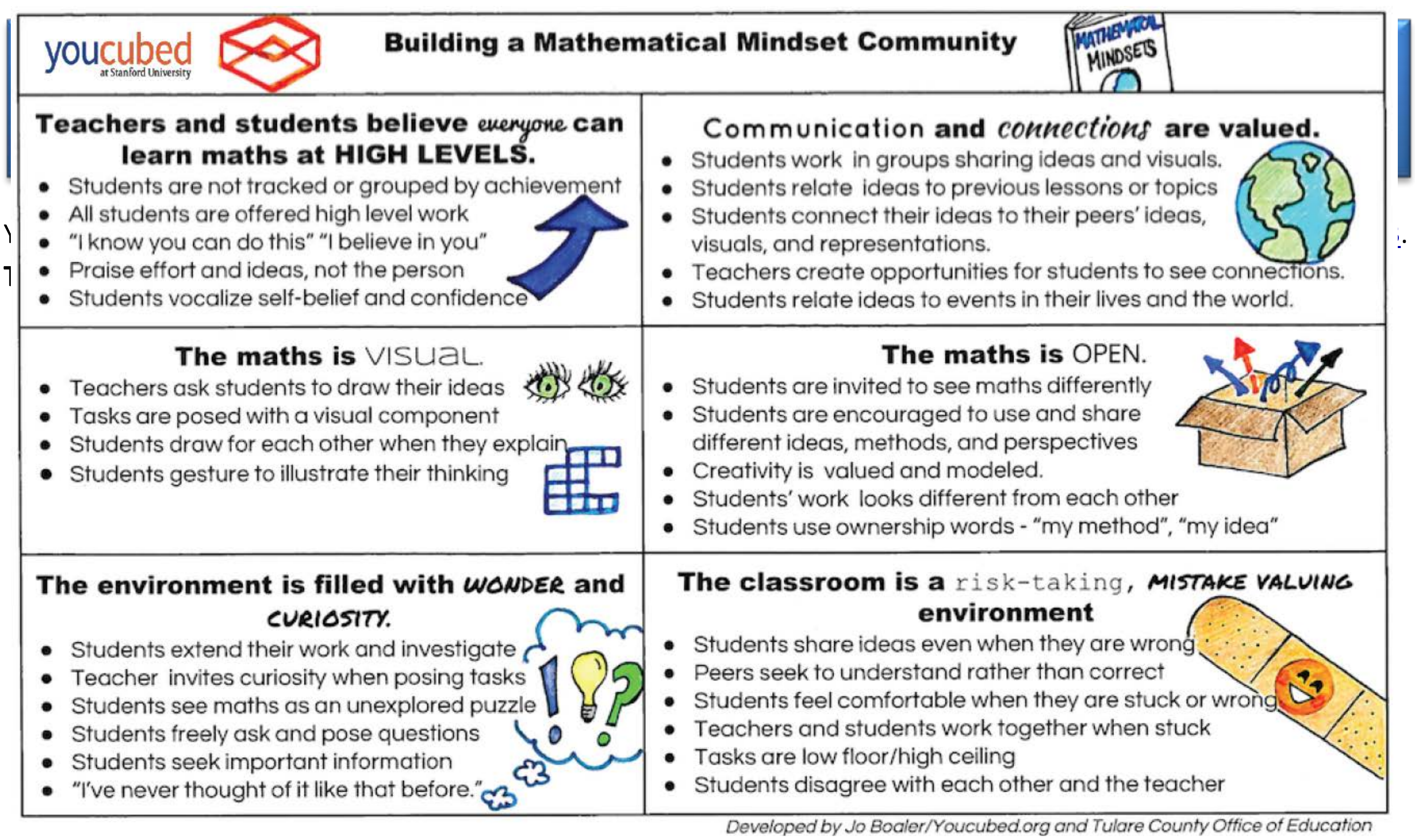
Growth Mindset

The term "growth mindset" comes from the groundbreaking work of Carol Dweck. She identified that everyone holds ideas about their own potential. Some people believe that their intelligence is more or less fixed in math – that you can do math or you can't, while others believe they can learn anything and that their intelligence can grow.

In a fixed mindset, people believe their basic qualities, like their intelligence or talent, are simply fixed traits. They spend their time documenting their intelligence or talent instead of developing it. They also believe that talent alone creates success—without effort. Students with a fixed mindset are those who are more likely to give up easily.

In a **growth mindset**, people believe that their most basic abilities can be developed through dedication and hard work—brains and talent are just the starting point. This view creates a love of learning and a resilience that is essential for great accomplishment. Students with a growth mindset are those who keep going even when work is hard, and who are persistent.

It is possible to change mindsets and to shift students' mindsets from fixed to growth and cause higher mathematics achievement and success in life. Watch this short video to get a better understanding of what Growth Mindset is and the benefits it can bring our students.



- Know that there are numbers that are not rational, and approximate them by rational numbers.
  NS.1 **NS.2**

Expressions and Equations (8.EE)

A. Work with radicals and integer exponents.
   EE.1 **EE.2** **EE.3**
B. Understand the connections between proportional relationships, lines, and linear equations.
   EE.4 **EE.5** **EE.6**
C. Analyze and solve linear equations and inequalities.
   EE.7

Functions (8.F)

A. Define, evaluate, and compare functions.
   F.1 **F.2** **F.3**
B. Use functions to model relationships between quantities.
   F.4 **F.5**

Standards for Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometry (8.G)

A. Geometric measurement: understand concepts of angle and measure angles.
   G.1 **G.2** **G.3** **G.4**
   G.5 **G.6**
B. Understand and apply the Pythagorean Theorem.
   G.7 **G.8** **G.9**
C. Solve real-world and mathematical problems involving measurement.
   G.10 **G.11** **G.12**

Statistics and Probability (8.SP)

A. Investigate patterns of association in **bivariate data**.
   SP.1 **SP.2** **SP.3**

Domain: The Number System (NS)

◆ **Cluster:** *Know that there are numbers that are not rational, and approximate them by rational numbers.*

Standard: Grade 8.NS.1

Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number. **(8.NS.1)**

Suggested Standards for Mathematical Practice (*MP*):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: 8.EE.4 & 8.EE.7b

This cluster is connected to:

- This cluster goes beyond the Grade 8 Critical Areas of Focus to address working with irrational numbers, integer exponents, and scientific notation.
- This cluster builds on previous understandings from Grades 6-7, The Number System.

Explanations and Examples:

Students distinguish between rational and irrational numbers. Any number that can be expressed as a fraction is a rational number. Students recognize that the decimal equivalent of a fraction will either terminate or repeat. Fractions that terminate will have denominators containing only prime factors of 2 and/or 5. This understanding builds on work in 7th grade when students used long division to distinguish between repeating and terminating decimals. Students convert repeating decimals into their fraction equivalent using patterns or algebraic reasoning. One method to find the fraction equivalent to a repeating decimal is shown below.

Change $0.\overline{4}$ to a fraction

- Let $x = 0.4444444 \ldots$
- Multiply both sides so that the repeating digits will be in front of the decimal. In this example, one digit repeats so both sides are multiplied by 10, giving $10x = 4.4444444 \ldots$
- Subtract the original equation from the new equation.

$$10x = 4.4444444 \ldots$$
$$x = 0.44444 \ldots$$
$$9x = 4$$

- Solve the equation to determine the equivalent fraction.

$$\frac{9x}{9} = \frac{4}{9}$$
$$x = \frac{4}{9}$$

Additionally, students can investigate repeating patterns that occur when fractions have a denominator of 9, 99, or 11. For example, $\frac{4}{9}$ is equivalent to $0.\overline{4}$, $\frac{5}{9}$ is equivalent to $0.\overline{5}$, etc.

A student made the following conjecture and found two examples to support the conjecture.

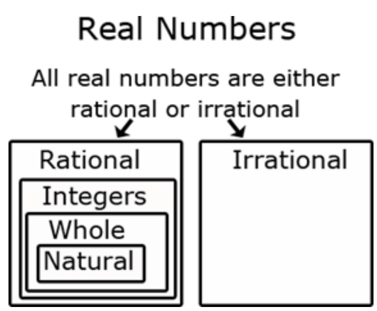
If a rational number is not an integer, then the square root of the rational number is irrational. For example, $\sqrt{3.6}$ *is irrational and* $\sqrt{\frac{1}{2}}$ is irrational.

Provide two examples of non-integer rational numbers that show that the conjecture is **false.**

Sample Response:

- Example 1: 2.25
- Example 2: $\frac{1}{4}$

Students can use graphic organizers to show the relationship between the subsets of the real number system.

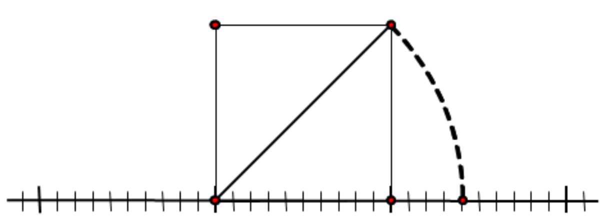


Instructional Strategies:

The distinction between rational and irrational numbers is an abstract distinction, originally based on ideal assumptions of perfect construction and measurement. In the real world, however, all measurements and constructions are approximate. Nonetheless, it is possible to see the distinction between rational and irrational numbers in their decimal representations.

A rational number is of the form $\frac{a}{b}$, where *a* and *b* are both integers, and *b* is not 0. In the elementary grades, students learned processes that can be used to locate any rational number on the number line: Divide the interval from 0 to 1 into *b* equal parts; then, beginning at 0, count out a of those parts. The surprising fact, now, is that there are numbers on the number line that cannot be expressed as $\frac{a}{b}$, with *a* and *b* both integers, and these are called irrational numbers.

Students construct a right isosceles triangle with legs of 1 unit. Using the Pythagorean Theorem, they determine that the length of the hypotenuse is $\sqrt{2}$. In the figure below, they can rotate the hypotenuse back to the original number line to show that indeed $\sqrt{2}$ is a number on the number line.



In the elementary grades, students become familiar with decimal fractions, most often with decimal representations that terminate a few digits to the right of the decimal point. For example, to find the exact decimal representation of $\frac{2}{7}$, students might use their calculator to find $\frac{2}{7} = 0.2857142857\ldots$ and they might guess that the digits 285714 repeat. To show that the digits do repeat, students in Grade 7 actually carry out the long division and recognize that the remainders repeat in a predictable pattern—a pattern that creates the repetition in the decimal representation (see 7.NS.2d).

Thinking about long division ask students what will happen if the remainder is 0 at some step. They can reason that the long division is complete, and the decimal representation terminates. If the reminder is never 0, in contrast, then the remainders will repeat in a cyclical pattern because at each step with a given remainder, the process for finding the next remainder is always the same. Thus, the digits in the decimal representation also repeat. When dividing by 7, there are 6 possible nonzero remainders, and students can see that the decimal repeats with a pattern of at most 6 digits. In general, when finding the decimal representation of $\frac{m}{n}$, students can reason that the repeating portion of decimal will have at most *n*-1 digits. The important point here is that students can see that the pattern will repeat, so they can imagine the process continuing without actually carrying it out.

Conversely, given a repeating decimal, students learn strategies for converting the decimal to a fraction. One approach is to notice that rational numbers with denominators of 9 repeat a single digit. With a denominator of 99, two digits repeat; with a denominator of 999, three digits repeat, and so on.

- $\frac{13}{99} = 0.13131313\ldots$
- $\frac{74}{99} = 0.74747474\ldots$
- $\frac{237}{999} = 0.237237237\ldots$
- $\frac{485}{999} = 0.485485485\ldots$

From this pattern, students can go in the other direction, conjecturing, for example, that the repeating decimal $0.285714285714\ldots = \frac{285714}{999999}$. And then they can verify that this fraction is equivalent to $\frac{2}{7}$.

Once students understand that (**1**) every rational number has a decimal representation that either terminates or repeats, and (**2**) every terminating or repeating decimal is a rational number, they can reason that on the number line, irrational numbers (i.e., those that are not rational) must have decimal representations that neither terminate nor repeat. Although students at this grade do not need to be able to prove that $\sqrt{2}$ is irrational, they need to know that $\sqrt{2}$ is irrational (see 8.EE.2), which means that its decimal representation neither terminates nor repeats.

Nonetheless, they can approximate $\sqrt{2}$ without using the square root key on the calculator.

Students can create tables like those shown to approximate $\sqrt{2}$ to one, two, and then three places to the right of the decimal point:

x	x^2
1.0	1.00
1.1	1.21
1.2	1.44
1.3	1.69
1.4	1.96
1.5	2.25
1.6	2.56
1.7	2.89
1.8	3.24
1.9	3.61
2.0	4.00

x	x^2
1.40	1.9600
1.41	1.9881
1.42	2.0164
1.43	2.0449
1.44	2.0736
1.45	2.1025
1.46	2.1316
1.47	2.1609
1.48	2.1904
1.49	2.2201
1.50	2.2500

x	x^2
1.410	1.988100
1.411	1.990921
1.412	1.993744
1.413	1.996569
1.414	1.999396
1.415	2.002225
1.416	2.005056
1.417	2.007889
1.418	2.010724
1.419	2.013561
1.420	2.016400

From knowing that $1^2 = 1$ and $2^2 = 4$, or from the tables on the previous page, students can reason that there is a number between 1 and 2 whose square is 2. In the first table above, students can see that between 1.4 and 1.5, there is a number whose square is 2. Then in the second table, they locate that number between 1.41 and 1.42. And in the third table they can locate $\sqrt{2}$ between 1.414 and 1.415.

Students can develop more efficient methods for this work. For example, from the picture above, they might have begun the first table with 1.4. And once they see that $1.42 > 2$, they do not need to generate the rest of the data in the second table.

Use set diagrams to show the relationships among real, rational, irrational numbers, integers, and counting numbers. The diagram should show that the all real numbers (numbers on the number line) are either rational or irrational.

Given two distinct numbers, it is possible to find both a rational and an irrational number between them.

Tools/Resources:

For detailed information, see Learning Progressions for The Number System 6-8.

Also see engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.NS.A
  - Estimating Square Roots
  - Calculating and Rounding Numbers
  - Calculating the square root of 2
- 8.NA.A.1
  - Converting Decimal Representations of Rational Numbers to Fraction Representations
  - Identifying Rational Numbers
  - Converting Repeating Decimals to Fractions

Common Misconceptions:

Some students are surprised that the decimal representation of *pi* does not repeat. Some students believe that if only we keep looking at digits farther and farther to the right, eventually a pattern will emerge.

A few irrational numbers are given special names (*pi* and *e*), and much attention is given to $\sqrt{2}$. Because we name so few irrational numbers, students sometimes conclude that irrational numbers are unusual and rare. However, irrational numbers are much more plentiful than rational numbers, in the sense that they are denser in the real number line.

Students may think that the number line only has the numbers that are labeled.

Students may confuse the radical sign with the division sign.

Students may forget that each rational number has a negative square root, as well as a principal (positive) square root.

Domain: The Number System (NS)

- **Cluster:** *Know that there are numbers that are not rational, and approximate them by rational numbers.*

Standard: Grade 8.NS.2

Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions *(e.g. π^2). For example, for the approximation of $\sqrt{68}$, show that $\sqrt{68}$ is between 8 and 9 and closer to 8.* **(8.NS.2)**

Suggested Standards for Mathematical Practice (*MP*):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

See: 8.NS.1; 8.G.8; 8.G.7

Explanations and Examples:

Students locate rational and irrational numbers on the number line. Students compare and order rational and irrational numbers. Additionally, students understand that the value of a square root can be approximated between integers and that non-perfect square roots are irrational. Students also recognize that square roots may be negative and written as $-\sqrt{28}$.

To find an approximation of $\sqrt{28}$, first determine the perfect squares 28 is between, which would be 25 and 36. The square roots of 25 and 36 are 5 and 6 respectively, so we know that $\sqrt{28}$ is between 5 and 6. Since 28 is closer to 25, an estimate of the square root would be closer to 5. One method to get an estimate is to divide 3 (the distance between 25 and 28) by 11 (the distance between the perfect squares of 25 and 36) to get 0.27. The estimate of $\sqrt{28}$ would be 5.27 (the actual is 5.29).

Students can approximate square roots by iterative processes.

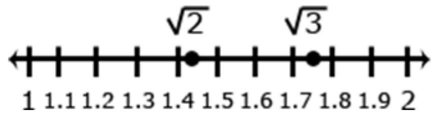
Examples:

Approximate the value of $\sqrt{5}$ to the nearest hundredth.

Solution:

Students start with a rough estimate based upon perfect squares. $\sqrt{5}$ falls between 2 and 3 because 5 falls between $2^2 = 4\ and\ 3^2 = 9$. The value will be closer to 2 than to 3. Students continue the iterative process with the tenths place value. $\sqrt{5}$ falls between 2.2 and 2.3 because 5 falls between $2.2^2 = 4.84\ and\ 2.3^2 = 5.29$. The value is closer to 2.2. Further iteration shows that the value of $\sqrt{5}$ is between 2.23 and 2.24 since 2.23^2 is 4.9729 and 2.24^2 is 5.0176

Compare $\sqrt{2}$ and $\sqrt{3}$ by estimating their values, plotting them on a number line, and making comparative statements.

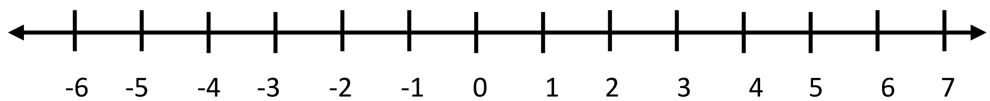


Solution:
Statements for the comparison could include:

- $\sqrt{2}$ is approximately 0.3 less than $\sqrt{3}$
- $\sqrt{2}$ is between the whole numbers 1 and 2
- $\sqrt{3}$ is between 1.7 and 1.8

Without using your calculator, label approximate locations for the following numbers on the number line.

a. π
b. $-\left(\frac{1}{2} \times \pi\right)$
c. $2\sqrt{2}$
d. $\sqrt{17}$



Solution:

a. π is slightly greater than 3.
b. $-\left(\frac{1}{2} \times \pi\right)$ is slightly less than -1.5
c. $\left(2\sqrt{2}\right)^2 = 4 \cdot 2 = 8 \; and \; 3^2 = 9, so \; 2\sqrt{2}$ is slightly less than 3.
d. $\sqrt{16} = 4$, so $\sqrt{17}$ is slightly greater than 4.

Instructional Strategies: See 8.NS.1.

Big ideas: Evaluate square roots and cube roots of perfect squares/cubes
Approximate square roots of non-perfect squares

Tools/Resources:

Also see engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.NS.A.2
    - Comparing Rational and Irrational Numbers
    - Irrational Numbers on the Number Line
    - Placing a square root on the number line

Common Misconceptions: See 8.NS.1.

Domain: Expressions and Equations (EE)

▶ **Cluster:** *Work with radicals and integer exponents.*

Standard: Grade 8.EE.1

Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of whole number perfect squares with solutions between 0 and 15 and cube roots of whole number perfect cubes with solutions between 0 and 5. Know that $\sqrt{2}$ is irrational. **(8.EE.2)**

Suggested Standards for Mathematical Practice (*MP*):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 8.G.7; 8.G.8

Explanations and Examples:

Students recognize that squaring a number and taking the square root $\sqrt{}$ of a number are inverse operations; likewise, cubing a number and taking the cube root $\sqrt[3]{}$ are inverse operations.

This understanding is used to solve equations containing square or cube numbers. Equations may include rational numbers such as $x^2 = \frac{1}{4}, x^2 = \frac{4}{9}\ or\ x^3 = \frac{1}{8}$ (Note: Both the numerator and denominators are perfect squares or perfect cubes.)

Students recognize perfect squares and cubes, understanding that non-perfect squares and non-perfect cubes are irrational. Students understand that in geometry a square root is the length of the side of a square and a cube root is the length of the side of a cube. The value of p for square root and cube root equations must be positive.
See this explanation from the Math Forum (This might make an excellent article to read and reinforce the pervasive nature of functions.)

Square Root of 100
From: Cody Cry
Subject: Why can't -10 be a solution to $\sqrt{100}$? When you take the square root of 100 you get 10. Why can't -10 be an answer? If you square -10 you get 100 also, so why can't 10 and -10 be the answers?
Thanks, Cody Cry

From: Doctor Rick
Subject: Re: Why can't -10 be a solution to 100 square rooted?
Hi, Cody! You've asked a good question. What's going on here, I think, is that we have two separate concepts that can easily be confused. One concept is the process of finding the root of a square; for instance, the number x such that $x^2 = 100$. You know that this has two solutions: 10 and -10 are both roots of this equation. The other concept is the square root FUNCTION. A function takes in one number and returns another number. The number that it returns must be UNIQUE since a function is by definition single-valued. You can't put in 100 and get out both 10 and -10. The square root function is therefore DEFINED so that sqrt(y) returns the NON-NEGATIVE root of $x^2 = y$. Then we say that the two roots of$x^2 = 100$ are $\pm\sqrt{100}$, that is, $\pm$ [non-negative] $\sqrt{100}$. Admittedly this is a rather arbitrary definition. But if functions were not single-valued, we'd have a mess. And when you use the square root function in real situations, it will make sense. For instance, the distance between the points (0, 0) and (x, y) is $\sqrt{x^2 + y^2}$, and it makes sense that the distance is a non-negative number. The answer I just gave is already in our Dr. Math Archives.

Here is something else from our Archives that might add a bit to my reasoning.
http://mathforum.org/dr.math/problems/ken.8.28.96.html
- Doctor Rick, The Math Forum http://mathforum.org/dr.math/

Examples:
$3^2 = 9$ and $\sqrt{9} = 3$

$\left(\frac{1}{3}\right)^3 = \left(\frac{1^3}{3^3}\right)$ and $\sqrt[3]{\frac{1}{27}} = \frac{\sqrt[3]{1}}{\sqrt[3]{27}} = \frac{1}{3}$

Solve: $x^2 = 9$

Solution:

$$x^2 = 9$$
$$\sqrt{\boldsymbol{x^2}} = \pm\sqrt{\mathbf{9}}$$
$$x = \pm 3$$

Solve $x^3 = 8$

Solution:

$$x^3 = 8$$
$$\sqrt[3]{x^3} = \sqrt[3]{8}$$
$$x = 2$$

Classify the numbers in the box as perfect squares and perfect cubes. To classify a number, drag it to the appropriate column in the chart. Numbers that are neither perfect squares nor perfect cubes should **not** be placed in the chart.

1 36 64 27 196 8		
Perfect Squares but NOT Perfect Cubes	Both Perfect Squares and Perfect Cubes	Perfect Cubes but NOT Perfect Squares

Solution:

Perfect Squares but NOT Perfect Cubes	Both Perfect Squares and Perfect Cubes	Perfect Cubes but NOT Perfect Squares
36 196	1 64	27 8

Use the numbers shown to make the equations true. Each number can be used only once.

To use a number, drag it to the appropriate box in an equation.

4	8	10	64	100	1,000

$\sqrt{\square} = \square$ $\qquad\qquad$ $\sqrt[3]{\square} = \square$

Sample Responses:

Equation 1: (64, 8) Equation 1: (100,10)

Equation 2: (1000,10) Equation 2: (64,4)

Ashley and Brandon have different methods for finding square roots.

Ashley's Method
To find the square root of *x,* find a number so that the product of the number and itself is *x*.
For example, $2 \bullet 2 = 4$, so the square root of 4 is 2.

Brandon's Method
To find the square root of *x*, multiply *x* by $\frac{1}{2}$. For example, $4 \cdot \frac{1}{2} = 2$, so the square root of 4 is 2.

Which student's method is **not** correct? Explain why the method you selected is **not** correct.

Sample Response:
Brandon's method is not correct.
Brandon's method works for the square root of 4, but it wouldn't work for the square root of 36. Half of 36 is 18, but the square root of 36 is 6 since 6 times 6 equals 36. Ashley describes the correct way to find the square root of a number.

Tools/Resources:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.NS.A
    - Estimating Square Roots
- 8.NS.A.2
    - Calculating the Square Root of 2
    - Placing a Square Root on the Number Line

Common Misconceptions: See 8.EE.1.

Domain: Expressions and Equations (EE)

▶ ***Cluster:*** *Work with radicals and integer exponents.*

Standard: Grade 8.EE.2

Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as* 3×10^8 *and the population of the world as* 7×10^9*, and determine that the world population is more than 20 times larger.* **(8.EE.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 8.EE.1; 8.G.7; 8.G.8

Explanations and Examples:

The goal of this standard is to continue to help build a student's number sense using scientific notation. In fourth and fifth grade students learn how to use the place value system, understanding a number's value is ten times larger than the number to its right and $\frac{1}{10}$ of the number to its left using whole numbers. Additionally, in fifth grade students expand their number sense using whole-number exponents (5.NBT.2). See picture below.

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
$10 \times 10 \times 10 = 10^3$	$10 \times 10 = 10^2$	10				
1,000	100	10	1	0.1 or $\frac{1}{10}$	0.01 or $\frac{1}{100}$	0.001 or $\frac{1}{1000}$

In sixth grade, students expand their understanding from whole numbers to integers (6.NS.5) and continue understanding whole-number exponents (6.EE.1, 6.NS.2c). In eighth grade, teachers must expand on the student's place value understanding to include integer exponents. Students can expand the pattern of whole-number exponents to integer exponents. See picture below.

Thousands	Hundreds	Tens	Ones	Tenths	Hundredths	Thousandths
$10 \times 10 \times 10 = 10^3$	$10 \times 10 = 10^2$	$1 \times 10 = 10^1$	10^0	10^{-1}	10^{-2}	10^{-3}
1,000	100	10	1	0.1 or $\frac{1}{10}$	0.01 or $\frac{1}{100}$	0.001 or $\frac{1}{1000}$

Using these understandings students can convert between standard and scientific notation and estimate numbers expressed in scientific notation **without using** of the properties of exponents. For instance, in the example mentioned above, "*For example, estimate the population of the United States as* 3×10^8 *and the population of the world as* 7×10^9*, and determine that the world population is more than 20 times larger.*" A student could solve this by knowing that 7 is more than two times as large as 3 and that 10^9 is ten times as large as 10^8. Therefore, knowing that $2 \times 10 = 20$ and our number is more than two times as large we can deduce that the population is more than 20 times larger.

Students may begin to notice patterns that would lead to the "discovery" of some of the properties of exponents. This is a natural progression in the student's building of number sense and expands to properties of exponents in high school (N.NRN.1).

Students express numbers in scientific notation. Students compare and interpret scientific notation quantities in the context of the situation. If the exponent increases by one, the value increases 10 times. Students understand the magnitude of the number expressed in scientific notation and choose an appropriate corresponding unit. For example, 3×10^8 is equivalent to 30 million, which represents a large quantity. Therefore, this value will affect the unit chosen.

Examples:

Example 1

How many times larger is 8×10^5 than 4×10^{-3}? Justify your reasoning.

Solution: Eight is twice as large at four and using the place value understanding I have to multiply by 10 eight times to get from 10^{-3} to 10^5. Therefore, 8×10^5 is $2 \times 100{,}000{,}000 = 200{,}000{,}000$ times as large as 4×10^{-3}.

Example 3

The average distance from Jupiter to the Sun is about 5×10^8 miles. The average distance from Venus to the Sun is about 7×10^7.

Based on these calculations is Jupiter or Venus closer to the sun? Justify your reasoning.

Solution: Venus. Explanations vary.

Tools/Resources:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.A.3
  - Ant and Elephant
  - Pennies to heaven
  - Orders of Magnitude

Common Misconceptions: See 8.EE.1.

Domain: Expressions and Equations (EE)

Cluster: *Work with radicals and integer exponents.*

Standard: Grade 8.EE.3

Read and write numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology. **(8.EE.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.

Connections: See 8.EE.2; 8.NS.1

Explanations and Examples:

There are multiple components to this standard.

1. Read and write scientific notation
2. Use scientific notation
3. Choose appropriate units for very large and small numbers
4. Interpret the output from technology when the answer is in scientific notation

Writing numbers in scientific notation should reinforce the magnitude of the number rather than memorizing procedures such as "moving the decimal point…"

When using scientific notation to solve problems, utilize technology to support the last part of the standard but include questions asking students to state the output from technology in an appropriate form.

Choosing units appropriate to the measurement is a difficult concept for students because there is not always a single correct answer and there is not an algorithm to be applied. However, this is an introduction to choosing appropriate units and will be explored further in high school. Therefore, it is recommended to make the choices fairly obvious. For example, choosing between $2.31X10^{19}\ cm$ vs. $23.1x10^{18}\ mm$ might be a matter of preference but $2.31X10^{-5}\ km$ is likely not the appropriate sized unit.

Students must also understand scientific notation as generated on various calculators or other technology. In working with calculators or spreadsheets, it is important that students recognize scientific notation. Students should recognize that the output of $2.45\ E23\ is\ 2.45 \times 10^{23}\ and\ 3.5\ E-4\ is\ 3.5 \times 10^{-4}$.

Students enter scientific notation using E or EE (scientific notation), * (multiplication), and ^ (exponent) symbols.

Examples:

1. Express each number in scientific notation:
   - 0.0259
   - 836
   - 34,121

2. Express each number in standard notation:
   - 7.452 X 10
   - 9.14 $X10^2$
   - 1.9824 $X10^{-4}$

3. Provide the scientific notation for each value:
   - 7,990,000
   - 3,600
   - 144,000
   - 0.000000432
   - 0.301

4. Mr. Griffin's class is studying the solar system. The circumference of the Earth at the equator is about 24,900 miles. Express this number in scientific notation.

5. The speed of light is approximately 6.71×10^8 miles per hour. Express this number in standard form.

6. Which is greater, 5.15×10^{-4}or 6.35×10^{-5}?

7. Compare the following
   - 2.56×10^5 and 4.2×10^{-7}
   - 4.3×10^4 and 1.6×10^6
   - 7.1×10^{-2} and 2.9×10^{-6}

Instructional Strategies: See 8.EE.2.

Resources/Tools:

See engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.A.4
  - Giantburgers
  - Ants versus humans
  - Pennies to heaven
  - Choosing appropriate units

Domain: Expressions and Equations (EE)

▶**Cluster:** *Understand the connections between proportional relationships, lines, and linear equations.*

Standard: Grade 8.EE.4

Graph proportional relationships, interpreting the unit rate as the slope (m) of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.* **(8.EE.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: 8.F.2, 8.F.3; 8.F.2

This cluster is connected to:

- Grade 8 Critical Area of Focus #1: **formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations.**
- Critical Area of Focus #3: **analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem**.

Explanations and Examples:

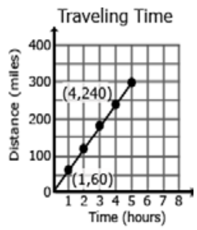
Students build on their work with unit rates from 6th grade and proportional relationships in 7th grade to compare graphs, tables and equations of proportional relationships. Students identify the unit rate (or slope) in graphs, tables and equations to compare two or more proportional relationships.

Using graphs of experiences that are familiar to students increases accessibility and supports understanding and interpretation of proportional relationship. Students are expected to both sketch and interpret graphs.

Examples:

Compare the scenarios to determine which represents a greater speed. Include a description of each scenario including the unit rates in your explanation.

Scenario 1:



Scenario 2:

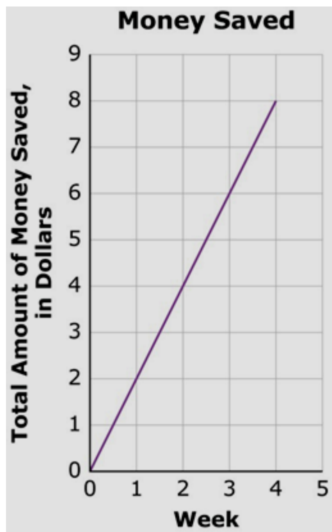
$y = 50x$

x is time in hours

y is distance in miles

Three students saved money for four weeks.

Antwan saved the same amount of money each week for 4 weeks. He made this graph to show how much money he saved.



Carla saved the same amount of money each week for 4 weeks. She made this table to show how much money she saved.

Week	Total Amount of Money Saved
1	\$1.75
2	\$3.50
3	\$5.25
4	\$7.00

Omar saved the same amount of money each week for 4 weeks. He wrote the equation below to show how much he saved. In the equation, S is the total amount of money saved, in dollars, and w is the number of weeks.

$$S = 2.5w$$

Identify the student who saved the greatest amount of money each week **and** the student who saved the least amount of money each week.

Solution:
Omar saved the greatest amount. Carla saved the least amount of money.

Instructional Strategies:

This cluster focuses on extending the understanding of ratios and proportions. Unit rates have been explored in Grade 6 as the comparison of two different quantities with the second unit a unit of one, (unit rate). In seventh grade unit rates were expanded to complex fractions and percents through solving multistep problems such as: discounts, interest, taxes, tips, and percent of increase or decrease. Proportional relationships were applied in scale drawings, and students should have developed an informal understanding that the steepness of the graph is the slope or unit rate. Now unit rates are addressed formally in graphical representations, algebraic equations, and geometry through similar triangles.

Distance time problems are notorious in mathematics. In this cluster, they serve the purpose of illustrating how the rates of two objects can be represented, analyzed and described in different ways: graphically and algebraically. Emphasize the creation of representative graphs and the meaning of various points. Then compare the same information when represented in an equation. By using coordinate grids and various sets of three similar triangles, students can prove that the slopes of the corresponding sides are equal, thus making the unit rate of change equal. After proving with multiple sets of triangles, students can be led to generalize the slope to $y = mx$ for a line through the origin and $y = mx + b$ for a line through the vertical axis at b.

Resources/Tools:

See engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.B
    - Find the Change
    - Equations of Lines
    - DVD Profits, Variation 1
    - Proportional relationships, lines, and linear equations
    - Stuffing Envelopes
    - Folding a Square into Thirds
- 8.EE.B.5
    - Coffee by the Pound
    - Peaches and Plums
    - Who Has the Best Job?
    - Comparing Speeds in Graphs and Equations
    - Sore Throats, Variation 2
    - Stuffing Envelopes

Domain: Expressions and Equations (EE)

▶**Cluster:** *Understand the connections between proportional relationships, lines, and linear equations.*

Standard: **Grade 8.EE.5**

Use similar triangles to explain why the slope (*m)* is the same between any two distinct points on a non-vertical line in the coordinate plane; and extend to include the use of the slope formula ($m = \frac{y_2 - y_1}{x_2 - x_1}$ when given two coordinate points (x_1, y_1) and (x_2, y_2)). Generate the equation $y = mx$ for a line through the origin (proportional) and the equation $y = mx + b$ for a line with slope *m* intercepting the vertical axis at y-intercept *b* (not proportional when $b \neq 0$). **(8.EE.6)**

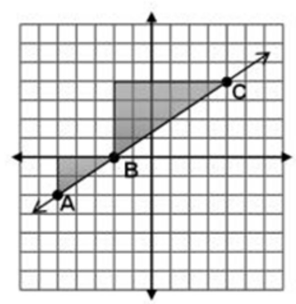
Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 8.EE.5; 8.F.3; 8.G.5 and 8.G.4

Explanations and Examples:

Triangles are similar when there is a constant rate of proportion between them. Using a graph, students construct triangles between two points on a line and compare the sides to understand that the slope (ratio of rise to run) is the same between any two points on a line.

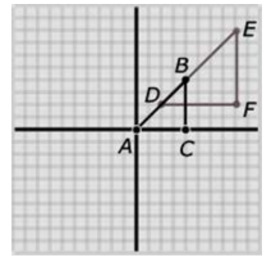


The triangle between A and B has a vertical height of 2 and a horizontal length of 3. The triangle between B and C has a vertical height of 4 and a horizontal length of 6. The simplified ratio of the vertical height to the horizontal length of both triangles is 2 to 3, which also represents a slope of $\frac{2}{3}$ for the line.

Students write equations in the form $y = mx + b$ for lines going through the origin, recognizing that *m* represents the slope of the line. Students write equations in the form $y = mx + b$ for lines not passing through the origin, recognizing that m represents the slope and b represents the *y*-intercept.

Examples:

Explain why ΔACB is similar to ΔDFE, and deduce that $\overline{AB}$ has the same slope as $\overline{BE}$. Express each line as an equation.



Mr. Perry's students used pairs of points to find the slopes of lines. Mr. Perry asked Avery how she used the pairs of points listed in this table to find the slope of a line.

x	*y*
8	18
20	45

Avery said, "The easiest way to find the slope is to divide *y* by *x*. The slope of this line is $\frac{18}{8}$or $\frac{9}{4}$."

Part A

Show another way to find the slope of the line that passes through the points listed in the table. Your way must be different than Avery's way.

Part B

Write an example that shows that Avery's "divide *y* by *x*" method will not work to find the slope of **any** line.

Sample Response:

Part A:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{45 - 18}{20 - 8} = \frac{27}{12} = \frac{9}{4}$$

Part B:

If Mr. Perry asked the class to find the slope of the line through (1, 1) and (2, 3), you can find the actual slope by using the formula and get $\frac{3-1}{2-1} = \frac{2}{1} = 2$, but Avery's method will not work because she would either say the slope is $\frac{1}{1} = 1$ or $\frac{3}{2} = 1.5$.

Instructional Strategies: See 8.EE.5.

Resources/Tools:

See engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.B.6
    - Slopes Between Points on a Line

Domain: Expressions and Equations (EE)

▶**Cluster:** *Understand the connections between proportional relationships, lines, and linear equations.*

Standard: **Grade 8.EE.6**

Describe the relationship between the proportional relationship expressed in $y = mx$ and the non-proportional linear relationship $y = mx + b$ as a result of a vertical translation. *Note: be clear with students that all linear relationships have a constant rate of change (slope), but only the special case of proportional relationships (line that goes through the origin) continue to have a* ***constant of proportionality****.* **(2017)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 8.EE.5, 8.F.2, 8.F.3; 8.G.5

- Also see Ratios and Proportional Relationships in grades 6 and 7 to see how these concepts are developed.

Explanations and Examples:

This standard might be a new math standard but it is really an explicit explanation of 8.EE.5. As students generate the equation $y = mx + b$, through an exploration of *y=mx* using similar triangles, they will be able to describe all linear functions as a translation of $y = mx$. There are many benefits to making this connection explicit. Many students incorrectly apply proportional reasoning, such as doubling the output when the input is doubled, to non-proportional relationships. Helping students see how the linear function is both the same and different than a proportional relationship will support the appropriate use of proportional thinking using the rate of change. To support these connections, students should graph proportional relationships, in context, and compare the graph to the same situation with a starting value.

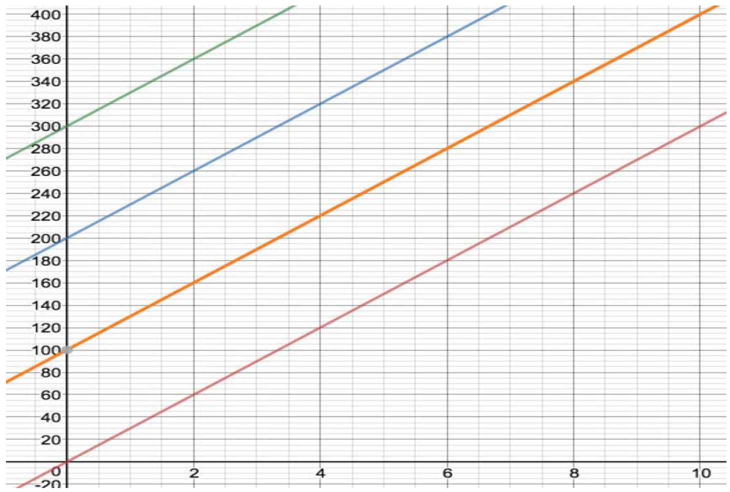
Another benefit that can result from this view of linear functions is to connect linear functions to the algebraic transformations that students will explore in high school. They will learn that all functions are vertically translated when a constant is added to the output. While students in 8^{th} grade do not need to understand transformations in 8^{th} grade, it is important for teachers to understand that this standard is the first layer of deeper knowledge to come. Teachers are *not* expected to teach geometric transformations. Students will see that there is a vertical shift without naming the shift a translation.

Examples:

Example 1:

The robotics club is taking a field trip to the University of Kansas robotics competition. Tickets to observe the engineering students compete are $30/student. Fill in the table and graph the results to illustrate the cost for the club to attend the competition.

Number of students	Total club cost
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	



Example of graphed solution.

How will the graph change if the club adds transportation costs to the total? Use the table to explore the variety of transportation costs explored. Graph each option on the plane provided above.

Number of students	Rental Van, $200 Total Club Cost	Bus Rental, $300 Total Club Cost	Individual Transportation, $100 Total Club Cost
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

Example 2:

Draw a Venn diagram comparing proportional and linear relationships.

Instructional Strategies:

Desmos is an excellent online dynamic graphing tool to explore how the graph changes as *b* changes. You can input any proportional equation, add the letter *b* and then select "add slider." This will create a slider that the students can adjust to see how the graph changes. The graphing calculator is another tool that allows students to input multiple equations and then compare the table of values for a variety of equations.

This standard is laying the foundation for studying the transformations of functions in Algebra 1 and Algebra 2. Students might be interested to learn that every function with a constant term added or subtracted will be vertically translated. Depending on your time, you might show them functions they will see in the future. For example, $y = x^2$ can be translated to $y = x^2 + 5$. Even functions they won't see for many years use the same rules learned with the standard. Trigonometric functions, such as $y = \sin x$ can be vertically translated with the equation $y = \sin x - 2$.

Resources/Tools:

Desmos

Graphing Calculator

MARS Mathematic Assessment Project

Domain: Expressions and Equations (EE)

▶**Cluster:** *Analyze and solve linear equations and inequalities.*

Standard: Grade 8.EE.7

8.EE.7. Fluently (efficiently, accurately, and flexibly) solve one-step, two-step, and multi-step linear equations and inequalities in one variable, including situations with the same variable appearing on both sides of the equal sign.

- 8.EE.7a. Give examples of linear equations in one variable with one solution ($x = a$), infinitely many solutions ($a = a$), or no solutions ($a = b$). Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a, a = a, or\ a = b$ results (where *a* and *b* are different numbers). **(8.EE.7a)**
- 8.EE.7b. Solve linear equations and inequalities with rational number coefficients, including equations/inequalities whose solutions require expanding and/or factoring expressions using the distributive property and collecting like terms. **(8.EE.7b)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: 8.F.3; 8.NS.1; 6.EE.B; 7.EE.4

This cluster is connected to:

- Grade 8 Critical Area of Focus #1: Formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations and.
- This cluster also builds upon the understandings in Grades 6 and 7 of Expressions and Equations, Ratios and Proportional Relationships, and utilizes the skills developed in the previous grade in The Number System.

Explanations and Examples:

Students solve one-variable equations with the variables being on both sides of the equals sign. Students recognize that the solution to the equation is the value(s) of the variable, which make a true equality when substituted back into the equation. Equations shall include rational numbers, distributive property and combining like terms.

Equations have one solution when the variables do not cancel out. For example, $10x - 23 = 29 - 3x$ can be solved to $x = 4$. This means that when the value of *x* is 4, both sides will be equal. If each side of the equation were treated as a linear equation and graphed, the solution of the equation represents the coordinates of the point where the two lines would intersect. In this example, the ordered pair would be $(4, 17)$:

$$10 \cdot 4 - 23 = 29 - 3 \cdot 4$$
$$40 - 23 = 29 - 12$$
$$17 = 17$$

Equations having no solution have variables that will cancel out and constants that are not equal. This means that there is not a value that can be substituted for x that will make the sides equal. For example, the equation $-x + 7 – 6x = 19 – 7x$, can be simplified to $-7x + 7 = 19 – 7x$. If $7x$ is added to each side, the resulting equation is $7 = 19$, which is not true. No matter what value is substituted for x the final result will be $7 = 19$. If each side of the equation were treated as a linear equation and graphed, the lines would be parallel.

An equation with infinitely many solutions occurs when both sides of the equation are the same. Any value of x will produce a valid equation. For example, the following equation, when simplified will give the same values on both sides.

$$-\frac{1}{2}(36a - 6) = \frac{3}{4}(4 - 24a)$$
$$-18a + 3 = 3 - 18a$$

If each side of the equation were treated as a linear equation and graphed, the graph would be the same line.

As students transform linear equations in one variable into simpler forms, they discover the equations can have one solution, infinitely many solutions, or no solutions.

When the equation has one solution, the variable has one value that makes the equation true as in $12 - 4y = 16$. The only value for y that makes this equation true is -1.

When the equation has infinitely many solutions, the equation is true for all real numbers as in $7x + 14 = 7(x + 2)$. As this equation is simplified, the variable terms cancel leaving $14 = 14$ or $0 = 0$. Since the expressions are equivalent, the value for the two sides of the equation will be the same regardless which real number is used for the substitution.

Examples:
Solve for x:

$$-3(x+7) = 4$$
$$3x - 8 = 4x - 8$$
$$3(x+1) - 5 = 3x - 2$$

Solve:

$$7(m-3) = 7$$
$$\frac{1}{4} - \frac{2}{3}y = \frac{3}{4} - \frac{1}{3}y$$

For each linear equation in this table, indicate whether the equation has no solution, one solution, or infinitely many solutions.

Equation	No Solution	One Solution	Infinitely Many Solutions
$7x + 21 = 21$			
$12x + 15 = 12x - 15$			
$-5x - 25 = 5x + 25$			

Solution:

1. One solution. This is designed to be an easy equation to solve to help students enter the problem. Answering this question correctly demonstrates minimal understanding.
2. No solution. Students may think there is no difference between adding 15 on the left-hand side and subtracting 15 on the right-hand side.
3. One solution. Students may think there are infinitely many solutions because the left-hand side is the negative of the right-hand side.

Three students solved the equation $3(5x - 14) = 18$ in different ways, but each student arrived at the correct answer. Select all of the solutions that show a correct method for solving the equation.

A.

$$\begin{aligned} 3(5x-14) &= 18 \\ 8x - 14 &= 18 \\ +14 \quad & \quad +14 \\ \frac{8x}{8} &= \frac{32}{8} \\ x &= 4 \end{aligned}$$

B.

$$\begin{aligned} \frac{1}{\cancel{3}} \cdot \cancel{3}(5x-14) &= 18 \cdot \frac{1}{3} \\ 5x - 14 &= 6 \\ +14 \quad & \quad +14 \\ \frac{5x}{5} &= \frac{20}{5} \\ x &= 4 \end{aligned}$$

C.

$$\begin{aligned} 3(5x-14) &= 18 \\ \frac{15x}{15} - \frac{42}{15} &= \frac{18}{15} \\ +\frac{42}{15} \quad & \quad +\frac{42}{15} \\ x &= \frac{60}{15} \\ x &= 4 \end{aligned}$$

Sample Response:

A. This solution is the simplest to follow, but the method is incorrect.

B. Although the method in this solution is correct, it is not the most commonly used method for solving equations like this, so students may think it is incorrect.

C. Although the method in this solution is correct, it is not the most commonly used method for solving equations like this, so students may think it is incorrect.

Consider the equation $3(2x + 5) = ax + b$.

Part A

Find one value for a and one value for b so that there is exactly one value of *x* that makes the equation true. Explain your reasoning.

Part B

Find one value for a and one value for b so that there are infinitely many values of x that make the equation true. Explain your reasoning.

Sample Response:

Part A

$a = 5; b = 16$ When you substitute these numbers in for a and b, you get a solution of $x = 1$.

Part B

$a = 6; b = 15$; When you substitute these numbers in for *a* and b, you get a solution of $0 = 0$, so there are infinitely many solutions, not just one.

Instructional Strategies:

In Grade 6, students applied the properties of operations to generate equivalent expressions, and identified when two expressions are equivalent. This cluster extends understanding to the process of solving equations and to their solutions, building on the fact that solutions maintain equality, and that equations may have only one solution, many solutions, or no solution at all. Equations with many solutions may be as simple as $3x = 3x, 3x + 5 = x + 2 + x + x + 3, or\ x(6 + 4)$ where both sides of the equation are equivalent once each side is simplified.

Table 3 in the appendix generalizes the properties of operations and serves as a reminder for teachers of what these properties are. Eighth graders should be able to describe these relationships with real numbers and justify their reasoning using words and not necessarily with the algebraic language of Table 3. In other words, students should be able to state that $3(-5) = (-5)3$ because multiplication is commutative and it can be performed in any order (it is commutative), or that $9(8) + 9(2) = 9(8 + 2)$ because the distributive property allows us to distribute multiplication over addition, or determine products and add them. Grade 8 is the beginning of using the generalized properties of operations, but this is not something on which students should be assessed.

Pairing contextual situations with equation solving allows students to connect mathematical analysis with real-life events. Students should experience analyzing and representing contextual situations with equations, identify whether there is one, none, or many solutions, and then solve to prove conjectures about the solutions. Through multiple opportunities to analyze and solve equations, students should be able to estimate the number of solutions and possible values(s) of solutions prior to solving. Rich problems, such as computing the number of tiles needed to put a border around a rectangular space or solving proportional problems as in doubling recipes, help ground the abstract symbolism to life.

Experiences should move through the stages of concrete, conceptual and algebraic/abstract. Utilize experiences with the pan balance model as a visual tool for maintaining equality (balance) first with simple numbers, then with pictures symbolizing relationships, and finally with rational numbers allows understanding to develop as the complexity of the problems increases. Equation solving in Grade 8 should involve multistep problems that require the use of the distributive property, collecting like terms, and variables on both sides of the equation.

This cluster builds on the informal understanding of slope from graphing unit rates in Grade 6 and graphing proportional relationships in Grade 7 with a stronger, more formal understanding of slope.

Provide opportunities for students to change forms of equations (from a given form to slope- intercept form) in order to compare equations.

Resources/Tools:

See engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.EE.C
  - Two Lines
- 8.EE.C.7
  - The Sign of Solutions
  - Coupon versus discount
  - Solving Equations
  - Sammy's Chipmunk and Squirrel Observations

Common Misconceptions:

Students think that only the letters x and y can be used for variables. Students think that you always need a variable to equal a constant as a solution. The variable is always on the left side of the equation.

Equations are not always in the slope intercept form, $y = mx + b$. Students confuse one-variable and two-variable equations.

Domain: Functions (F)

▶**Cluster:** *Define, evaluate, and compare functions.*

Standard: Grade 8.F.1

Explain that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output. (Function notation is not required in Grade 8.) **(8.F.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This Cluster is connected to:

- Grade 8 Critical Area of Focus #2: Grasping the concept of a function and using functions to describe quantitative relationships.
- Expressions and Equations: Linear equations in two variables can be used to define linear functions, and students can use graphs of functions to reason toward solutions to linear equations.
- Expressions and Equation: Similar triangles are used to show that the slope of a line is constant.
- Statistics and Probability: Bivariate data can often be modeled by a linear function.

Explanations and Examples:

Students distinguish between functions and non-functions, using equations, graphs, and tables. Non-functions occur when there is more than one *y*-value is associated with any *x*-value.

For example, the rule that takes *x* as input and gives $x^2 + 5x + 4$ as output is a function. Using y to stand for the output we can represent this function with the equation $y = x^2 + 5x + 4$, and the graph of the equation is the graph of the function. Students are not yet expected use function notation such as $f(x) = x^2 + 5x + 4$.

Examples:

Fill in each *x*-value and *y*-value in the table below to create a relation that is **not** a function.

x	y

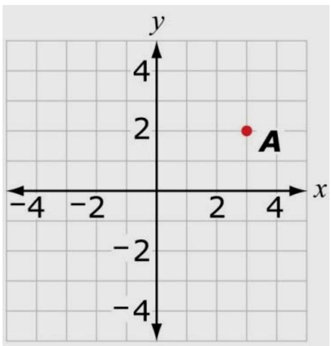
Sample Response:

x	y
4	0
4	1
4	2
4	3
4	4

Point ***A*** is plotted on the *xy*-coordinate plane below. You must determine the location of point ***C*** given the following criteria:

- Point *C* has integer coordinates.
- The graph of line $\overleftrightarrow{AC}$ is **not** a function.

Place a point on the *xy*-coordinate plane that could represent point ***C***.



Sample Responses:
(3, 5) , (3, 4) , (3, 3) , (3, 1) , (3, 0) , (3, −1) , (3, −2) , (3, −3) , (3, −4) , or (3, −5)

Instructional Strategies:

In grade 6, students plotted points in all four quadrants of the coordinate plane. They also represented and analyzed quantitative relationships between dependent and independent variables. In Grade 7, students decided whether two quantities are in a proportional relationship. In Grade 8, students begin to call relationships functions when each input is assigned to exactly one output. Also, in Grade 8, students learn that proportional relationships are part of a broader group of linear functions, and they are able to identify whether a relationship is linear. Nonlinear functions are included for comparison. Later, in high school, students use function notation and are able to identify types of nonlinear functions.

To determine whether a relationship is a function, students should be expected to **reason from a context**, a graph, or a table, after first being clear which quantity is considered the input and which is the output. When a relationship is not a function, students should produce a counterexample: an “input value” with at least two “output values.” If the relationship is a function, the students should explain how they verified that for each input there was exactly one output. The “vertical line test” should be avoided because (1) it is too easy to apply without thinking, (2) students do not need an efficient strategy at this point, and (3) it creates misconceptions for later mathematics, when it is useful to think of functions more broadly, such as whether *x* might be a function of *y*.

“Function machine” pictures are useful for helping students imagine input and output values, with a rule inside the machine by which the output value is determined from the input.

Notice that the standards explicitly call for exploring functions numerically, graphically, verbally, and algebraically (symbolically, with letters). This is sometimes called the “rule of four.” For fluency and flexibility in thinking, students need experiences translating among these. In Grade 8, the focus is on linear functions, and students begin to recognize a linear function from its form $y = mx + b$. Students also need experiences with nonlinear functions, including functions given by graphs, tables, or verbal descriptions but for which there is no formula for the rule, such as a girl’s height as a function of her age.

In the elementary grades, students explore number and shape patterns (sequences), and they use rules for finding the next term in the sequence. At this point, students describe sequences both by rules relating one term to the next and also by rules for finding the *n*th term directly. (In high school, students will call these recursive and explicit formulas.) Students express rules in both words and in symbols. Instruction should focus on additive and multiplicative sequences as well as sequences of square and cubic numbers, considered as areas and volumes of cubes, respectively.

When plotting points and drawing graphs, students should develop the habit of determining, based upon the context, whether it is reasonable to “connect the dots” on the graph. In some contexts, the inputs are discrete, and connecting the dots can be misleading. For example, if a function is used to model the height of a stack of *n* paper cups, it does not make sense to have 2.3 cups, and thus there will be no ordered pairs between $n = 2$ and $n = 3$.

Provide multiple opportunities to examine the graphs of linear functions and use graphing calculators or computer software to analyze or compare at least two functions at the same time. Illustrate with a slope triangle where the run is "1" that slope is the "unit rate of change." Compare this in order to compare two different situations and identify which is increasing/decreasing as a faster rate.

Students can compute the area and perimeter of different-size squares and identify that one relationship is linear while the other is not by either looking at a table of value or a graph in which the side length is the independent variable (input) and the area or perimeter is the dependent variable (output).

Resources/Tools:

For detailed information, see Learning Progressions Functions.

See engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.F.A.1
    - The Customers
    - Foxes and Rabbits
    - US Garbage, Version 1
    - Pennies to heaven
    - Function Rules
    - Introducing Functions

Common Misconceptions:

Some students will mistakenly think of a straight line as horizontal or vertical only.
Students may mistakenly believe that a slope of zero is the same as "no slope" and then confuse a horizontal line (slope of zero) with a vertical line (undefined slope).

Confuse the meaning of "domain" and "range" of a function.

Some students will mix up *x*- and *y*-axes on the coordinate plane, or mix up the ordered pairs. When emphasizing that the first value is plotted on the horizontal axes (usually *x,* with positive to the right) and the second is the vertical axis (usually called *y*, with positive up), point out that this is merely a convention: It could have been otherwise, but it is very useful for people to agree on a standard customary practice.

Domain: Functions (F)

◆**Cluster:** *Define, evaluate, and compare functions.*

Standard: Grade 8.F.2

Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change, the greater y-intercept, or the point of intersection.* **(8.F.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: See 8.F.1; 8.EE.5

Explanations and Examples:

Students compare functions from different representations.

For example, compare the following functions to determine which has the greater rate of change:

Function 1: $y = 2x + 4$

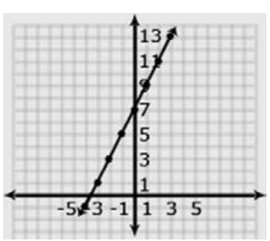
Function 2: (table)

x	*y*
-1	-6
0	-3
2	3

Examples:

Compare the two linear functions listed below and determine which equation represents a greater rate of change.

Function 1:



Function 2: The function whose input *x* and output *y* are related by: $y = 3x + 7$

Compare the two linear functions listed below and determine which has a negative slope.

Function 1: Gift Card

Samantha starts with $20 on a gift card for the bookstore. She spends $3.50 per week to buy a magazine. Let y be the amount remaining as a function of the number of weeks, *x*.

Function 2:

The school bookstore rents graphing calculators for $5 per month. It also collects a non-refundable fee of $10.00 for the school year. Write the rule for the total cost (*c*) of renting a calculator as a function of the number of months (*m*).

Solution:

Function 1 is an example of a function whose graph has negative slope. Samantha starts with $20 and spends money each week. The amount of money left on the gift card decreases each week. The graph has a negative slope of -3.5, which is the amount the gift card balance decreases with Samantha's weekly magazine purchase.

Function 2 is an example of a function whose graph has positive slope. Students pay a yearly nonrefundable fee for renting the calculator and pay $5 for each month they rent the calculator. This function has a positive slope of 5 which is the amount of the monthly rental fee. An equation for Function 2 could be $c = 5m + 10$.

Instructional Strategies: See 8.F.1

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.F.A.2
    - Battery Charging

Common Misconceptions: See 8.F.1

Domain: Functions (F)

▶**Cluster:** *Define, evaluate, and compare functions.*

Standard: Grade 8.F.3

Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.* **(8.F.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 8.F.1; 8.EE.5; 8.EE7;

Explanations and Examples:

Students use equations, graphs and tables to categorize functions as linear or non-linear. Students recognize that points on a line will have the same rate of change between any two of the points.

Examples:

Determine which of the functions listed below are linear and which are not linear and explain your reasoning.

- $y = -2x^2 + 3$ non linear
- $y = 2x$ linear
- $A = \pi r^2$ non linear
- $y = 0.25 + 0.5(x - 2)$ linear

Samir was assigned to write an example of a linear functional relationship. He wrote this example for the assignment.

The relationship between the year and the population of a county when the population increases by 10% each year

Part A

Complete the table below to create an example of the population of a certain county that is increasing by 10% each year.

Year	Population of a Certain County
0	
1	
2	
3	
4	

Part B

State whether Samir's example represents a linear functional relationship. Explain your reasoning.

Sample Response:

Part A

Year	Population of a Certain County
0	100,000
1	110,000
2	121,000
3	133,100
4	146,410

Part B

Samir's example is not a linear functional relationship. The population does not increase by the same amount each year, so the relationship is not linear.

Instructional Strategies: See 8.F.1.

Resources/Tools:

See engageNY Modules.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.F.A.3
    - 8.F Introduction to Linear Functions

Common Misconceptions: See 8.F.1.

Domain: Functions (F)

▶**Cluster**: *Use functions to model relationships between quantities.*

Standard: Grade 8.F.4

Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (*x*, *y*) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values. **(8.F.4)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections: 8.SP.2; 8.SP.3; 8.EE.5

This Cluster is connected to:

- Grade 8 Critical Area of Focus #2: Grasping the concept of a function and using functions to describe quantitative relationships.
- Expressions and Equations: Linear equations in two variables can be used to define linear functions, and students can use graphs of functions to reason toward solutions to linear equations.
- Expressions and Equations Similar triangles are used to show that the slope of a line is constant.
- Statistics and Probability: Bivariate data can often be modeled by a linear function.

Explanations and Examples:

Students identify the rate of change (slope) and initial value (y-intercept) from tables, graphs, equations or verbal descriptions.

Students recognize that in a table the *y*-intercept is the *y*-value when *x* is equal to 0. The slope can the determined by finding the ratio $\frac{y}{x}$ between the change in two *y*-values and the change between the two corresponding *x*-values.

The *y*-intercept in the table below would be (0, 2). The distance between 8 and -1 is 9 in a negative direction; the distance between -2 and 1 is 3 in a positive direction. The slope is the ratio of rise to run or $\frac{y}{x}$ or $\frac{-9}{3} = -3$

x	***y***
-2	8
0	2
1	-1

Using graphs, students identify the *y*-intercept as the point where the line crosses the *y*-axis and the slope as the rise to run. In a linear equation the coefficient of *x* is the slope and the constant is the *y*-intercept. Students need to be given the equations in formats other than $y = mx + b$, such as $y = ax + b$ (format from graphing calculator), $y = b + mx$ (often the format from contextual situations), etc. Note that point-slope form and standard forms are not expectations at this level.

In contextual situations, the *y*-intercept is generally the starting value or the value in the situation when the independent variable is 0. The slope is the rate of change that occurs in the problem. Rates of change can often occur over years. In these situations it is helpful for the years to be "converted" to 0, 1, 2, etc. For example, the years of 1960, 1970, and 1980 could be represented as 0 (for 1960), 10 (for 1970) and 20 (for 1980).

Students use the slope and y-intercepts to write a linear function in the form $y = mx + b.$ Situations may be given as a verbal description, two ordered pairs, a table, a graph, or rate of change and another point on the line. Students interpret slope and *y*-intercept in the context of the given situation.

Examples:
The table below shows the cost of renting a car. The company charges $45 a day for the car as well as charging a one-time $25 fee for the car's navigation system (GPS). Write an expression for the cost in dollars, *c,* as a function of the number of days, *d*.

Students might write the equation $c = 45d + 25$ using the verbal description or by first making a table.

Days (*d*)	**Cost (*c*) in dollars**
1	70
2	115
3	160
4	205

Students should recognize that the rate of change is 45 (the cost of renting the car) and that initial cost (the first day charge) also includes paying for the navigation system. Classroom discussion about one-time fees vs. recurrent fees will help students model contextual situations.

When scuba divers come back to the surface of the water, they need to be careful not to ascend too quickly. Divers should not come to the surface more quickly than a rate of 0.75 ft per second. If the divers start at a depth of 100 feet, the equation $d = 0.75t - 100$ shows the relationship between the time of the ascent in seconds (*t)* and the distance from the surface in feet (*d*).

Will they be at the surface in 5 minutes? How long will it take the divers to surface from their dive?

Make a table of values showing several times and the corresponding distance of the divers from the surface. Explain what your table shows. How do the values in the table relate to your equation?

You work for a video streaming company that has two plans to choose from:

Plan 1: A flat rate of \$7 per month plus \$2.50 per video viewed
Plan 2: \$4 per video viewed

a. What type of function models this situation? Explain how you know.
b. Define variables that make sense in the context and write an equation representing a function that describes each plan.
c. How much would 3 videos in a month cost for each plan? 5 videos?
d. Compare the two plans and explain what advice you would give to a customer trying to decide which plan is best for them, based on their viewing habits.

Sample Response:

a. Each plan can be modeled by a linear function since the constant rate per video indicates a linear relationship.
b. We let C_1 be the total cost per month of Plan 1, C_2 the total cost per month of Plan 2, and V the number of videos viewed in a month.
   Then $C_1(V) = 7 + 2.5V$
   $C_2(V) = 4V$
c. 3 videos on Plan 1: $C_1(3) = 7 + 2.5(3) = \$14.50$
   5 videos on Plan 1: $C_1(5) = 7 + 2.5(5) = \$19.50$
   3 videos on Plan 2: $C_2(3) = 4(3) = \$12$
   5 videos on Plan 2: $C_2(5) = 4(5) = \$20$
d. Plan 1 costs less than Plan 2 for 5 or fewer videos per month. A customer who watches more than 5 videos per month should choose Plan 2.

A customer who watches 5 or fewer videos per month should choose Plan 1.

Instructional Strategies:

In Grade 8, students focus on linear equations and functions. Nonlinear functions are used for comparison.

Students will need many opportunities and examples to figure out the meaning of $y = mx + b$.

What does *m* mean? What does *b* mean? They should be able to "see" *m* and *b* in graphs, tables, and formulas or equations, and they need to be able to interpret those values in contexts. For example, if a function is used to model the height of a stack of *n* paper cups, then the rate of change, *m*, which is the slope of the graph, is the height of the "lip" of the cup: the amount each cup sticks above the lower cup in the stack. The "initial value" in this case is not valid in the context because 0 cups would not have a height, and yet a height of 0 would not fit the equation. Nonetheless, the value of *b* can be interpreted in the context as the height of the "base" of the cup: the height of the whole cup minus its lip.

Use graphing calculators and web resources to explore linear and non-linear functions. Provide context as much as possible to build understanding of slope and y-intercept in a graph, especially for those patterns that do not start with an initial value of 0.

Give students opportunities to gather their own data or graphs in contexts they understand. Students need to measure, collect data, graph data, and look for patterns, then generalize and symbolically represent the patterns. They also need opportunities to draw graphs (qualitatively, based upon experience) representing real-life situations with which they are familiar. Probe student thinking by asking them to determine which input values make sense in the problem situations given.

Provide students with a function in symbolic form and ask them to create a problem situation in words to match the function. Given a graph, have students create a scenario that would fit the graph. Ask students to sort a set of "cards" to match a graphs, tables, equations, and problem situations. Have students explain their reasoning to each other.

From a variety of representations of functions, students should be able to classify and describe the function as linear or non-linear, increasing or decreasing. Provide opportunities for students to share their ideas with other students and create their own examples for their classmates.

Use the slope of the graph and similar triangle arguments to call attention to not just the change in *x* or *y*, but also to the rate of change, which is a ratio of the two.

Emphasize key vocabulary. Students should be able to explain what key words mean: e.g., model, interpret, initial value, functional relationship, qualitative, linear, non-linear. Use of a "word wall" or "mind map" will help reinforce vocabulary.

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.F.B
  - Modeling with a Linear Function
  - Heart Rate Monitoring
- 8.F.B.4
  - Downhill
  - Video Streaming
  - High School Graduation
  - Chicken and Steak, Variation 1
  - Baseball Cards
  - Chicken and Steak, Variation 2
  - Distance across the channel
  - Delivering the Mail, Assessment Variation

NCTM Principles to Actions Toolkit: NCTM has many great resources available to educators; some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flipbooks requires membership access, check with your school/district to see if they have an institutional membership, which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- The Case of Elizabeth Brovey and the Calling Plans 2 Task
- The Case of Peter Dubno and the Counting Cubes Task

Georgia Department of Education

- "Is it Fair?": Students play the game "Is It Fair?" and record their information using probability to determine whether they feel the game is fair or not. Predictions are made before the game begins. Based on their trials, students determine all outcomes, create tree diagrams and determine the theoretical chance of winning for each player.

See Dan Meyer Lesson:

- Linear Function: Stacking Cups

Common Misconceptions:

Students often confuse the rule to describe a pattern with the formula for a function. For example, after identifying that a linear function shows an increase of 2 in the values of the output for every change of 1 in the input, some students will represent the equation as $y = x + 2$ instead of realizing that this means $y = 2x + b$. When tables are constructed with increasing consecutive integers for input values, then the distinction between the recursive and explicit formulas is about whether you are reasoning vertically or horizontally in the table. Both types of reasoning—and both types of formulas—are important for developing proficiency with functions.

When input values are not increasing consecutive integers (e.g., when the input values are decreasing, when some integers are skipped, or when some input values are not integers), some students have more difficulty identifying the pattern and calculating the slope. It is important that all students have experience with such tables, so as to be sure that they do not overgeneralize from the easier examples.

Some students may not pay attention to the scale on a graph, assuming that the scale units are always "one."

When making axes for a graph, some students may not use equal intervals to create the scale.

Some students may infer a cause and effect between independent and dependent variables, but this is often not the case.

Some students graph incorrectly because they do not understand that *x* usually represents the independent variable and *y* represents the dependent variable. Emphasize that this is a convention that makes it easier to communicate.

Domain: Functions (F)

▶**Cluster:** *Use functions to model relationships between quantities.*

Standard: Grade 8.F.5

Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., *where the function is increasing or decreasing, linear or nonlinear*). Sketch a graph that exhibits the qualitative features of a function that has been described verbally. **(8.F.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 8.F.4.

Explanations and Examples:

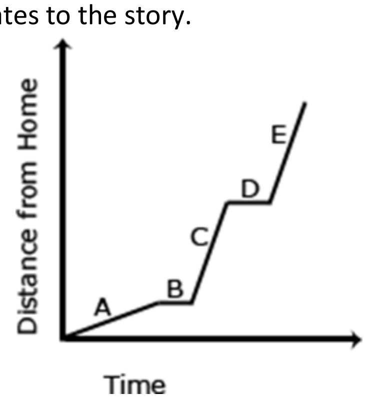
Given a verbal description of a situation, students sketch a graph to model that situation. Given a graph of a situation, students provide a verbal description of the situation.

Students learn that graphs tell stories and have to be interpreted by carefully thinking about the quantities shown.

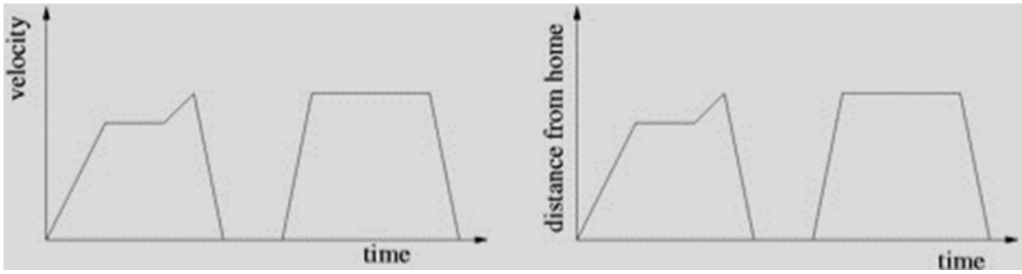
Examples:

The graph below shows a student's trip to school. This student walks to his friend's house and, together, they ride a bus to school. The bus stops once before arriving at school.

Describe how each part A-E of the graph relates to the story.



Below are two graphs that look the same. Note from the axis labels that the first graph shows the velocity of a car as a function of time and the second graph shows the distance of the car from home as a function of time. Describe what someone who observes the car's movement would see in each case.



Sample Responses:
For a velocity function, output values tell us how fast the car is moving. For the distance function, output values tell us how far from home the car is. Since we don't have scales on either axis, we can't talk about specific values of time, velocity and distance, but we can make qualitative statements about velocity and distance.

Velocity Graph: The car starts at rest and speeds up at a constant rate. When the graph becomes a horizontal line, the car is maintaining its speed for a while before speeding up for a short time and then quickly slowing down until it comes to a complete stop. It stays stationary for a little while where the graph is on the horizontal axis. Then the car speeds up, goes at a constant speed for a while and then slows down and comes to a complete stop.

Distance Graph: The car starts its trip at home. It moves away from home at a constant speed. When the graph is horizontal, the car's distance from home is not changing, which probably means it has come to a stop for awhile. Then the car moves farther away from home before turning around and coming back home. After staying at home for a time, the car moves away from home at a constant speed. It comes to a stop for a while before coming back home.

*If the distance from home is not changing, it is also possible that the car is driving along a circle with the driver's home at the center, although this does not seem very likely.

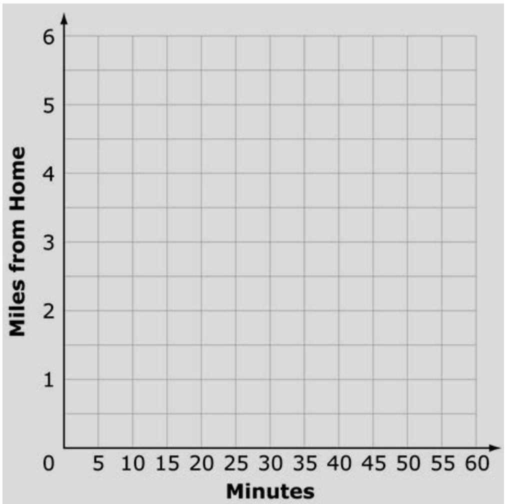
Carla rode her bike to her grandmother's house. The following information describes her trip:

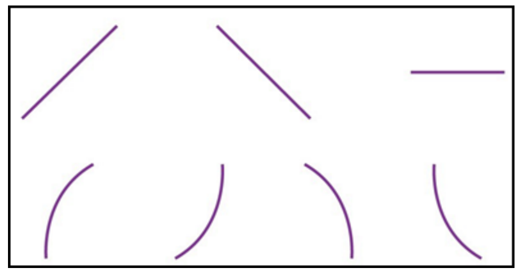
- For the first 5 minutes, Carla rode fast and then slowed down. She rode 1 mile.
- For the next 15 minutes, Carla rode at a steady pace until she arrived at her grandmother's house. She rode 3 miles.
- For the next 10 minutes, Carla visited her grandmother.
- For the next 5 minutes, Carla rode slowly at first but then began to ride faster. She rode 1 mile.
- For the last 10 minutes, Carla rode fast. She rode 3 miles at a steady pace.

Graph each part of Carla's trip. To graph part of her trip, first click the correct line type in the box. Then click in the graph to add the starting point and the ending point for that part of her trip. Repeat these steps until a graph of Carla's entire trip has been created.

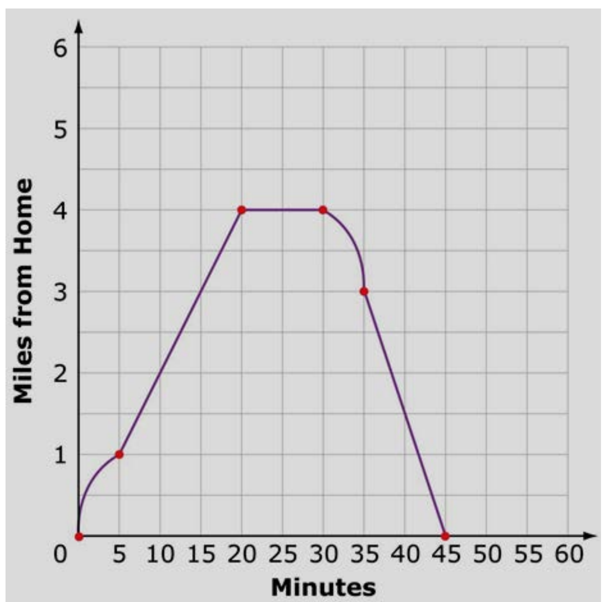
Graph each part of Carla's trip. To graph part of her trip, first click the correct line type in the box. Then click in the graph to add the starting point and the ending point for that part of her trip.

Repeat these steps until a graph of Carla's entire trip has been created.





Sample Solution:



Instructional Strategies: See 8.F.4.

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.F.B.5
    - Tides
    - Distance
    - Bike Race
    - Riding by the Library

NCTM Principles to Actions Toolkit: NCTM has many great resources available to educators; some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flipbooks requires membership access, check with your school/district to see if they have an institutional membership, which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- The Case of Elizabeth Brovey and the Calling Plans 2 Task

Common Misconceptions: See 8.F.4.

Domain: Geometry (G)

Cluster: *Geometric measurement: understand concepts of angle and measure angles.*

Standard: Grade 8.G.1

Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

8.G.1a An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the **circular arc** between the points where the two rays intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle," and can be used to measure angles. **(4.MD.5a)**

8.G.1b An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees. **(4.MD.5b)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

Connections: 8.G.A

This whole cluster deals with the concept of angles. This is the first time that students have worked with the geometric features of circles

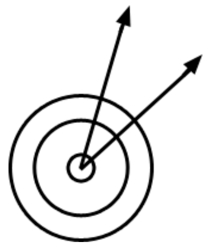
Explanations and Examples:

An angle seems like a simple concept to define but students continue to struggle with measuring and identifying angles in the midst of a problem. The reason for these difficulties is because there are actually three different viewpoints of an angle and different views are used based on the type of question asked. The first viewpoint sees an angle as a shape and can use the same rigid transformations that other shapes use. There is no motion involved in the creation of this shape. The angle measurement is simply measuring the space between the rays.

Another viewpoint used to understand angles views an angle as a portion of a circle. Where the angle intersects with the circle creates the sector to be measured. One benefit of this viewpoint is that is directly addresses the student difficulty that students have understanding that an angle made of short line segments can be congruent to an angle made of long line segments, even though the distance between the ends of the segments are significantly different. For example, $\frac{1}{5}$ of a circle is 72°. The angle cuts off $\frac{1}{5}$ of a small circle in the same proportion as $\frac{1}{5}$ of a large circle. This is also a good opportunity to apply proportional thinking.

This standard calls for students to explore the connection between angles (measure of rotation) and circular measurement (360 degrees).

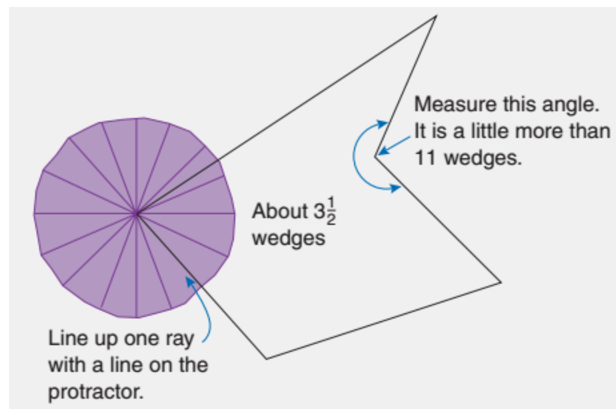
The diagram below will help students understand that an angle measurement is not related to an area since the area between the 2 rays is different for both circles, yet the angle measure is the same.



Students explore an angle as a series of "one-degree turns." A water sprinkler rotates one-degree at each interval. If the sprinkler rotates a total of 100 degrees, how many one-degree turns has the sprinkler made?

Instructional Strategies:

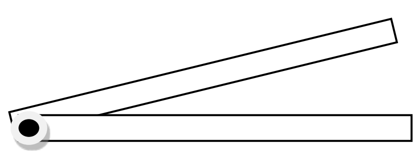
Patty paper is a great tool to help students work with an angle as a shape because it can be moved around (rigid transformation) as one whole piece. There are lessons available online to help students understand what it means to be a proportion of a circle. These lessons typically ask students to create their own rudimentary protractor by folding patty paper multiple times and cutting the corner to make it look circular. Students will use the wedge to measure the size of the angle. After using the wedge to measure the angle, the students are ready to discuss a standardized measure unit of measure, i.e. $\frac{1}{360}$ of the circle. Throughout this unit, continue to be explicit about the viewpoint the angle that you are using and explain why that particular view is the tool chosen. When multiple views work well, encourage students to discuss how they viewed the problem; discussing the pro's and con's for each view. (Graphic came from Van de Walle, John A.; Karp, Karen S.; Lovin, LouAnn H.; Bay-Williams, Jennifer M.. Teaching Student-Centered Mathematics: Developmentally Appropriate Instruction for Grades 3-5 (Volume II): 2 (Student Centered Mathematics Series) (Page 365). Pearson Education. Kindle Edition.)



Angles are geometric shapes composed of two rays that are infinite in length. Students can understand this concept by using two rulers held together near the ends. The rulers can represent the rays of an angle. As one ruler is rotated, the size of the angle is seen to get larger.

Ask questions about the types of angles created. Responses may be in terms of the relationship to right angles. Introduce angles as acute (less than the measure of a right angle) and obtuse (greater than the measure of a right angle). Have students draw representations of each type of angle. They also need to be able to identify angles in two-dimensional figures.

Students can also create an angle explorer (two strips of cardboard attached with a brass fastener) to learn about angles.



They can use the angle explorer to get a feel of the relative size of angles as they rotate the cardboard strips around.

Students can compare angles to determine whether an angle is acute or obtuse. This will allow them to have a benchmark reference for what an angle measure should be when using a tool such as a protractor or an angle ruler.

Provide students with four pieces of straw, two pieces of the same length to make one angle and another two pieces of the same length to make an angle with longer rays. Each set of straws can be attached with two jointed paper clips.

Another way to compare angles is to place one angle over the other angle. Provide students with a transparency to compare two angles to help them conceptualize the spread of the rays of an angle.

Students can make this comparison by tracing one angle and placing it over another angle. The side lengths of the angles to be compared need to be different.

Students are ready to use a tool to measure angles once they understand the difference between an acute angle and an obtuse angle. Angles are measured in degrees. There is a **relationship** between the number of degrees in an angle and circle which has a measure of 360 degrees.

Students are to use a protractor to measure angles in whole-number degrees. They can determine if the measure of the angle is reasonable based on the relationship of the angle to a right angle. They also make sketches of angles of specified measure.

Resources/Tools:

Paper folding a protractor.

- https://kera.pbslearningmedia.org/resource/vtl07.math.geometry.pla.lpangle/introduction-to-angle-measure/#.WslsnNPwbOQ
- https://www.raymondgeddes.com/measuring-all-angles

See engageNY Module 4:
http://www.learner.org/series/modules/express/videos/video_clips.html?type=1&subject=math&practice=structure

"What's My Angle, Figure This Challenge #10", FigureThis.org. Students can estimate the measures of the angles between their fingers when they spread out their hand.

Common Misconceptions:

A common misconception is that the larger the distance between the rays of the angle, the larger the angle. Tracing a small angle (small as in short line segments) with patty paper and placing it on top of the large angle (large as in long line segments) so that students can see that they angles are the same size. Another misconception is that angles must measure less than 180°. Asking them to measure larger angles, using full circle protractors, or asking them to label both the major and minor angle are some strategies to address this misconception.

Domain: Geometry (G)

▶**Cluster**: *Geometric measurement: understand concepts of angle and measure angles.*

Standard: **Grade 8.G.2**

Measure angles in whole-number degrees using a protractor. Draw angles of specified measure using a protractor and straight edge. **(4.MD.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.7 Look for and make use of structure.

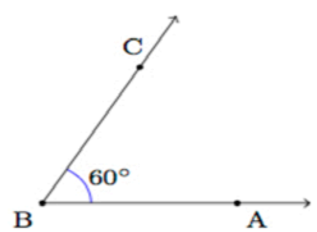
Connections: See 8.G.1

Explanations and Examples:

See 8.G.1

Example (from Illustrated Math: 4.MD.6 Measuring Angles)

a. Draw an angle that measures 60 degrees like the one shown here:



b. Draw another angle that measures 25 degrees. It should have the same vertex and share side $\overrightarrow{BA}$.

c. How many angles are there in the figure you drew? What are their measures?

d. Make a copy of your 60 degree angle. Draw a different angle that measures 25 degrees and has the same vertex and also shares side $\overrightarrow{BA}$.

e. How many angles are there in the figure you drew? What are their measures?

Instructional Strategies:

Establish solid conceptual understanding for what an angle is (see above) before proceeding to measuring angles and then start with estimating and using homemade protractors. Once students begin using protractors, have multiple tools available for students to select the tool best for them (standard protractor, full circle protractor, angle ruler). A classroom discussion asking students to speculate about the advantages of a protractor with numbers that increase clockwise and counterclockwise might help students select the correct measurement. The best strategy to support students is to teach them to ask themselves every time "Is this a reasonable answer?" Focusing on acute and obtuse, partnered with this question, will help students select the proper choice between 110° and 70°.

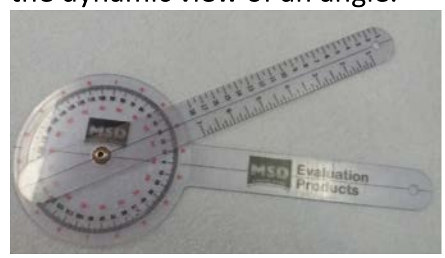
Resources/Tools:

Protractor Tools:

Standard Protractor: Measuring tool that students are likely to encounter in other settings

Full Circle Protractor: Useful to connect the angle measure to the portion of a circle view.

Angle Ruler: Useful to connect to the dynamic view of an angle.



Illustrative Mathematics **Grade 4** tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.C.6
    - Measuring Angles

Desmos:

- Measuring Angles
- Laser Challenge

Common Misconceptions:

Students struggle with selecting the appropriate measurement and lining up the vertex and ray correctly. Extending rays to a length that can be measured is also difficult for many students.

Domain: Geometry (G)

▶**Cluster:** *Geometric measurement: understand concepts of angle and measure angles.*

Standard: Grade 8.G.3

Recognize angle measure as additive. When an angle is **decomposed** into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real world and mathematical problems, *e.g. by using an equation with a symbol for the unknown angle measure.* **(4.MD.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP 2 Reason abstractly and quantitatively.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 8.G.1.

Explanations and Examples:

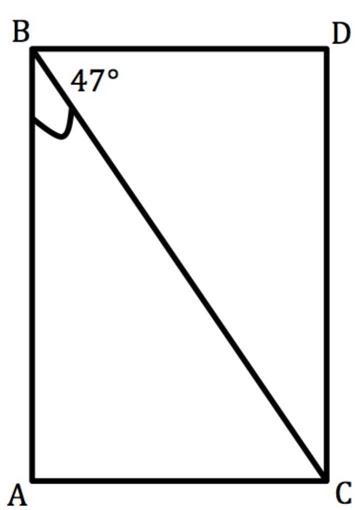
Seeing that multiple angles can be put together to create a new angle is critical to geometric thinking. The definition of an adjacent angle is wordy and requires creating a clear picture that the angles must have a common side and vertex and must not overlap. Using manipulatives, such as fraction circles or patty paper can help students construct examples and non-examples.

This standard also integrates algebraic thinking at the application level of thinking. Students need to understand that, while two angles add to make the whole, but to find the unknown angle requires subtraction. Multiple views of an angle are used to help students navigate these types of problems.

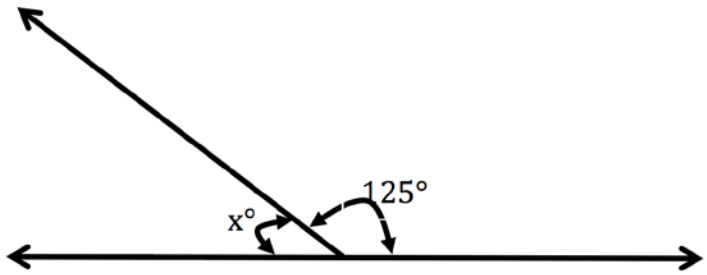
Examples:

Example 1:

In the rectangle *ABCD*, $\angle CBD = 31°$. Find the measure of $\angle ABC$.



Example 2:
Calculate the value of angle x and describe the relationship between the two angles.



Instructional Strategies:
There are a variety of strategies to help students' ability to "see" the angle in question but all use the same basic strategy: physically highlighting and removing or combining the angles. Thinking about moving from concrete to representational to abstract is a good instructional sequence to apply. Carefully selected problems can utilize fraction circles. For example, students can use the $\frac{1}{8}$ piece for 45° and the $\frac{1}{4}$ piece for 90°. They can put two 45° angles together to equal one 90° angle or they can take one 45° away to represent subtraction. After seeing the concrete representation, students can use patty paper to illustrate the representational. The benefit of patty paper is that it is see-through, can be built up into layers and then have layers taken away. Students should put one angle on each piece of patty paper. Be sure to include the larger angle that encompasses the smaller angles. Also, students should not measure with a protractor, since the goal for this standard is based in geometric reasoning. Finally, students are ready for abstract reasoning but will still benefit from scaffolding. Tracing the problem onto patty paper and using colored pencils to color the separate angles provides scaffolding that continues to focus students' attention on the angle in question but is slowly removing the additional supports.

Resources/Tools:

Illustrative Mathematics **Grade 4** tasks: **Scroll to the appropriate section to find named tasks.**

- 4.MD.C.7
  - Finding an Unknown Angle
  - Measuring Angles

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flipbooks requires membership access, check with your school/district to see if they have an institutional membership, which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- Angle Sums

- https://hcpss.instructure.com/courses/107/pages/4-dot-md-dot-7-assessment-tasks

Domain: Geometry (G)

▶**Cluster:** *Geometric measurement: understand concepts of angle and measure angles.*

Standard: Grade 8.G.4

Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and use them to solve simple equations for an unknown angle in a figure. **(7.G.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.3 Construct viable arguments and critique the reasoning of others.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

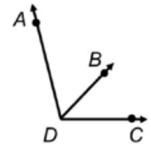
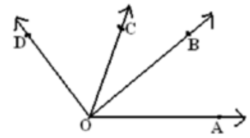
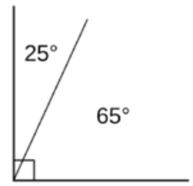
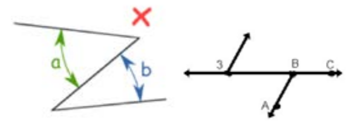
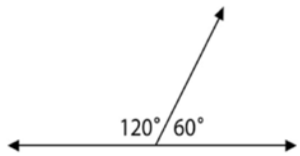
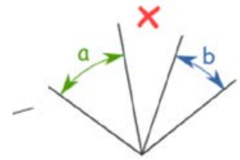
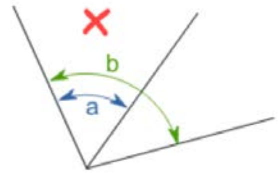
Connections: See 8.G.A, 8.EE.7

Explanations and Examples:

Before students will be able to use these facts, they need to learn and understand the facts. This is the first time students have studied these terms. While technology can be used to explore these angle relationships, it is easy to become lost in manipulated technology and lose the relationships. Patty paper makes an excellent tool for exploration.

- Supplementary angles add to 180°. They must be a pair of angles. They do not have to be adjacent. There are multiple instructional approaches to explore this. Using patty paper, move the two angles so that they are adjacent and inside a circle to show students that supplementary angles split the circle into two equal parts. Another strategy is to visualize the movement of the rays as they form the first angle. Once that angle stops, how much more would it need to rotate to make a straight line.
- Complementary angles add to 90°. They must be a pair of angles. They do not have to be adjacent. They can be studied in a similar way but, while students have an intuitive understanding of straight lines, they struggle to visualize a precise 90° angle. Visualizing 90° as $\frac{1}{4}$ of the circle $\frac{1}{4}$ seems to be a helpful strategy.
- Students are often confused about adjacent angles and the definition can seem more difficult to process than the other relationships because it has multiple criteria that must be met to consider the angles "adjacent." However, being adjacent or non-adjacent is included as criteria for many other definitions so it is critical that students be able to use this term accurately. Instructionally, using examples and non-examples to have students write their own definition, revise it based on critiques for accuracy, until they have identified all the critical elements is an excellent strategy. For example, begin with 4 or 5 examples and 4 or 5 non- examples and ask students to come up with a rule to place a new angle in the correct category. The teacher generates the new angles to use for testing but the students develop the rules. For example, students might initially say that adjacent angles have a common side. The teacher can then add an angle with a common side but not a common vertex in the non-example category. Students can then go back and revise their answer. Another helpful strategy is to use the viewpoint of an angle as a shape. Trace each angle with a separate colored pencil and shade the space between. This helps students identify if there is any overlap, if there is a common side, and if there is a common vertex. The importance of studying adjacent angles is because of its connection to other Geometric definitions, as well as supporting part/part/whole equations in the context of a situation.

  Adjacent angles are

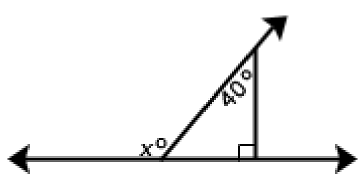
Criteria	Example	Non-example
• Pair of angles		
• Common vertex		
• Common side		
• Do not overlap		

- Similarly, vertical angles have multiple criteria in the definition. The exploration done with adjacent angles is worth the investment of time because students also struggle with identification of vertical angles.

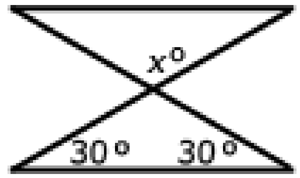
Criteria
Formed from exactly 2 intersecting lines
---Intersecting lines are straight
Pair of angles
Non-adjacent
Common vertex

The remainder of the standard asks students to write and solve equations using facts about these angle relationships. This provides an opportunity to merge Algebraic standards with Geometric standards but also checks to make sure students understand the angle relationships.

Examples:
1. Write and solve an equation to find the measure of angle *x.*



2. Write and solve an equation to find the measure of angle *x.*



3. Determine if each of the following statements is always true, sometimes true, or never true.
    1. The sum of the measures of two complementary angles is 90°.
    2. Vertical angles are also adjacent angles.
    3. Two adjacent angles are complementary.
    4. If the measure of an angle is represented by *x*, then the measure of its supplement is represented by $180 - x$.
    5. If two lines intersect, each pair of vertical angles are supplementary.

For each statement you chose as “sometimes true,” provide one example of when the statement is true and one example of when the statement is not true. Your examples should be a diagram with the angle measurements labeled. If you did not choose any statement as “sometimes true,” write “none”.

Sample Response:

1: Always true

2: Never true

3: Sometimes true

4: Always true

5: Sometimes true

<u>*Statement 3*</u>:

Example of True - (two adjacent angles that have a sum of 90°)

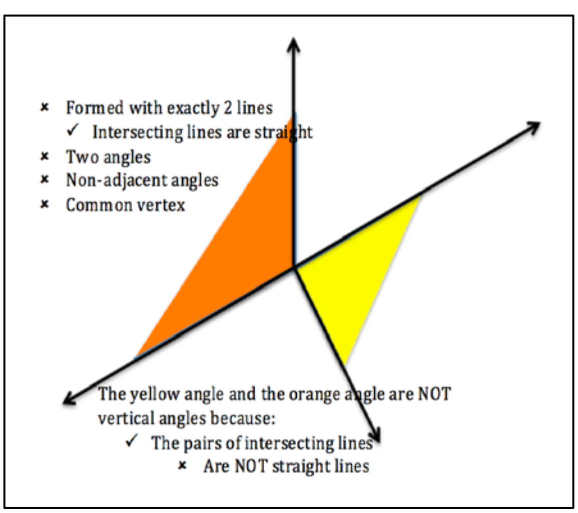
Example of Not true - (two adjacent angles that have a sum of 80°)

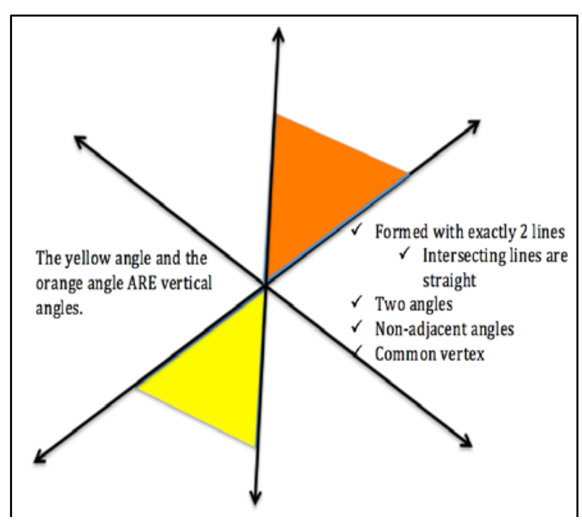
<u>*Statement 5*</u>*:*

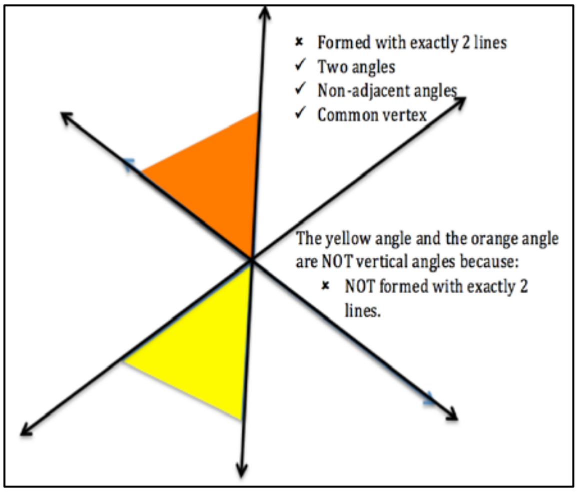
Example of True - (two intersecting lines with all angle measurements of 90°)

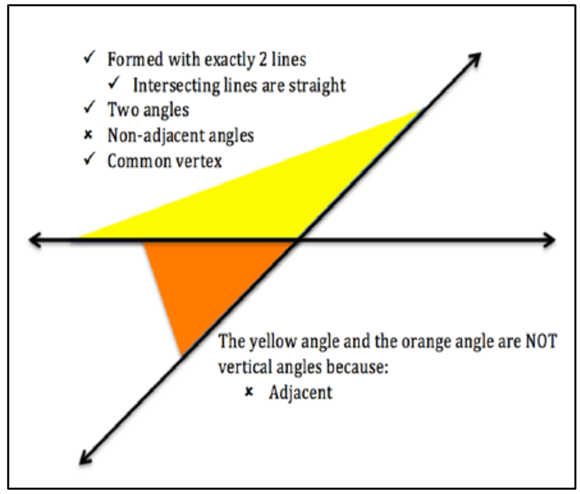
Example of Not true - (two lines that intersect with no right angles)

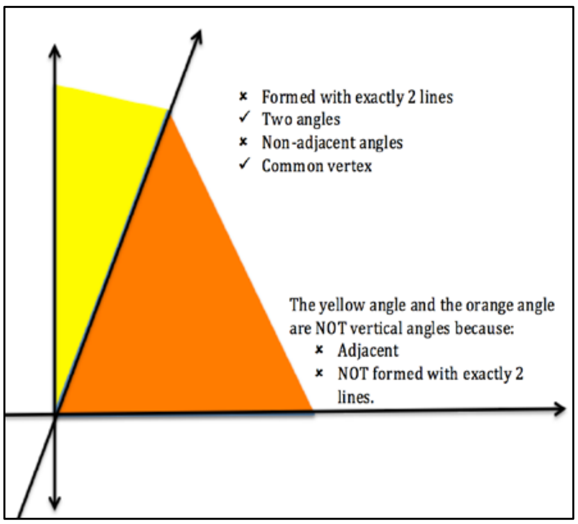
Below are additional examples illustrating how students an use a criteria checklist to determine if a pair of angles meets the conditions given in the definition.

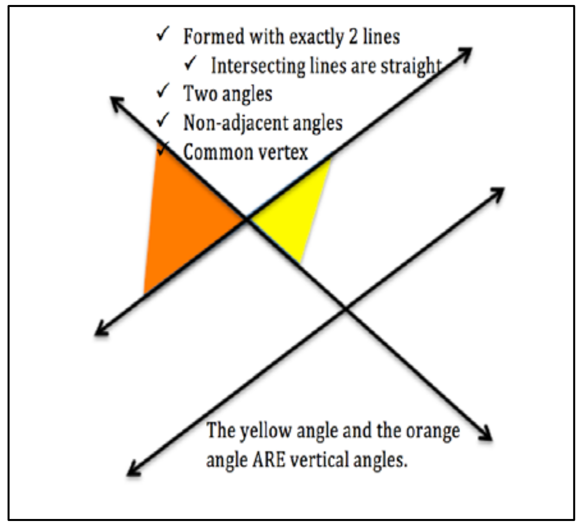


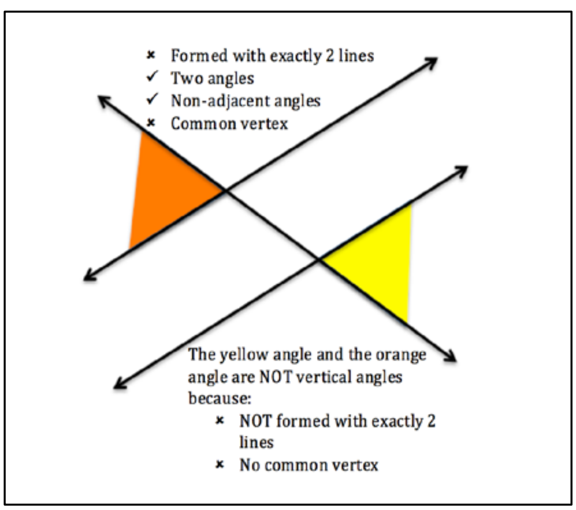


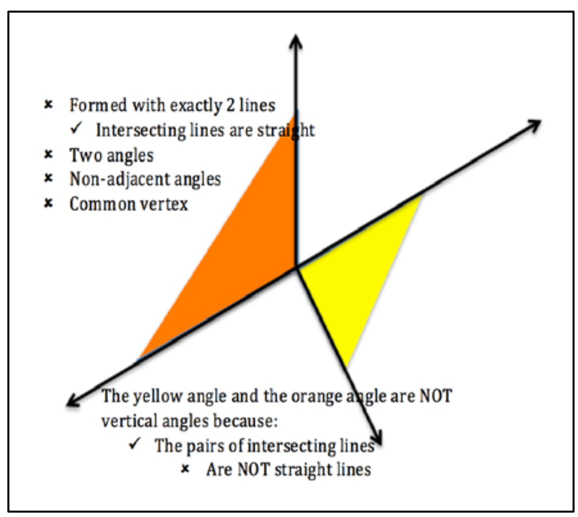












Instructional Strategies:
See suggestions included in explanations above.

Resources/Tools:

See engageNY Modules.

Domain: Geometry (G)

▶**Cluster:** *Understand congruence and similarity using physical models, transparencies, or geometry software.*

Standard: Grade 8.G.5

Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.* **(8.G.5)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: This standard is connected to 8.G.6. It might make more instructional sense for 8.G.6 to be taught prior to 8.G.5.

Explanations and Examples:

Students use exploration and deductive reasoning to determine relationships that exist between:

a) angle sums and exterior angle sums of triangles
b) angles created when parallel lines are cut by a transversal
c) the angle- angle criterion for similarity of triangle.

Students construct various triangles and find the measures of the interior and exterior angles. Students make conjectures about the relationship between the measure of an exterior angle and the other two angles of a triangle. (The measure of an exterior angle of a triangle is equal to the sum of the measures of the other two interior angles) and the sum of the exterior angles (360°). Using these relationships, students use deductive reasoning to find the measure of missing angles.

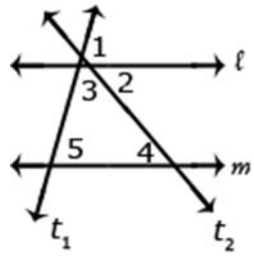
Students construct parallel lines and a transversal to examine the relationships between the created angles. Students recognize vertical angles, adjacent angles and supplementary angles and can build on these relationships to identify other pairs of congruent angles. Using these relationships, students can use reasoning to find the measure of missing angles.

Students construct various triangles having line segments of different lengths but with two corresponding congruent angles. Comparing ratios of sides will produce a constant scale factor, meaning the triangles are similar.

Students can informally prove relationships with transversals.

Examples:

Show that $m\angle 3 + m\angle 4 + m\angle 5 = 180^{\circ}$ if l and *m* are parallel lines and t_1 & t_2 are transversals.

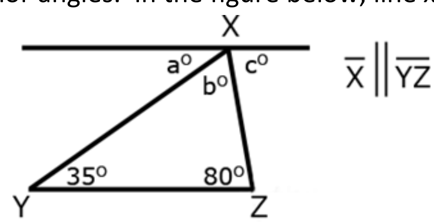


Sample Response:

$\angle 1 + \angle 2 + \angle 3 = 180^{\circ}$. Angle 1 and Angle 5 are congruent because they are corresponding angles ($\angle 5 \cong \angle 1$). $\angle 1$ can be substituted for $\angle 5$. $\angle 4 \cong \angle 2$: because alternate interior angles are congruent. $\angle 4$ can be substituted for $\angle 2$

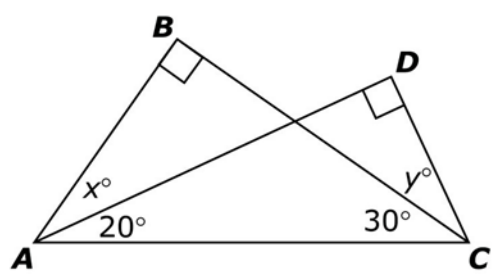
Therefore $m\angle 3 + m\angle 4 + m\angle 5 = 180^{\circ}$

Students can informally conclude that the sum of a triangle is 180º (the angle-sum theorem) by applying their understanding of lines and alternate interior angles. In the figure below, line x is parallel to line *yz*:



Angle *a* is 35° because it alternates with the angle inside the triangle that measures 35°. Angle *c* is 80° because it alternates with the angle inside the triangle that measures 80°. Because lines have a measure of 180°, and angles *a* + *b* + *c* form a straight line, then angle *b* must be 65° (180 – 35 + 80 = 65). Therefore, the sum of the angles of the triangle are $35^{\circ} + 65^{\circ} + 80^{\circ}$

Right triangle *ABC* and right triangle *ACD* overlap as shown below. Angle *DAC* measures 20º and angle *BCA* measures 30º.



not drawn to scale

What are the values of *x* and *y*?

Solution:

$x = 40^{\circ}$ and $y = 40^{\circ}$

Students need to use the fact that the sum of the angles of a triangle is 180 degrees to find the correct values of *x* and *y*. Students may incorrectly assume that *x* + 20 must equal *y* + 30.

Instructional Strategies: See 8.G.5

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.A.5
  - Find the Angle
  - Find the Missing Angle
  - Tile Patterns II: hexagons
  - Tile Patterns I: octagons and squares
  - Triangle's Interior Angles
  - Congruence of Alternate Interior Angles via Rotations
  - Street Intersections
  - Rigid motions and congruent angles

Common Misconceptions: See 8.G.1

Domain: Geometry (G)

▶**Cluster:** *Geometric measurement: understand concepts of angle and measure angles.*

Standard: **Grade 8.G.6**

Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on drawing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle. **(7.G.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.6 Attend to precision.

Connections: See 7.G.A.

Explanations and Examples:

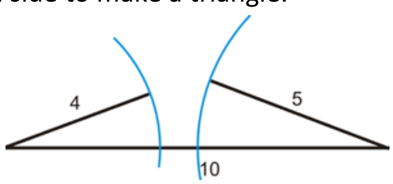
This standard has three types of problems that must be explored:

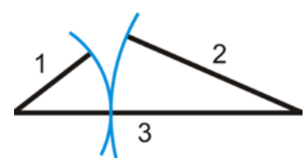
1. What conditions make a unique triangle?
2. What conditions make more than one triangle?
3. What conditions make no triangle?

During the exploration, students are provided with three measures that could be three angles, three sides, or any combination of the three.

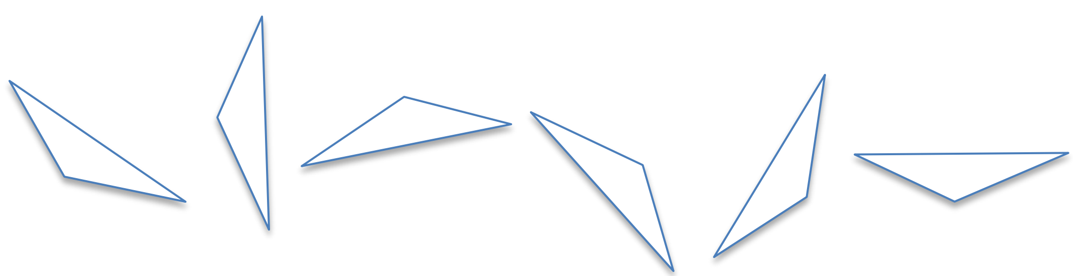
Three measurements of sides:

Providing the measurement for three sides of a triangle might make no triangle or a unique triangle. It is a surprising fact when students learn that three lines will not always make a triangle. The sum of the two shorter sides must be greater than the third side to make a triangle.





The second possibility is that the lengths of a triangle will make a unique triangle. If you provided 30 students three lengths of the same size (assuming those lengths make a triangle), they will all make exactly the same triangle. Pay careful attention to development of geometric thinking here. Students in early development are distracted by the orientation of the object so students might not realize they made the same triangle without re-orienting the triangles to look the same. For example, each of the triangles below have side lengths that are the same.



Three measurements of angles:

Students previously learned that the three angles of a triangle add to 180°. However, students might not have thought about this fact being sufficient criteria to guarantee a triangle. Using patty paper or technology, students can explore a variety of angle measures to discover that the only way to create a triangle is if the angles add to 180°. This is a powerful advancement in their development of geometric thinking. All triangles have angles that add to 180° and if you have three angles that add to 180°, you can always make a triangle.

After this realization is reached, students can focus their exploration on being given three angle measures that together add to 180°. Under these conditions, students will discover that many (infinitely many, actually) triangles are created. Depending on sequencing of the standards, you can either connect this to previously learned proportional triangles or use these standards to lead into proportional triangles.

Examples:

Is it possible to draw a triangle with a 90˚ angle and one leg that is 4 inches long and one leg that is 3 inches long? If so, draw one. Is there more than one such triangle?

Draw a triangle with angles that are 60 degrees. Is this a unique triangle? Why or why not?

Can you draw a triangle with sides that are 13 cm, 5 cm and 6 cm?

Draw a quadrilateral with all sides the same length.

Instructional Strategies:

Constructions facilitate understanding of geometry. Provide opportunities for students to physically construct triangles with straws, sticks, or geometry apps prior to using rulers and protractors to discover and justify the side and angle conditions that will form triangles.

Explorations should involve giving students: three side measures or three angle measures to determine if a unique triangle, no triangle or an infinite set of triangles results. Through discussion of their exploration results, students should conclude that triangles cannot be formed by any three arbitrary side or angle measures. They may realize that for a triangle to result the sum of any two side lengths must be greater than the third side length, or the sum of the three angles must equal 180 degrees. Students should be able to transfer from these explorations to reviewing measures of three side lengths or three angle measures and determining if they are from a triangle justifying their conclusions with both sketches and reasoning.

Further construction work can be replicated with quadrilaterals, determining the angle sum, noticing the variety of polygons that can be created with the same side lengths but different angle measures, and ultimately generalizing a method for finding the angle sums for regular polygons and the measures of individual angles.

For example, subdividing a polygon into triangles using a vertex $(N-2)180^{\circ}$ or subdividing polygons into triangles using an interior point $180^{\circ}N - 360^{\circ}$ where N = the number of sides in the polygon. An extension would be to realize that the two equations are equal.

Resources/Tools:

See engageNY Modules.

Common Misconceptions:

Be careful to precisely explain that when you say sides with the given length, you mean that the end point of one side must meet with the endpoint of another side. Some students will attempt to make a triangle such as figure 1 below. Another misconception might develop from the exploration, itself. The physical objects might impair accuracy. If the lengths are “almost” a triangle but the object they are physically manipulating is “fat” (such as a straw or pipe cleaner) it might appear to make a triangle. For example, figure 2 and 3 are the same shape but with thicker lines. Notice that the figure 3 *just* touches on the inside. Be aware that this is a possible error.

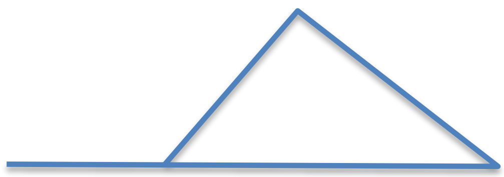


Figure 2

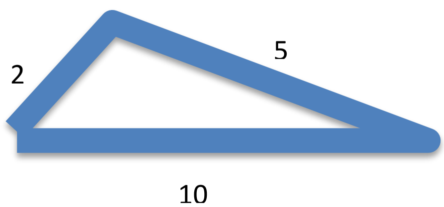


Figure 2

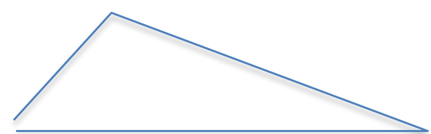


Figure 3

Domain: Geometry (G)

▶**Cluster**: *Understand and apply the Pythagorean Theorem.*

Standard: Grade 8.G.7

Explain a proof of the Pythagorean Theorem and its converse. **(8.G.6)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP 1 Solve problems and persevere in solving them.
- ✓ MP 3 Construct viable arguments and critique the reasoning of others.
- ✓ MP 4 Model with mathematics.
- ✓ MP 6 Attend to precision
- ✓ MP 7 Look for and make use of structure.

Connections:

This cluster is connected to:

- Grade 8 Critical Area of Focus #3: analyzing two- and three- dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem.

Explanations and Examples:

Students explain the Pythagorean Theorem as it relates to the area of squares coming off of all sides of a right triangle. Students also understand that given three side lengths with this relationship forms a right triangle.

Students should verify, using a model, that the sum of the squares of the legs is equal to the square of the hypotenuse in a right triangle. Students should also understand that if the sum of the squares of the 2 smaller legs of a triangle is equal to the square of the third leg, then the triangle is a right triangle.

Instructional Strategies:

Previous standards about triangles are extended with the introduction of right triangles. Students should be given the opportunity to explore right triangles to determine the relationships between the measures of the legs and the measure of the hypotenuse. Experiences should involve using grid paper to draw right triangles from given measures and representing and computing the areas of the squares on each side. Data should be recorded in a chart such as the one below, allowing for students to conjecture about the relationship among the areas within each triangle.

Triangle	Measure of Leg 1	Measure of Leg 2	Area of Square on Leg 1	Area of Square on Leg 2	Area of square on Hypotenuse
1					

Students should then test out their conjectures, then explain and discuss their findings. Finally, the Pythagorean Theorem should be introduced and explained as the **pattern** they have explored. Time should be spent analyzing several proofs of the Pythagorean Theorem to develop a beginning sense of the process of deductive reasoning, the significance of a theorem, and the purpose of a proof. Students should be able to justify a simple proof of the Pythagorean Theorem or its converse.

Previously, in 8.G.7, students have discovered that not every combination of side lengths will create a triangle. Now they need situations that explore using the Pythagorean Theorem to test whether or not side lengths represent right triangles. (Recording could include Side length a, Side length b, Sum of $a^2 + b^2$, c^2, $a^2 + b^2 = c^2$, Right triangle? Through these opportunities, students should realize that there are Pythagorean (triangular) triples such as (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41) that always create right triangles, and that their multiples also form right triangles. Students should see how similar triangles can be used to find additional triples. Students should be able to explain why a triangle is or is not a right triangle using the Pythagorean Theorem.

The Pythagorean Theorem should be applied to finding the lengths of segments on a coordinate grid, especially those segments that do not follow the vertical or horizontal lines, as a means of discussing the determination of distances between points. Contextual situations, created by both the students and the teacher, that apply the Pythagorean Theorem and its converse should be provided.

For example, apply the concept of similarity to determine the height of a tree using the ratio between the student's height and the length of the student's shadow. From that, determine the distance from the tip of the tree to the end of its shadow and verify by comparing to the computed distance from the top of the student's head to the end of the student's shadow, using the ratio calculated previously.

Challenge students to identify additional ways that the Pythagorean Theorem is or can be used in real world situations or mathematical problems, such as finding the height of something that is difficult to physically measure, or the diagonal of a prism.

Tools/Resources:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.B
    - Applying the Pythagorean Theorem in a mathematical context
    - Bird and Dog Race
    - Is this a rectangle?
    - A rectangle in the coordinate plane
    - Sizing up Squares
- 8.G.B.6
    - Converse of the Pythagorean Theorem

- Maryland Geometry Lessons

- Video Tutorial with Graph Paper (Pythagorean Theorem with 3-4-5 Triangle

Domain: Geometry (G)

▶**Cluster**: *Understand and apply the Pythagorean Theorem.*

Standard: Grade 8.G.8

Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. *For example: Finding the slant height of pyramids and cones.* **(8.G.7)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 8.G.6; 8.NS.2

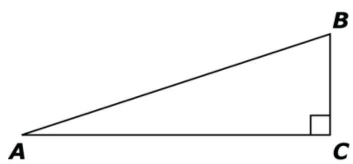
Explanations and Examples:

Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Through authentic experiences and exploration, students should use the Pythagorean Theorem to solve problems. Problems can include working in both two and three dimensions. Students should be familiar with the common Pythagorean triplets.

Examples:

In right triangle *ABC*, side *AC* is longer than side *BC.* The boxed numbers represent the possible side lengths of triangle *ABC*.



not drawn to scale

7	**8**
15	**17**
18	**20**
24	**25**

Identify three boxed numbers that could be the side lengths of triangle *ABC*.

1a. *BC* =

1b. *AC* =

1c. *AB* =

Solutions: BC = 7, AC = 24, AB = 25 **or** BC = 15, AC = 20, AB = 25 or BC = 8, AC = 15, AB = 17

Part A

Triangle *STV* has sides with lengths of 7, 11, and 14 units. Determine whether this triangle is a right triangle. Show all work necessary to justify your answer.

Part B

A right triangle has a hypotenuse with a length of 15. The lengths of the legs are whole numbers. What are the lengths of the legs?

Sample Response:

Part A

72 + 112 does not equal 142 because 49 + 121 = 170, not 196.
Therefore, it is not a right triangle because the side lengths do not satisfy the Pythagorean Theorem.

Part B

9, 12

Students in a class are using their knowledge of the Pythagorean Theorem to make conjectures about triangles. A student makes the conjecture shown below.

A triangle has side lengths *x*, *y, z*. If $x < y < z$ and $x^2 + y^2 < z^2$, the triangle is an obtuse triangle.

Use the Pythagorean Theorem to develop a chain of reasoning to justify or refute the conjecture. You must demonstrate that the conjecture is always true or that there is at least one example in which the conjecture is not true.

Sample Response:

Picture the triangle with the side of length *x* on the bottom, the side of length *y* on the left, and the side of length *z* on the top. If $x^2 + y^2 = z^2$ the triangle is a right triangle. Since $x^2 + y^2 < z^2$ if the sides of length *x* and *y* were left so they made a right angle and the side of length *z* started at the other end of the side of length *x*, it would extend past the other end of the side of length *y*. So the end of the side of length *y* has to swing out to the left so the ends of all the segments can connect to form a triangle. When the side of length *y* swings out to the left, the measure of the angle between that side and the side of length *x* increases, so the triangle is an obtuse triangle. The conjecture is true.

Instructional Strategies: See 8.G.6.

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.B.7
  - Glasses
  - Spiderbox
  - Running on the Football Field
  - Two Triangles' Area
  - Area of a Trapezoid
  - Points from Directions
  - Areas of Geometric Shapes with the Same Perimeter
  - Circle Sandwich

Domain: Geometry (G)

▶**Cluster**: *Understand and apply the Pythagorean Theorem.*

Standard: Grade 8.G.9

Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

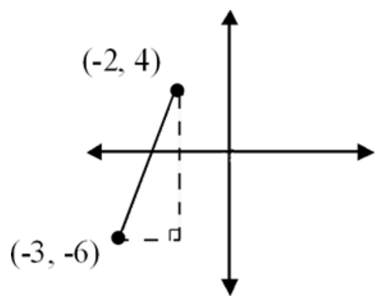
Connections: See 8.G.6; 8.NS.2

One application of the Pythagorean Theorem is finding the distance between two points on the coordinate plane. Students build on work from 6th grade (finding vertical and horizontal distances on the coordinate plane) to determine the lengths of the legs of the right triangle drawn connecting the points. Students understand that the line segment between the two points is the length of the hypotenuse. The use of the distance formula is not an expectation.

Explanations and Examples:

Examples:

Students will create a right triangle from the two points given (as shown in the diagram below) and then use the Pythagorean Theorem to find the distance between the two given points.



What is the distance between (0,0) and (8,15) on the *xy*-coordinate plane?

Solution: 17 units

Common Misconceptions:
Students may subtract 8 from 15 to find the distance. Students may think that the distance is equal to the *x* value of the point that is not at the origin. Students may add 8 and 15 to find the distance.

Instructional Strategies: See 8.G.6.

Resources/Tools:
Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.B.8
  - Finding isosceles triangles
  - Finding the distance between points

Domain: Geometry (G)

▶**Cluster:** *Solve real-world and mathematical problems involving measurement.*

Standard: **Grade 8.G.10**

Use the formulas or informal reasoning to find the arc length, areas of sectors, surface areas and volumes of pyramids, cones, and spheres. *For example, given a circle with a 60° central angle, students identify the arc length as* $\frac{1}{6}$ *of the total circumference* $\left(\frac{1}{6} = \frac{60}{360}\right)$. **(8.G.9)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

Connections:

This cluster is connected to:

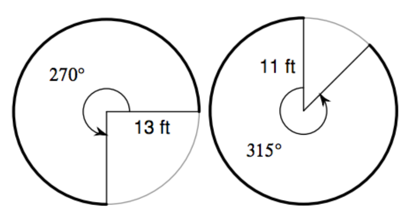
- Grade 8 Critical Area of Focus #3: Analyzing two- and three- dimensional space and figures using distance, angle, similarity, and understanding and applying the Pythagorean Theorem.

Explanations and Examples:

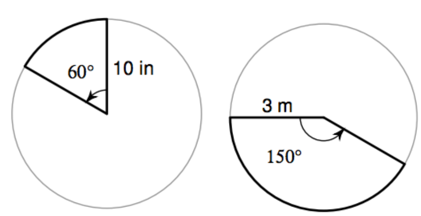
Students build on understandings of circles and volume from 7th grade to find the volume of cylinders, cones and spheres. Students understand the relationship between the volume of a) cylinders and cones and b) cylinders and spheres to the corresponding formulas.

Examples:

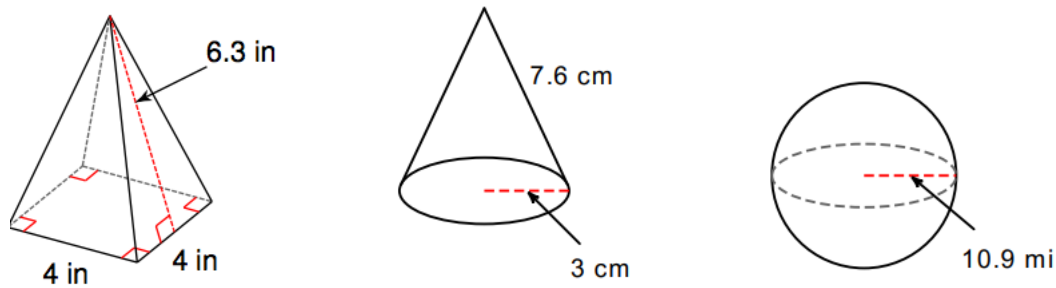
Find the length of each arc. Round your answers to the nearest tenth.



Find the area of each sector. Round your answers to the nearest tenth.



Calculate the surface and volume for the following shapes:



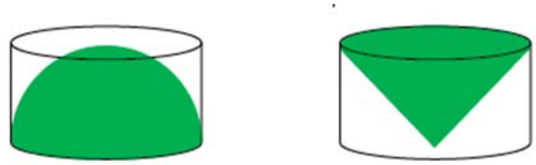
Instructional Strategies:

Begin by recalling the formula, and its meaning, for the volume of a right rectangular prism: $V = l \times w \times h$. Then ask students to consider how this might be used to make a conjecture about the volume formula for a cylinder

Most students can be readily led to the understanding that the volume of a right rectangular prism can be thought of as the area of a base times the height, and so because the area of the base of a cylinder is πr^2 the volume of a cylinder is $V_c = \pi r^2 h$.

To motivate the formula for the volume of a cone, use cylinders and cones with the same base and height. Fill the cone with rice or water and pour into the cylinder. Students will discover/experience that 3 cones full are needed to fill the cylinder. This non-mathematical derivation of the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, will help most students remember the formula.

In a drawing of a cone inside a cylinder, students might see that that the triangular cross-section of a cone is $\frac{1}{2}$ the rectangular cross-section of the cylinder. Ask them to reason why the volume (three dimensions) turns out to be *less* than $\frac{1}{2}$ the volume of the cylinder. It turns out to be $\frac{1}{3}$.



For the volume of a sphere, it may help to have students visualize a hemisphere "inside" a cylinder with the same height and base. || The radius of the circular base of the cylinder is also the radius of the sphere and the hemisphere. The area of the base of the cylinder and the area of the section created by the division of the sphere into a hemisphere is πr^2. The height of the cylinder is also *r* so the volume of the cylinder is πr^3. Students can see that the volume of the hemisphere is less than the volume of the cylinder and more than half the volume of the cylinder. Illustrating this with concrete materials and rice or water will help students see the relative difference in the volumes. At this point, students can reasonably accept that the volume of the hemisphere of radius *r* is $\frac{2}{3}\pi r^3$and therefore volume of a sphere with radius *r* is twice that or $\frac{4}{3}\pi r^3$. There are several websites with explanations for students who wish to pursue the reasons in more detail. (Note that in the pictures above, the hemisphere and the cone together fill the cylinder.)

Students should experience many types of real-world applications using these formulas. They should be expected to explain and justify their solutions.

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.G.C.9
  - Comparing Snow Cones
  - Glasses
  - Flower Vases
  - Shipping Rolled Oats

NCTM Illuminations – NCTM has many great resources available to educators, some of these resources (i.e. interactives) are open to any educator while others (i.e. lessons) require an individual or institutional membership. If you find that a resource referenced in the flipbooks requires membership access, check with your school/district to see if they have an institutional membership, which would grant you access all NCTM documents. If they do not have a membership, this would be a valuable resource to request.

- Cubes.

Common Misconceptions:

A common misconception among middle grade students is that volume is a number that results from substituting other numbers into a formula. For these students there is no recognition that volume is a measure related to the amount of space occupied. If a teacher discovers that students do not have an understanding of volume as a measure of space, it is important to provide opportunities for hands on experiences where students fill three-dimensional objects. Begin with right rectangular prisms and fill them with cubes will help students understand why the units for volume are cubed.

Domain: Geometry (G)

▶**Cluster:** *Solve real-world and mathematical problems involving measurement.*

Standard: **Grade 8.G.11**

Investigate the relationship between the formulas of three-dimensional geometric shapes;

a. Generalize the volume formula for pyramids and cones ($V = \frac{1}{3}Bh$). **(G.GMD.3)**
b. Generalize surface area formula of pyramids and cones ($SA = B + \frac{1}{2}Pl$). **(G.GMD.3)**

Suggested Standards for Mathematical Practice (MP):

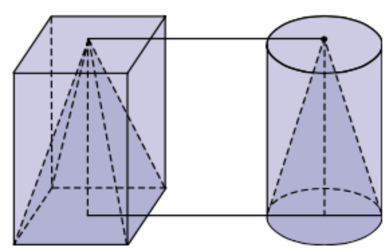
- ✓ MP 1 Solve problems and persevere in solving them
- ✓ MP 2 Reason abstractly and quantitatively.
- ✓ MP 3 Construct viable arguments and critique the reasoning of others.
- ✓ MP 4 Model with mathematics.
- ✓ MP 5 Use appropriate tools strategically
- ✓ MP 7 Look for and make use of structure

Connections: 7.G.5, 7.G.6, 8.G.B

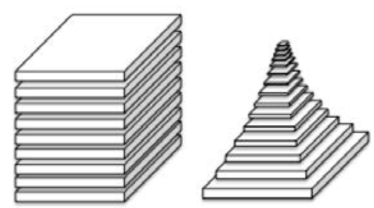
Explanations and Examples:

The goal for this standard is for students investigate the formulas and not *derive* the formulas. There are two types of relationships between 3-d geometric shapes that should be explored.

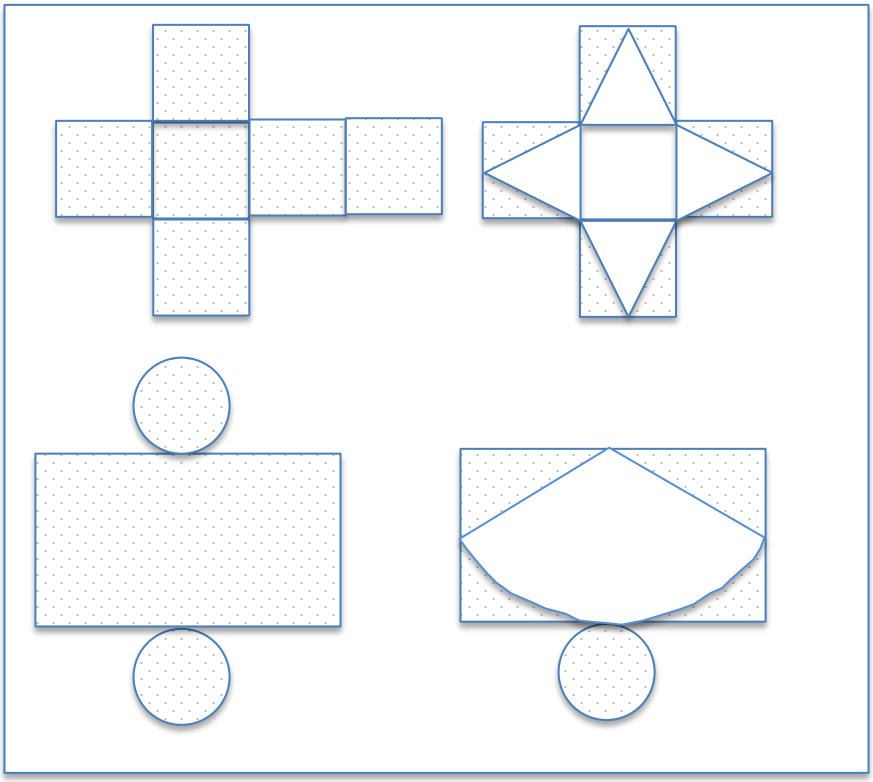
1. Explore the relationship between the formulas for prisms/pyramids and cylinders/cones.
2. Explore how the formula is generalized, rather than having a different formula for each different shaped base.



First, we want students to recognize that the volume formulas between prisms and pyramids, for example, follow a similar pattern. One investigation to explore this relationship uses material that is easily cut and the displacement definition of volume. For example, florist green foam can be purchased as rectangular prisms and cylinders. Cutting off the sides to create a prism, students can measure the volume of the original shape and the volume of the new shape. Students should be able to see that the volume of the new shape is approximately $\frac{1}{3}$ of the original shape. If this lab is done after statistics, you could draw a scatterplot and informally fit a line to the data. If sequencing does not have statistics prior to this unit, the teacher could draw students attention to the $\frac{1}{3}$ relationship. Giving students different shapes will help them see that, regardless of the shape of the base, the volume of a prism is always $\frac{1}{3}$ the volume of the pyramid.



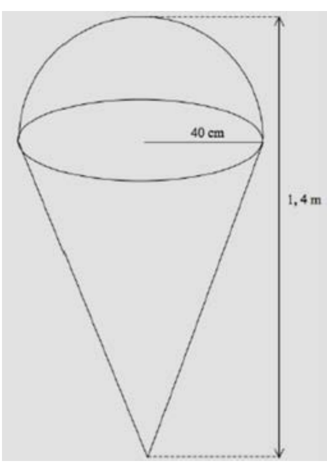
The students likely did an investigation in 7th grade exploring $V = Bh$ can be reviewed and expanded here to explain the connection between the base and height of the prism and the volume formula. Each successive layer is smaller but the shape is still the same and the height is still the same so the formula is similar. It is important for students to understand how the base and height are related and that it is smaller, but the actual derivation of the number $\frac{1}{3}$ is not critical at this stage.



The formulas for surface area can investigated using nets, as they were in 7th grade. Since the entire volume formula is $\frac{1}{3}$ less for a prism, students might want to make the entire surface area formula $\frac{1}{3}$ less. The net can help the student understand why the area of the base is unchanged. Superimposing the net for the pyramid onto the net for the prism can help student visual why 1) there is only one base, 2) the area of the faces for pyramid is $\frac{1}{2}$ the area of the faces for the prism, and 3) the height of the pyramid is changed to the slant height.

Examples:

- Determine the volume of the figure below.



- Find the volume of a cylindrical oatmeal box.

- Given a three-dimensional object, compute the effect on volume of doubling or tripling one or more dimension(s). For example, how is the volume of a cone affected by doubling the height?

- Janine is planning on creating a water-based centerpiece for each of the 30 tables at her wedding reception. She has already purchased a cylindrical vase for each table. The radius of the vases is 6 cm. and the height is 28cm.

She intends to fill them half way with water and then add a variety of colored marbles until the waterline is approximately three-quarters of the way up the cylinder. She can buy bags of 100 marbles in 2 different sizes, with radii of 9mm or 12mm. A bag of 9mm marbles costs $3, and a bag of 12mm marbles costs $4.

a. If Janine only bought 9 mm marbles how much would she spend on marbles for the whole reception? What if Janine only bought 12 mm marbles? (Note: $1cm^2 = 1ml$)
b. Janine wants to spend at most d dollars on marbles. Write a system of equalities and/or inequalities that she can use to determine how many marbles of each type she can buy.
c. Based on your answer to part *b*. How many bags of each size marble should Janine buy if she has $180 and wants to buy as many small marbles as possible?

Instructional Strategies:

Use hands on strategies such as building the shape from solid material and building the shape from a net to compare and contrast the difference between the geometric shapes.

Resources/Tools:

Mathematics Assessment Project:

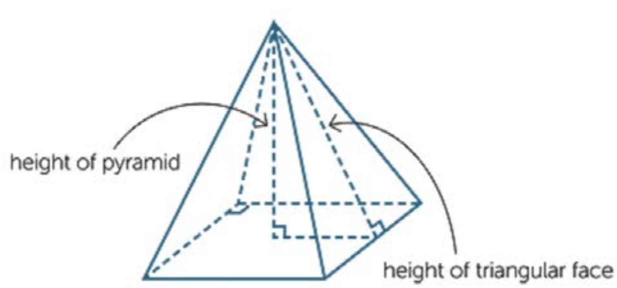
- "Calculating Volumes of Compound Objects" – This lesson unit is intended to help you assess how well students solve problems involving measurement, and in particular, to identify and help students who have the following difficulties:
  - Computing measurements using formulas.
  - Decomposing compound shapes into simpler ones.
  - Using right triangles and their properties to solve real-world problems.

Illustrative Mathematics **High School Geometry** tasks: **Scroll to the appropriate section to find named tasks.**

- G-GMD.A.3
  - Doctor's Appointment
  - Centerpiece

Common Misconceptions:

Students confuse slant height and height and understanding the relationship between the two. The two are related by using the Pythagorean Theorem. However, this requires students to visualize a triangle that is half of the cross section of the pyramid.



Domain: Geometry (G)

●**Cluster:** *Solve real-world and mathematical problems involving measurement.*

Standard: Grade 8.G.12

Solve real-world and mathematical problems involving arc length, area of two-dimensional shapes including sectors, volume and surface area of three-dimensional objects including pyramids, cones and spheres. **(8.G.9)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.1 Make sense of problems and persevere in solving them.
- ✓ MP 2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.
- ✓ MP.8 Look for and express regularity in repeated reasoning.

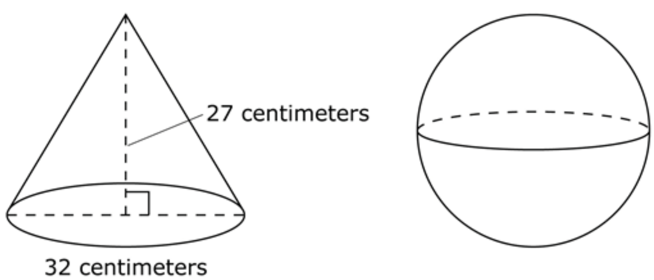
Connections: See 8.G.10

Explanations and Examples:

The primary difference between 8.G.10 and 8.G.12 occurs in the first few words. The earlier standard asks students to "use the formulas or informal reasoning…" while the later standard says "solve real-world and mathematical problems…" The difference is subtle but 8.G.10 is focused on using the formula and 8.G.12 is focused on problem solving, either in context or in a more complicated math situation.

Examples:

This cone and sphere have equal volumes.

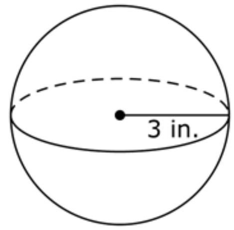


not drawn to scale

What is the radius of the sphere?

Solution: 12 centimeters

This sphere has a 3-inch radius.



What is the volume, in cubic inches, of the sphere?

Solution: 36π cu in. (or any number between 113 and 113.1)

Instructional Strategies: See 8.G.10.

Resources/Tools: See 8.G.10.

Common Misconceptions: See 8.G.10.

Domain: Statistics and Probability (SP)

◆**Cluster:** *Investigate patterns of association in bivariate data.*

Standard: Grade 8.SP.1

Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association. **(8.SP.1)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections:

This Cluster is connected to:

Grade 8 Critical Area of Focus #1: Formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations.

Explanations and Examples:

Bivariate data refers to two variable data, one to be graphed on the x-axis and the other on the *y*-axis. Students represent measurement (numerical) data on a scatter plot, recognizing patterns of association. These patterns may be linear (positive, negative or no association) or non-linear.

Students build on their previous knowledge of scatter plots examine relationships between variables.

They analyze scatterplots to determine positive and negative associations, the degree of association, and type of association. Students examine outliers to determine if data points are valid or represent a recording or measurement error. Students can use tools such as those at the National Center for Educational Statistics to create a graph or generate data sets.

Examples:

Data for 10 students' Math and Science scores are provided in the chart. Describe the association between the Math and Science scores.

Student	1	2	3	4	5	6	7	8	9	10
Math	64	50	85	34	56	24	72	63	42	93
Science	68	70	83	33	60	27	74	63	40	96

Data from a local fast food restaurant is provided showing the number of staff members and the average time for filling an order are provided in the table below. Describe the association between the number of staff and the average time for filling an order.

Number of staff	3	4	5	6	7	8
Average time to fill order (seconds)	180	138	120	108	96	84

The chart below lists the life expectancy in years for people in the United States every five years from 1970 to 2005. What would you expect the life expectancy of a person in the United States to be in 2010, 2015, and 2020 based upon this data? Explain how you determined your values.

Date	1970	1975	1980	1985	1990	1995	2000	2005
Life Expectancy (in years)	70.8	72.6	73.7	74.7	75.4	75.8	76.8	77.4

Instructional Strategies:

Building on the study of statistics using univariate data in Grades 6 and 7, students are now ready to study bivariate data. Students will extend their descriptions and understanding of variation to the graphical displays of bivariate data.

Scatter plots are the most common form of displaying bivariate data in Grade 8. Provide scatter plots and have students practice informally finding the *line of best fit*. Students should create and interpret scatter plots, focusing on outliers, positive or negative association, linearity or curvature. By changing the data slightly, students can have a rich discussion about the effects of the change on the graph. Have students use a graphing calculator to determine a linear regression and discuss how this relates to the graph.

Students should informally draw a line of best fit for a scatter plot and informally measure the strength of fit. Discussion should include "What does it mean to be above the line, below the line?"

The study of the line of best fit ties directly to the algebraic study of slope and intercept.

Students should interpret the slope and intercept of the line of best fit in the context of the data.

Then students can make predictions based on the line of best fit.

Resources/Tools:

For detailed information see Learning Progression for Statistics & Probability.

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.SP.A.1
    - Birds' Eggs
    - Texting and Grades I
    - Hand span and height
    - Animal Brains

Common Misconceptions:

Students may believe bivariate data is only displayed in scatter plots. Grade 8.SP.3 in this cluster provides the opportunity to display bivariate, categorical data in a table.

In general, students think there is only one correct answer in mathematics. Students may mistakenly think their lines of best fit for the same set of data will be exactly the same. Because students are informally drawing lines of best fit, the lines will vary slightly. To obtain the exact line of best fit, students would use technology to find the line of regression.

Domain: Statistics and Probability (SP)

◆**Cluster:** *Investigate patterns of association in bivariate data.*

Standard: Grade 8.SP.2

Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line. **(8.SP.2)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

Connections: See 8.SP.1; 8.EE.5; 8.F.3

Explanations and Examples:

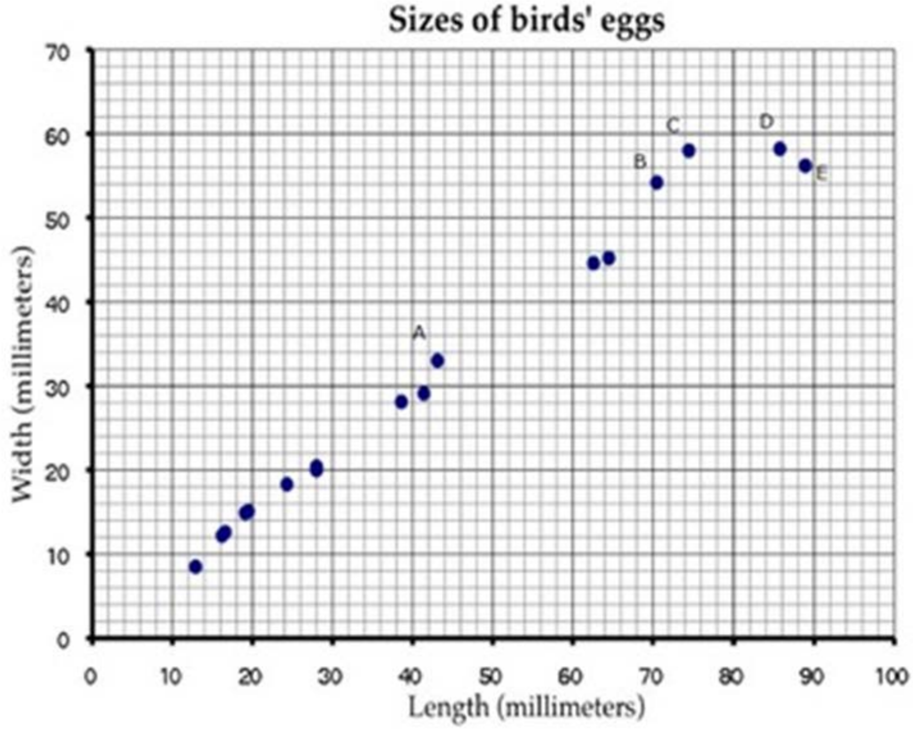
Students understand that a straight line can represent a scatter plot with linear association. The most appropriate linear model is the line that comes closest to most data points. The use of linear regression is not expected.

Examples:

The capacity of the fuel tank in a car is 13.5 gallons. The table below shows the number of miles traveled and how many gallons of gas are left in the tank. Describe the relationship between the variables. If the data is linear, determine a line of best fit. Do you think the line represents a good fit for the data set? Why or why not? What is the average fuel efficiency of the car in miles per gallon?

Miles Traveled	0	75	120	160	250	300
Gallons Used	0	2.3	4.5	5.7	9.7	10.7

This scatter diagram shows the lengths and widths of the eggs of some American birds.



1. A biologist measured a sample of one hundred Mallard duck eggs and found they had an average length of 57.8 millimeters and average width of 41.6 millimeters. Use an X to mark a point that represents this on the scatter diagram.

2. What does the graph show about the relationship between the lengths of birds' eggs and their widths?

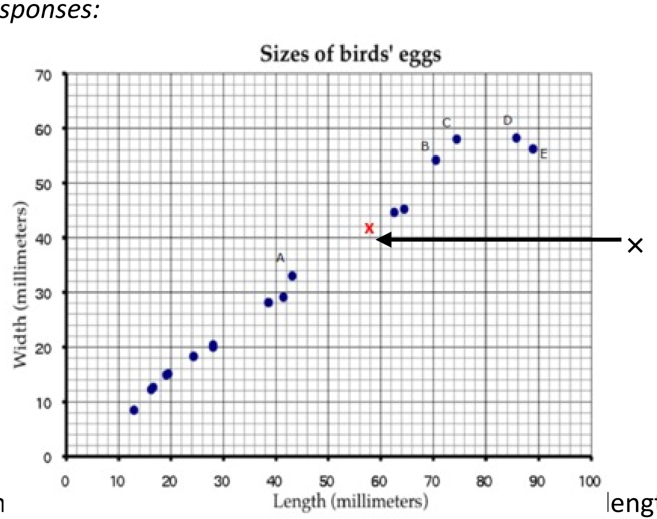
3. Another sample of eggs from similar birds has an average length of 35 millimeters. If these bird eggs follow the trend in the scatter plot, about what width would you expect these eggs to have, on average?

4. Describe the differences in shape of the two eggs corresponding to the data points marked C and D in the plot.

5. Which of the eggs A, B, C, D, and E has the greatest ratio of length to width? Explain how you decided.

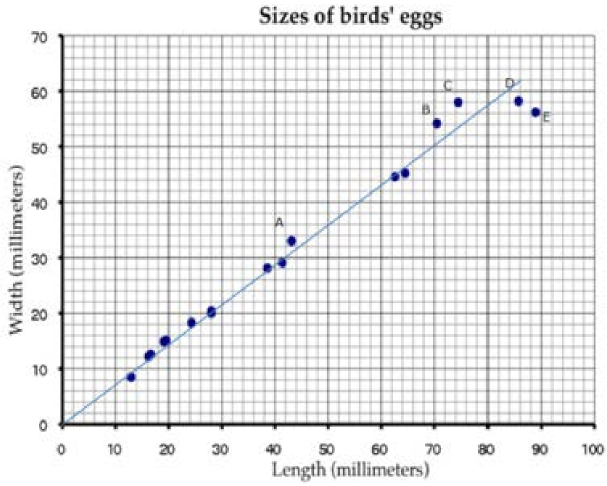
Sample Responses:

1.



There seen length and width of the eggs.

3. The line below appears to fit the data fairly well: (Connects to 8.SP.2)



Since it passes through (0,0) and (50,36), it's slope is $\frac{36}{50} = 0.72$, so the equation of the line is *y*=0.72*x*

If x = 35, then our line would predict that $y = 0.72 \cdot 35 = 25.2$. So we would expect the width of these eggs to be, on average, about 25 mm. Answers using different lines can vary up to 1 mm in either direction.

4. Without reading off precise numerical values from the plot, we can see that eggs C and D have very nearly the same width, but egg D is about 12 millimeters longer than egg C.

5. First we note that egg E certainly has a higher length-to-width ratio than C or D, since it is both longer and narrowed. Similarly, E has a higher ratio than B because it is significantly longer, and only a tad wider. It is harder to visually identify the difference between A and E, we compute their respective length-to-width ratios numerically, which turn out to be approximately 1.3 for A and 1.6 for E. So E has the greatest ratio of length to width.

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.SP.A.2
  - Birds' Eggs
  - Animal Brains
  - Laptop Battery Charge

Also see engageNY Modules.

Common Misconceptions: See 8.SP.1

Domain: Statistics and Probability (SP)

◆**Cluster:** *Investigate patterns of association in bivariate data.*

Standard: Grade 8.SP.3

Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height*. **(8.SP.3)**

Suggested Standards for Mathematical Practice (MP):

- ✓ MP.2 Reason abstractly and quantitatively.
- ✓ MP.4 Model with mathematics.
- ✓ MP.5 Use appropriate tools strategically.
- ✓ MP.6 Attend to precision.
- ✓ MP.7 Look for and make use of structure.

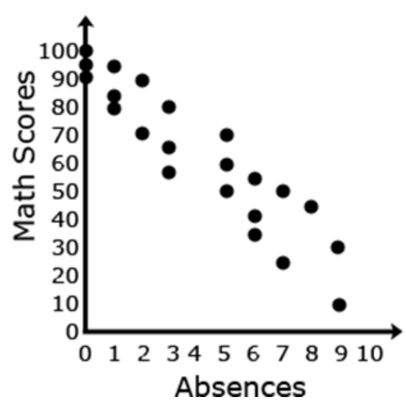
Connections: See 8.SP.1; 8.EE.5; 8.F.3; 8.F.4;

Explanations and Examples:

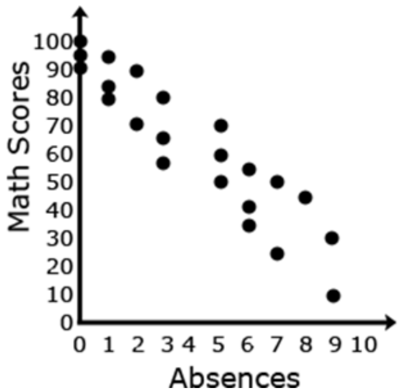
Given data from students' math scores and absences, make a scatterplot.

Absences	Math Scores
3	65
5	50
1	95
1	85
3	80
6	34
5	70
3	56
0	100
7	24
8	45
2	71
9	30
0	95
6	55
6	42
2	90
0	92
5	60
7	50
9	10
1	80

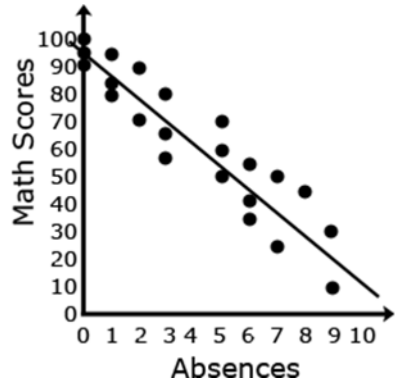
Solution:



Draw a line of best fit, paying attention to the closeness of the data points on either side of the line.



Solution:



From the line of best fit, determine an approximate linear equation that models the given data.

Solution: (about $y = -\frac{25}{3}x + 95$)

Students should recognize that 95 represents the y intercept and $-\frac{25}{3}$ represents the slope of the line.

Students can use this linear model to solve problems. For example, through substitution, they can use the equation to determine that a student with 4 absences should expect to receive a math score of about 62. They can then compare this value to their line.

Instructional Strategies:

Scatter plots are the most common form of displaying bivariate data in Grade 8. Provide scatter plots and have students practice informally finding the *line of best fit*. Students should create and interpret scatter plots, focusing on outliers, positive or negative association, linearity or curvature. By changing the data slightly, students can have a rich discussion about the effects of the change on the graph. Have students use a graphing calculator to determine a linear regression and discuss how this relates to the graph.

Students should informally draw a line of best fit for a scatter plot and informally measure the strength of fit. Discussion should include "What does it mean to be above the line, below the line?"

The study of the line of best fit ties directly to the algebraic study of slope and intercept. Students should interpret the slope and intercept of the line of best fit in the context of the data. Then students can make predictions based on the line of best fit.

Resources/Tools:

Illustrative Mathematics Grade 8 tasks: **Scroll to the appropriate section to find named tasks.**

- 8.SP.A.3
    - US Airports, Assessment Variation

Georgia Department of Education

- "Walk the Graph" - In this task, students will use CBR™ motion detectors to create real-time graphs displaying lines with positive, negative, and zero slopes.

Common Misconceptions: See 8.SP.1

APPENDIX: TABLE 1. Common Addition and Subtraction Situations

Shading taken from OA progression

	Result Unknown	Change Unknown	Start Unknown
Add to	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? 2 + 3 = ?	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? 2 + ? = 5	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? ? + 3 = 5
Taken from	Five apples were on the table. I ate two apples. How many apples are on the table now? 5 – 2 = ?	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? 5 – ? = 3	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? ? – 2 = 3
	Total Unknown	**Addend Unknown**	**Both Addends Unknown[1]**
Put Together/ Take Apart[2]	Three red apples and two green apples are on the table. How many apples are on the table? 3 + 2 = ?	Five apples are on the table. Three are red and the rest are green. How many apples are green? 3 + ? = 5, 5 – 3 = ?	Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? 5 = 0 + 5, 5 = 5 + 0 5 = 1 + 4, 5 = 4 + 1 5 = 2 + 3, 5 = 3 + 2
	Difference Unknown	**Bigger Unknown**	**Smaller Unknown**
Compare[3]	("How many more?" version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? ("How many fewer?" version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? 2 + ? = 5, 5 – 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? 2 + 3 = ?, 3 + 2 = ?	(Version with "more"): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with "fewer"): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? 5 – 3 = ?, ? + 3 = 5

Blue shading indicates the four Kindergarten problem subtypes. Students in grades 1 and 2 work with all subtypes and variants (blue and green). Yellow indicates problems that are the difficult four problem subtypes or variants that students in Grade 1 work with but do not need to master until Grade 2.

[1]These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as.

[2]Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10.

[3]For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

TABLE 2. Common Multiplication and Division Situations

Grade level identification of introduction of problem situations taken from OA progression

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	3 x 6 = ?	3 x ? = 18; 18 ÷ 3 = ?	? x 6 = 18; 18 ÷ 6 = ?
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? *Measurement example.* You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? *Measurement example.* You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? *Measurement example.* You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays[4], Area[5]	There are 3 rows of apples with 6 apples in each row. How many apples are there? *Area example.* What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? *Area example.* A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? *Area example.* A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs $6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? *Measurement example.* A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs $18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? *Measurement example.* A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat? *Measurement example.* A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$, and $p \div a = ?$	$? \times b = p$, and $p \div b = ?$

Multiplicative compare problems appear first in Grade 4 (green), with whole number values and with the "times as much" language from the table. In Grade 5, unit fractions language such as "one third as much" may be used. Multiplying and unit language change the subject of the comparing sentence ("A red hat costs n times as much as the blue hat" results in the same comparison as "A blue hat is 1/n times as much as the red hat" but has a different subject.)

TABLE 3. The Properties of Operations

Name of Property	Representation of Property	Example of Property, Using Real Numbers
Properties of Addition		
Associative	$(a+b)+c=a+(b+c)$	$(78+25)+75=78+(25+75)$
Commutative	$a+b=b+a$	$2+98=98+2$
Additive Identity	$a+0=a\ \ and\ \ 0+a=a$	$9875+0=9875$
Additive Inverse	For every real number a, there is a real number $-a$ such that $a+-a=-a+a=0$	$-47+47=0$
Properties of Multiplication		
Associative	$(a\times b)\times c=a\times(b\times c)$	$(32\times 5)\times 2=32\times(5\times 2)$
Commutative	$a\times b=b\times a$	$10\times 38=38\times 10$
Multiplicative Identity	$a\times 1=a\ \ and\ \ 1\times a=a$	$387\times 1=387$
Multiplicative Inverse	*For every real number a,* $a\neq 0$, *there is a real number* $\frac{1}{a}$ *such that* $a\times\frac{1}{a}=\frac{1}{a}\times a=1$	$\frac{8}{3}\times\frac{3}{8}=1$
Distributive Property of Multiplication over Addition		
Distributive	$a\times(b+c)=a\times b+a\times c$	$7\times(50+2)=7\times 50+7\times 2$

(Variables a, b, and c represent real numbers.)
Excerpt from NCTM's ***Developing Essential Understanding of Algebraic Thinking****, grades 3-5 p. 16-17*

TABLE 4. The Properties of Equality

Name of Property	Representation of Property	Example of property
Reflexive Property of Equality	$a = a$	$3{,}245 = 3{,}245$
Symmetric Property of Equality	$If\ a = b, then\ b = a$	$2 + 98 = 90 + 10, then\ 90 + 10 = 2 + 98$
Transitive Property of Equality	$If\ a = b\ and\ b = c, then\ a = c$	$If\ 2 + 98 = 90 + 10\ and\ 90 + 10 = 52 + 48$ *then* $2 + 98 = 52 + 48$
Addition Property of Equality	$If\ a = b, then\ a + c = b + c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} + \frac{3}{5} = \frac{2}{4} + \frac{3}{5}$
Subtraction Property of Equality	$If\ a = b, then\ a - c = b - c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} - \frac{1}{5} = \frac{2}{4} - \frac{1}{5}$
Multiplication Property of Equality	$If\ a = b, then\ a \times c = b \times c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \times \frac{1}{5} = \frac{2}{4} \times \frac{1}{5}$
Division Property of Equality	$If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$	$If\ \frac{1}{2} = \frac{2}{4}, then\ \frac{1}{2} \div \frac{1}{5} = \frac{2}{4} \div \frac{1}{5}$
Substitution Property of Equality	If $a = b$, then b may be substituted for a in any expression containing a.	$If\ 20 = 10 + 10$ *then* $90 + 20 = 90 + (10 + 10)$

(Variables a, b, and c can represent any number in the rational, real, or complex number systems.)

TABLE 5. The Properties of Inequality

Exactly one of the following is true: $a < b, a = b, a > b$.

$If\ a > b\ and\ b > c\ then\ a > c.$

$If\ a > b, then\ b < a.$

$If\ a > b, then - a < -b.$

$If\ a > b, then\ a \pm c > b \pm c.$

$If\ a > b\ and\ c > 0, then\ a \times c > b \times c.$

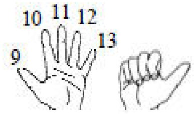
$If\ a > b\ and\ c < 0, then\ a \times c < b \times c.$

$If\ a > b\ and\ c > 0, then\ a \div c > b \div c.$

$If\ a > b\ and\ c < 0, then\ a \div c < b \div c.$

Here a, b, and c stand for arbitrary numbers in the rational or real number systems.

TABLE 6. Development of Counting in K-2 Children

Levels	8 + 6 = 14	14 – 8 = 6
Level 1: Count all	Count All a 1 2 3 4 5 6 7 8 b 1 2 3 4 5 6 1 2 3 4 5 6 7 8 9 10 11 12 13 14 c	Take Away a 1 2 3 4 5 6 7 8 9 10 11 12 13 14 1 2 3 4 5 6 7 8 1 2 3 4 5 6 b c
Level 2: Count on	Count On 8 → 8 9 10 11 12 13 14	To solve 14 – 8 I count on 8 + ? = 14 I took away 8  9 10 11 12 13 14 8 to 14 is 6 so 14 – 8 = 6
Level 3: Recompose Make a ten (general): one addend breaks apart to make 10 with the other addend Make a ten (from 5's within each addend)	Recompose: Make a Ten 10 + 4 10 + 4	14 – 8: I make a ten for 8 + ? = 14 8 + 2 + 4 6 8 + 6 = 14
Doubles ± n	6 + 8 = 6 + 6 + 2 = 12 + 2 = 14	

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on; it makes subtraction as easy as addition.

Beginning--A child can count very small collections (1-4) collection of items and understands that the last word tells "how many" even. Beyond on small numbers the number words may be said without the lack of numerical understanding. This is often referred to as rote counting.

Level 1—The child uses one to one correspondence to find the number of objects in two sets. Even if the quantity is known for the first set, the child will start with the first set and continue counting on into the second set. The child begins the count with one. This also connects to Piaget's Hierarchical Inclusion – that in order to have 7 – you have to have 6, 5, 4, etc.

Level 2 – At this level the student begins the counting, starting with the known quantity of the first set and "counts on" from that point in the sequence to establish how many. This method is used to find the total in two sets. This counting is not rote. This level of counting requires several connections between cardinality and counting meanings of the number words.

Level 3 - At this level the student begins using known facts to solve for unknown facts. For example, the student uses "make a ten" where one addend breaks apart to make 10 with another addend OR a doubles plus/minus one strategy. Students begin to implicitly use the properties of operations.

Table 7. Cognitive Rigor Matrix/Depth of Knowledge (DOK)

Kansas Math Standards require high-level cognitive demand asking students to demonstrate deeper conceptual understanding through the application of content knowledge and skills to new situations and sustained tasks. For each Assessment Target the depth(s) of knowledge (DOK) that the student needs to bring to the item/task will be identified, using the Cognitive Rigor Matrix shown below.

Depth of Thinking (Webb)+ Type of Thinking (Revised Bloom)	DOK Level 1 Recall & Reproduction	DOK Level 2 Basic Skills & Concepts	DOK Level 3 Strategic Thinking & Reasoning	DOK Level 4 Extended Thinking
Remember	• Recall conversions, terms, facts			
Understand	• Evaluate an expression • Locate points on a grid or number on number line • Solve a one-step problem • Represent math relationships in words, pictures, or symbols	• Specify, explain relationships • Make basic inferences or logical predictions from data/observations • Use models/diagrams to explain concepts • Make and explain estimates	• Use concepts to solve non-routine problems • Use supporting evidence to justify conjectures, generalize, or connect ideas • Explain reasoning when more than one response is possible • Explain phenomena in terms of concepts	• Relate mathematical concepts to other content areas, other domains • Develop generalizations of the results obtained and the strategies used and apply them to new problem situations
Apply	• Follow simple procedures • Calculate, measure, apply a rule (e.g., rounding) • Apply algorithm or formula • Solve linear equations • Make conversions	• Select a procedure and perform it • Solve routine problem applying multiple concepts or decision points • Retrieve information to solve a problem • Translate between representations	• Design investigation for a specific purpose or research question • Use reasoning, planning, and supporting evidence • Translate between problem & symbolic notation when not a direct translation	• Initiate, design, and conduct a project that specifies a problem, identifies solution paths, solves the problem, and reports results
Analyze	• Retrieve information from a table or graph to answer a question • Identify a pattern/trend	• Categorize data, figures • Organize, order data • Select appropriate graph and organize & display data • Interpret data from a simple graph • Extend a pattern	• Compare information within or across data sets or texts • Analyze and draw conclusions from data, citing evidence • Generalize a pattern • Interpret data from complex graph	• Analyze multiple sources of evidence or data sets
Evaluate			• Cite evidence and develop a logical argument • Compare/contrast solution methods • Verify reasonableness	• Apply understanding in a novel way, provide argument or justification for the new application
Create	• Brainstorm ideas, concepts, problems, or perspectives related to a topic or concept	• Generate conjectures or hypotheses based on observations or prior knowledge and experience	• Develop an alternative solution • Synthesize information within one data set	• Synthesize information across multiple sources or data sets • Design a model to inform and solve a practical or abstract situation

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